

# Forward Bhabha Scattering – Theoretical Problems



Tord Riemann, DESY, Zeuthen

FCAL Collaboration Meeting

Tel-Aviv University, 18–19 Sep 2005

A project in collaboration with

**Michal Czakon** Univ. Würzburg (and Katowice)

**Janusz Gluza** DESY (and Katowice)

See also: • NPB(PS) 135 (2004), hep-ph/0406203

• PRD 71 (2005), hep-ph/0412164

• <http://www-zeuthen.desy.de/theory/research/bhabha/>

- **What do we need?**  $\rightarrow 10^{-4}$  for  $d\sigma/d\cos\vartheta$  at small  $\vartheta$
- **Higher Order Corrections – Status**
- **Summary**

## The Physics Needs

For more details see e.g.:

**K. Mönig, "Bhabha scattering at the ILC"**

talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005

[/afs/afh.de/user/m/moenig/public/www/bhabha\\_ilc.pdf](/afs/afh.de/user/m/moenig/public/www/bhabha_ilc.pdf)

**ILC** – Need Bhabha cross-sections with **3–4 significant digits**.

Why?

- **ILC**:  $e^+e^- \rightarrow W^+W^-, f\bar{f}$  with  $O(10^6)$  events  $\rightarrow 10^{-3}$
- **GigaZ**: relevant physics derived from  $Z \rightarrow \text{hadrons}, l^+l^-$ , the latter with  $O(10^8)$  events  $\rightarrow 10^{-4}$ , the systematic errors (**luminosity!**) influence this
- **ILC**:  $e^+e^- \rightarrow e^+e^-$ , a probe for New Physics with  $O(10^5)$  events/year  $\rightarrow 10^{-3}$

Conclude: will need  $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

**The luminosity comes from very forward Bhabha scattering.**

## Some Kinematics

Need a cross-section prediction with **5 significant digits**.

**Perturbative orders:**

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

**Kinematics:**

$$\sqrt{s} = 90 \dots 1000 \text{ GeV}$$

$$\vartheta = 26 \dots 82 \text{ mrad}$$

$$\cos \vartheta \sim 0.999\ 66 \dots 0.996\ 64$$

$$T = \frac{s}{2}(1 - \beta^2 \cos \vartheta) > 1.36 \text{ GeV}|_{GigaZ}, 42.2 \text{ GeV}|_{ILC500}$$

**Conclude:**

- $t$ -channel exchange of  $\gamma$  dominates everything else
- $m_e^2/s < m_e^2/T \leq 10^{-5} \dots 10^{-7}$
- **Calculate:** 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung

The **1-loop electroweak** corrections (plus some leading higher order terms) are well-known, with rising technical precision, since about 1988/91.

**Böhm, Denner, Hollik; Bardin, TR 1991**

→ Fig. 2004 **Lorca, TR**

## 2-loop Bhabha scattering: What to be done?

- **Calculate:**

$$\sigma = (2 \rightarrow 2) + (2 \rightarrow 3) + (2 \rightarrow 4)$$

$$\begin{aligned} \sigma = & |\mathbf{Born} + \mathbf{1-loop} + \mathbf{2-loop}|^2 \\ & + |(\mathbf{Born} + \mathbf{1-\gamma}) + (\mathbf{1-loop} + \mathbf{1-\gamma})|^2 \\ & + |(\mathbf{Born} + \mathbf{2-\gamma})|^2 \end{aligned}$$

- Do **not** include:  $|\mathbf{2-loop}|^2$   
 $|(\mathbf{1-loop} + \mathbf{1-\gamma})|^2$

Status by end of 2004

## Established: $10^{-3}$ MC programs for LEP, ILC

Introduction to **NLLBHA** by Trentadue and to **BHLUMI** by Jadach in:  
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP",  
hep-ph/0306083 [1]

- **BHLUMI** v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]
- **NLLBHA**: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- **SAMBHA**: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2] → **Conclude**:

The nonlogarithmic  $O(\alpha^2)$  terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

**Beware:**

$$m_e, m_\gamma, (d-4), E_\gamma$$

## Results: Numerical comparison in all $ff$

**Bhabha**  $e^-e^+ \rightarrow e^-e^+(\gamma)$  at LC:  $\sqrt{s} = 500$  GeV,  $E_{\max}(\gamma_{\text{soft}}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$[\frac{d\sigma}{d\cos\theta}]_{\text{Born}}$ (pb)	$[\frac{d\sigma}{d\cos\theta}]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
-0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	aITALC
-0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	Feyn.Arts
-0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	aITALC
-0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	Feyn.Arts
-0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
+0.0	0.59814 23072 50330 3	0.54667 71794 69423 1	aITALC
+0.0	0.59814 23072 50329 4	0.54667 71794 69421 8	Feyn.Arts
+0.0	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
+0.9	0.18916 03223 32270 6 · 10 <sup>3</sup>	0.17292 83490 66507 2 · 10 <sup>3</sup>	aITALC
+0.9	0.18916 03223 32270 6 · 10 <sup>3</sup>	0.17292 83490 66508 0 · 10 <sup>3</sup>	Feyn.Arts
+0.9	0.18916 03223 31848 5 · 10 <sup>3</sup>	0.17292 83490 61347 4 · 10 <sup>3</sup>	$m_e = 0$
+0.9999	0.20842 90676 46142 9 · 10 <sup>9</sup>	0.19140 17861 11341 6 · 10 <sup>9</sup>	aITALC
+0.9999	0.20842 90676 46436 4 · 10 <sup>9</sup>	0.19140 17861 11979 0 · 10 <sup>9</sup>	Feyn.Arts

**Great independent agreement up to 14 digits! : limit in double precision**

Previous agreement with FeynArts: 11 digits [hep-ph/0307132](https://hep-ph/0307132), SANC: 10 digits [hep-ph/0207156](https://hep-ph/0207156)

Thanks to **T. Hahn**, numbers supplied with FeynArts + FormCalc + LoopTools

**Table 2:**

The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.

$M_Z = 91.16 \text{ GeV}$ ,  $m_t = 150 \text{ GeV}$ ,  $M_H = 100 \text{ GeV}$ .

Upper rows:  $DZ$ , lower rows:  $H$ .

$\delta_m$ : largest relative deviation in per mille.

$\sqrt{s}$ (GeV)	60	89	91.16	93	200
$\theta$					
15°	129.6	65.11	57.93	49.00	11.82
	129.6	65.11	57.93	49.00	11.82
45°	1.451	1.376	1.755	.4833	11.67
	1.451	1.377	1.756	.4837	11.68
60°	.4303	.6124	1.125	.2697	.03075
	.4305	.6129	1.126	.2699	.03077
75°	.1717	.3627	.8718	.2232	.01072
	.1718	.3630	.8720	.2233	.01072
90°	.08873	.2768	.7790	.2088	.004862
	.08876	.2769	.7787	.2087	.004855
105°	.05917	.2690	.8082	.2157	.002858
	.05918	.2690	.8074	.2157	.002853
120°	.04906	.3053	.9323	.2429	.002077
	.04906	.3051	.9309	.2426	.002074
135°	.04671	.3626	1.111	.2838	.001743
	.04672	.3624	1.109	.2833	.001742
165°	.04839	.4638	1.425	.3590	.001539
	.04839	.4635	1.422	.3584	.001540
$\delta_m$	0.6	0.8	1.8	2.0	1.7

## Bhabha scattering

Bardin, Hollik, T.R., Z.PhysikC49(1991)485

Status 2005

Know the constant term ( $m_e = 0$ )  
from 2-loop Bhabha scattering

A. Penin, **Two-Loop Corrections to Bhabha Scattering**, hep-ph/0501120 v.3, → PRL

Transform the **massless 2-loop results** of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by  $D = 4 - 2\epsilon$  into the **on-mass-shell renormalization** with  $m_e \rightarrow 0$  and IR regulation by  $\lambda = m_\gamma \neq 0$

Use **IR-properties of amplitudes** (see Penin):

[A ] **Exponentiation** of the IR logarithms (Sudakov 1956,...)

[B ] **Factorization** of the collinear logarithms into external legs (Frenkel, Taylor 1976)

[C ] **Non-renormalization** of the IR exponents (YFS 1961, ....)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

**If all this is correct, the constant term in  $m_e$  is known for the MCs** (but the radiative one-loops with 5-point functions).

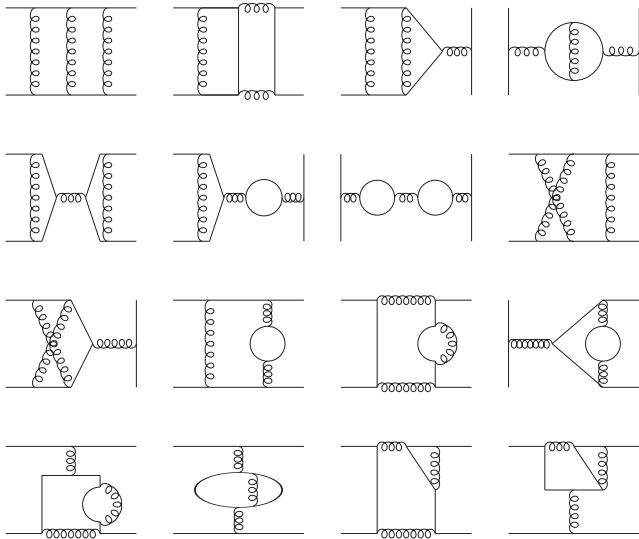


$$m = 0$$

### Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute  $e^+e^- \rightarrow \mu^+\mu^-$ , since it's closely related but has less diagrams.

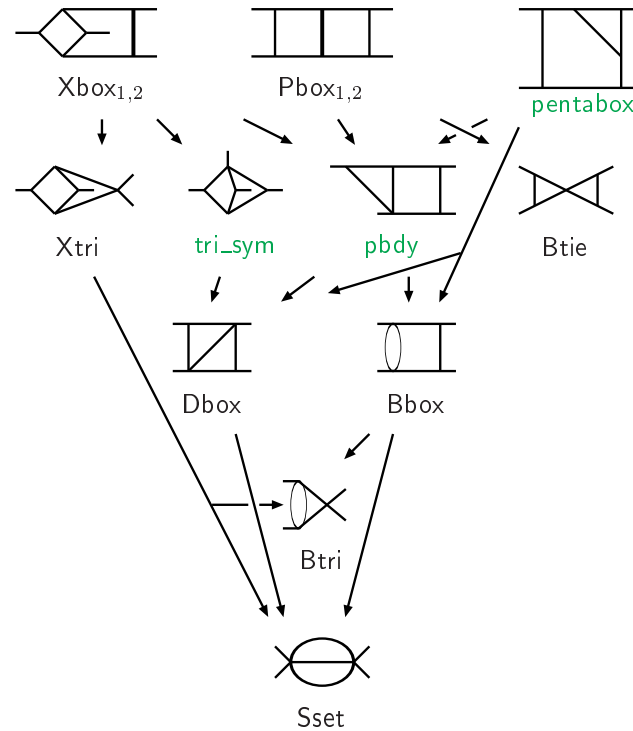
There are 47 QED diagrams contributing to  $e^+e^- \rightarrow \mu^+\mu^-$ .



In this calculation all particles massless.

The Bhabha scattering amplitude can be obtained from  $e^+e^- \rightarrow \mu^+\mu^-$  simply by summing it with the crossed amplitude (including fermi minus sign).

### Two-loop integral inheritance chart



## The massive 2-loop contributions

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \rightarrow e^+e^-)$$

We are using a scheme with

- (1)  $m_e \neq 0$  (**good** with the MC's)
- (2)  $m_\gamma = 0$  (**bad** with the MC's;  $\rightarrow$  **Mastrolia, Remiddi 2003**)
- (3) **dim.reg.** for UV and **IR** divergences

Also:

**Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij:** all but many **2-boxes**  
**Heinrich, Smirnov:** Calculation of selected complicated Feynman integrals

### There are few topologies only:

- 1-loop: 5
- 2-loop self energies: 5 (3 for external legs)
- 2-loop vertices: 5
- 2-loop boxes: 6 → next slide

The many Feynman integrals may be reduced to 'few' **master integrals** by sophisticated methods (**Remiddi-Laporta algorithm, 1996/2000** → **IdSolver** (Czakon 2003) ).

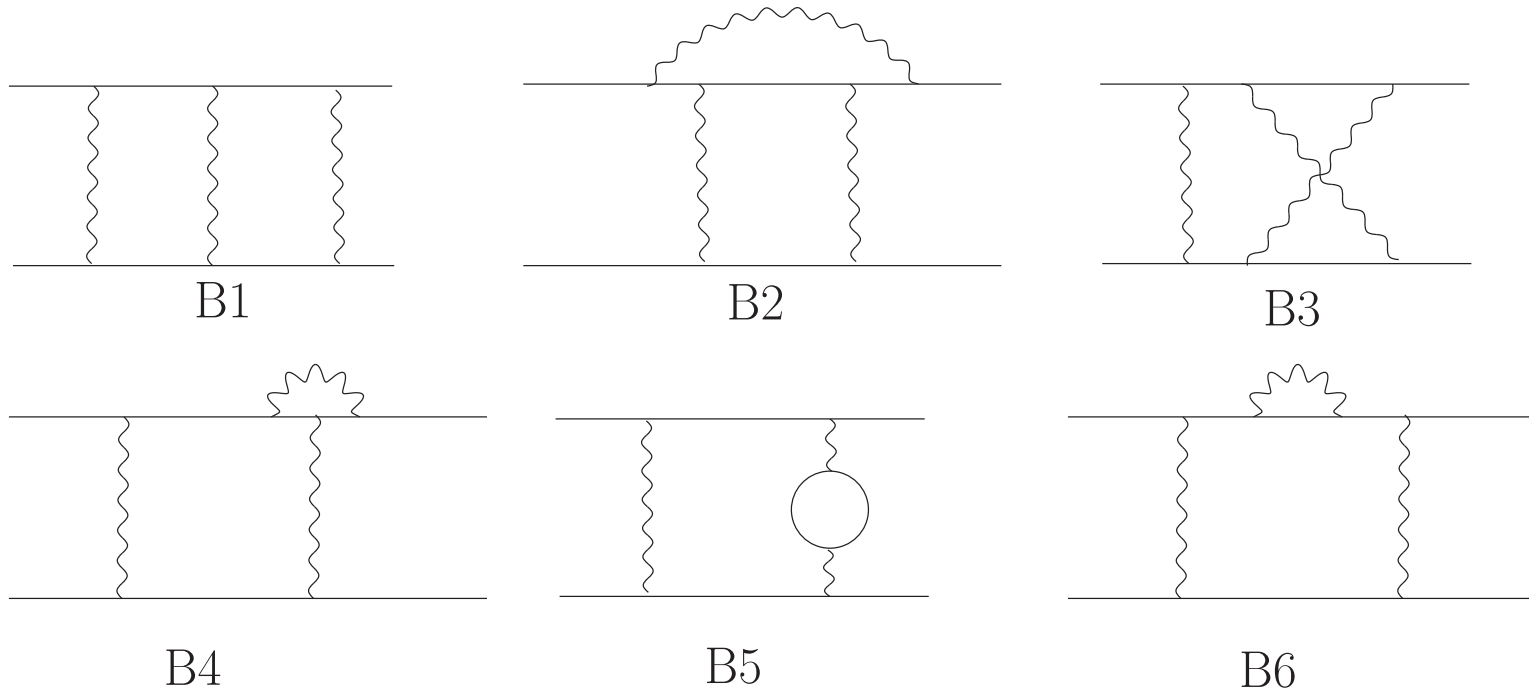
### The massive diagrams (See also webpage)

Assume 3 leptonic flavors, do not look at loops in external legs.

### Not too many QED diagrams:

- Born diagrams: 2
- 1-loop diagrams: 14
- 2-loop diagrams: 154 (with 68 double-boxes)      interfere with Born

## The two-loop box diagrams for massive Bhabha scattering



- **B5**: Completely known (2004)  
Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321  
Czakon, Gluza, Riemann: <http://www-zeuthen.desy.de/.../MastersBhabha.m> (unpubl.)
- **B1–B3**: Few masters known (Smirnov, Heinrich 2002,2004)
- **B4, B6**: Not much known (Czakon et al. 2004)

The basic planar 2-box master of **B1**, **B7|4m**, was a breakthrough

## The two-loop Feynman integrals

One has to solve **many, very complicated** Feynman integrals with  $L = 2$  loops and  $N \leq 7$  internal lines:

$$G(\mathbf{X}) = \frac{1}{(i\pi^{d/2})^2} \int \frac{d^D k_1 d^D k_2 \mathbf{X}}{(q_1^2 - m_1^2)^{\nu_1} \dots (q_j^2 - m_j^2)^{\nu_j} \dots (q_N^2 - m_N^2)^{\nu_N}},$$

$$\mathbf{X} = 1, (k_1 P), (k_1 k_2), (k_2 P), \dots$$

where  $P$  is some external momentum:  $p_1, \dots, p_4$

A **completely numerical approach** might be possible **Passarino 2004**.

For **checks in the Euclidean region** ( $s < 0, t < 0$ ) this has been proven to be a powerful tool **Binoth, Heinrich 2000/03**

We prefer to calculate the integrals analytically (where possible)

Derive a minimal set of so-called **master integrals** and **algebraic expressions** in terms of them for all the other Feynman integrals

So we need a **A table of master integrals**

We use **IdSolver** with the Laporta/Remiddi algorithm:

Derive with integration-by-parts (and Lorentz-invariance) identities a system of *algebraic* equations for the Feynman integrals and solve the system.

- 1-loop: **5** masters (all known)
- 2-loop self energies: **6** masters (all known)
- 2-loop vertices: **19** masters (all known)
- 2-loop boxes: **33** masters  $\rightarrow$  (**O(5) published**, maybe more known) see table

The **calculation of the master integrals** is mainly done with two methods:

- derive and solve (systems of) **differential equations** (with boundary conditions)
- derive and solve (up to 8-dimensional) **Mellin-Barnes integral representations** for single Feynman integrals

From Czakon et al., PRD 71 (2004): 4-point MIs entering basic two-loop box diagrams. An asterisk denotes one-loop MI. MIs with a dagger: know singular parts only

MI	B1	B2	B3	B4	B5	B6	ref.
B714m1	+	-	-	-	-	-	Smirnov:2001cm
B714m1N	+	-	-	-	-	-	Heinrich:2004iq
B714m2	-	+	-	-	-	-	Heinrich:2004iq <sup>†</sup>
B714m2[d1--d3]	-	+	-	-	-	-	
B714m3	-	-	+	-	-	-	Heinrich:2004iq <sup>†</sup>
B714m3[d1--d2]	-	-	+	-	-	-	
B613m1	+	-	+	-	-	-	
B613m1d	+	-	+	-	-	-	
B613m2	-	+	-	+	-	-	
B613m2d	-	+	-	+	-	-	
B613m3	-	-	+	-	-	-	
B613m3[d1--d5]	-	-	+	-	-	-	
B512m1	+	-	+	-	-	-	Czakon:2004tg
B512m2	-	+	-	+	-	+	Sec. III E1 <sup>†</sup>
B512m2[d1--d2]	-	+	-	+	-	+	Sec. III E1 <sup>†</sup>
B512m3	+	-	+	-	-	-	
B512m3[d1--d3]	+	-	+	-	-	-	Sec. III E1 <sup>†</sup>
B513m	-	+	+	+	-	-	
B513m[d1--d3]	-	+	+	+	-	-	
B514m	-	+	+	+	+	-	Bonciani:2003cj
B514md	-	+	+	+	+	-	Sec. III E
B412m <sup>*</sup>	-	-	-	+	+	+	'tHooft:1972fi,Bonciani:2003cj
total = 33+1 <sup>*</sup>	9	15	22	11+1 <sup>*</sup>	2+1 <sup>*</sup>	3+1 <sup>*</sup>	

## A simple example: A class of scalar self-energy integrals

$$\text{SE312m}(a, b, c, d) = -\frac{e^{2\epsilon\gamma_E}}{\pi^D} \int \frac{d^D k_1 d^D k_2 (k_1 k_2)^{-d}}{[(k_1 + k_2 - p)^2 - m^2]^b [k_1^2]^a [k_2^2 - m^2]^c}.$$

The two Master Integrals are:

$$\text{SE312m} = \text{SE312m}(1, 1, 1, 0)$$

$$\text{SE312md} = \text{SE312m}(1, 1, 2, 0)$$

In [Bonciani et al. 2003](#) it is used instead as a master integral:

$$\text{SE312mN} = \text{SE312m}(1, 1, 1, -1)$$

By an algebraic relation, valid for  $m^2 = 1$  and  $p^2 = s$ ,

$$\text{SE312md} = \frac{-(1+s) + \epsilon(2+s)}{s-4} \text{SE312m} + \frac{2(1-\epsilon)}{s-4} (\text{T111m}^2 + 3 \text{SE312mN}),$$

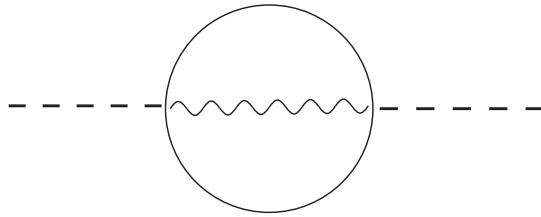
one may derive then SE312md.

Because the integral SE312md is one of our masters, we reproduce it here explicitly:

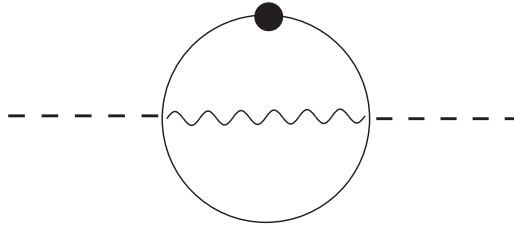
$$\text{SE312md}(x) = \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left( \frac{1-\zeta_2}{2} + \frac{1+x}{1-x} H[0, x] + \frac{1+x^2}{(1-x)^2} H[0, 0, x] \right) + \mathcal{O}(\epsilon).$$

At our webpage, there is a file with all the master integrals we have determined so far.

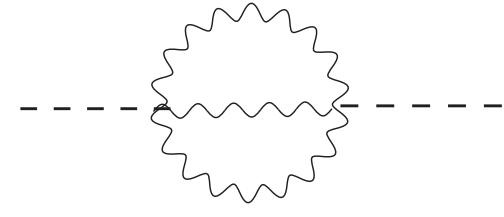




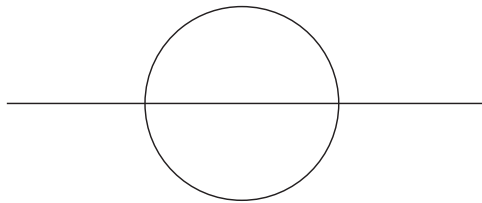
SE3l2m



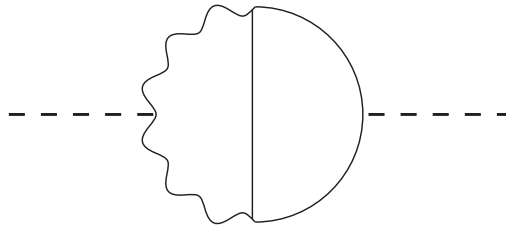
SE3l2md



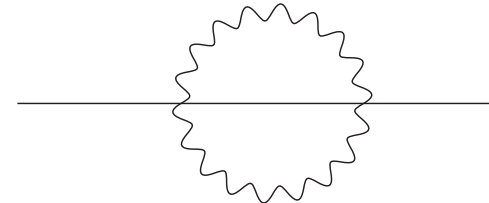
SE3l0m



SE3l3m

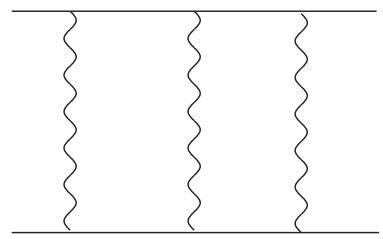


SE5l3m

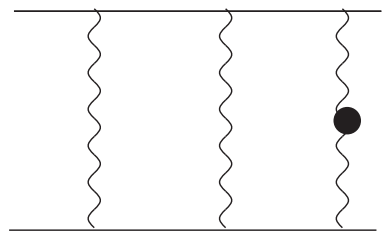


SE3l1m

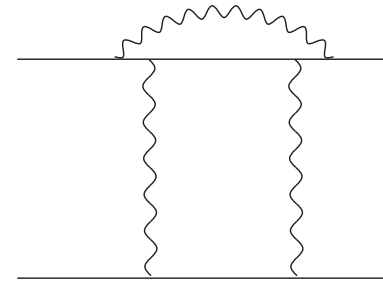
The six two-loop 2-point MIs. External solid (dashed) lines describe on (off) -shell momenta.



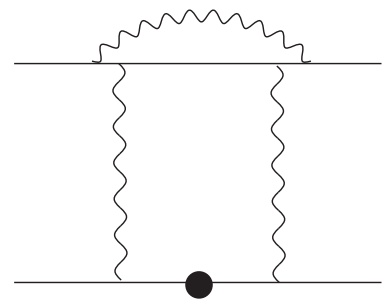
B7l4m1



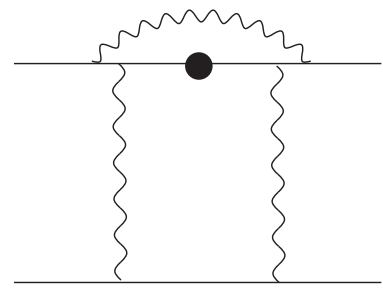
B7l4m1d



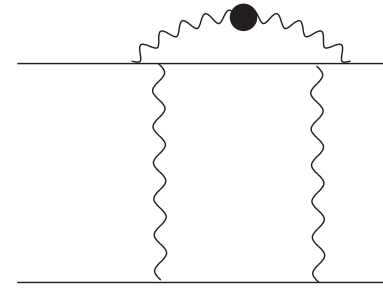
B7l4m2



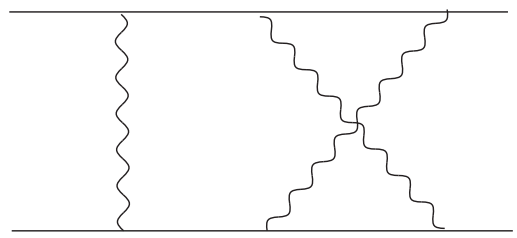
B7l4m2d1



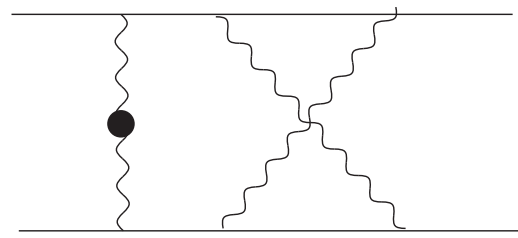
B7l4m2d2



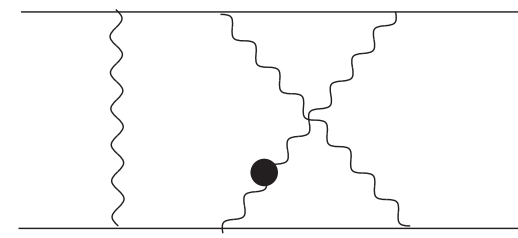
B7l4m2d3



B7l4m3



B7l4m3d1

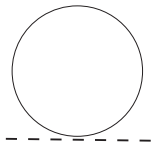


B7l4m3d2

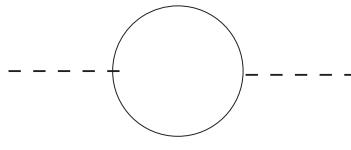
The nine two-loop box MIs with seven internal lines.

The simplest diagram is the **tadpole**:

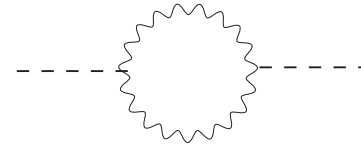
$$\begin{aligned} T_{111m} &= \frac{e^{\epsilon\gamma_E}}{i\pi^{D/2}} \int \frac{d^D q}{q^2 - 1} \\ &= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right) \epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right) \epsilon^2 + \dots \end{aligned}$$



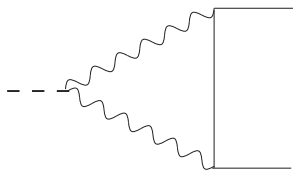
T111m



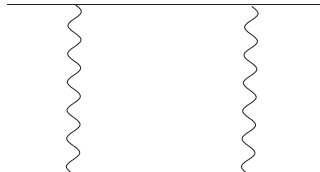
SE2l2m



SE2l0m



V3l1m



B4l2m

## How to calculate 2-loop Bhabha masters?

- Self-energies and vertices and (very few) 2-boxes:  
use **differential equations** and **Harmonic Polylogarithms**, introduced by Remiddi, Vermaseren, plus ... )
- Some 7-line 2-boxes  
use **Mellin-Barnes technique**, sum up **multiple series**, use numerical checks in Euclidean space ( $s, t$  negative)
- For the unsolved 2-boxes:  
**Combination of both methods**: present study

There are other methods not used here:

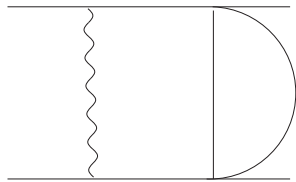
**difference equations**

**pure numerical approaches**

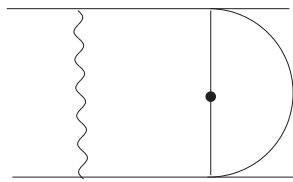
...

## The 2-boxes with 5 lines

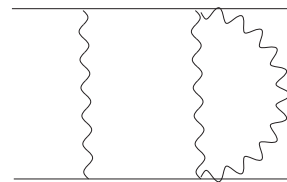
The completely known 2-boxes with 5 lines are B5l4m (Bonciani et al., Czakon et al. 2004), B5l2m1 (Czakon et al. 2004) :



B5l4m1

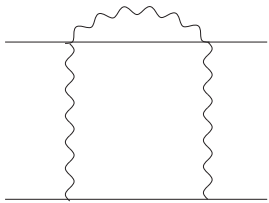


B5l4m1d1

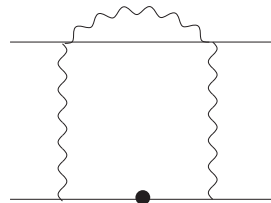


B5l2m1

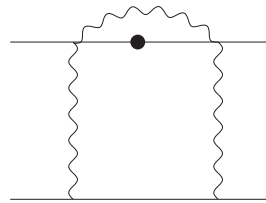
The divergent parts of the B5l2m2 and B5l2m3 type are known (Czakon et al. 2004):



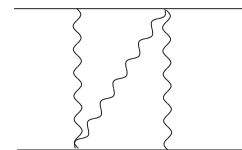
B5l2m2



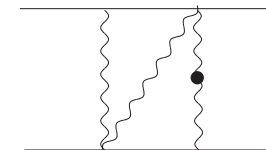
B5l2m2d1



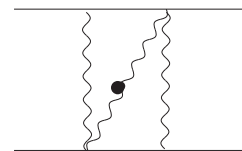
B5l2m2d2



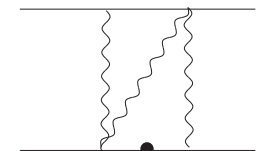
B5l2m3



B5l2m3d1

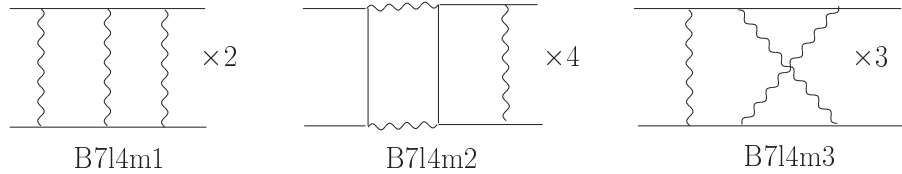


B5l2m3d2

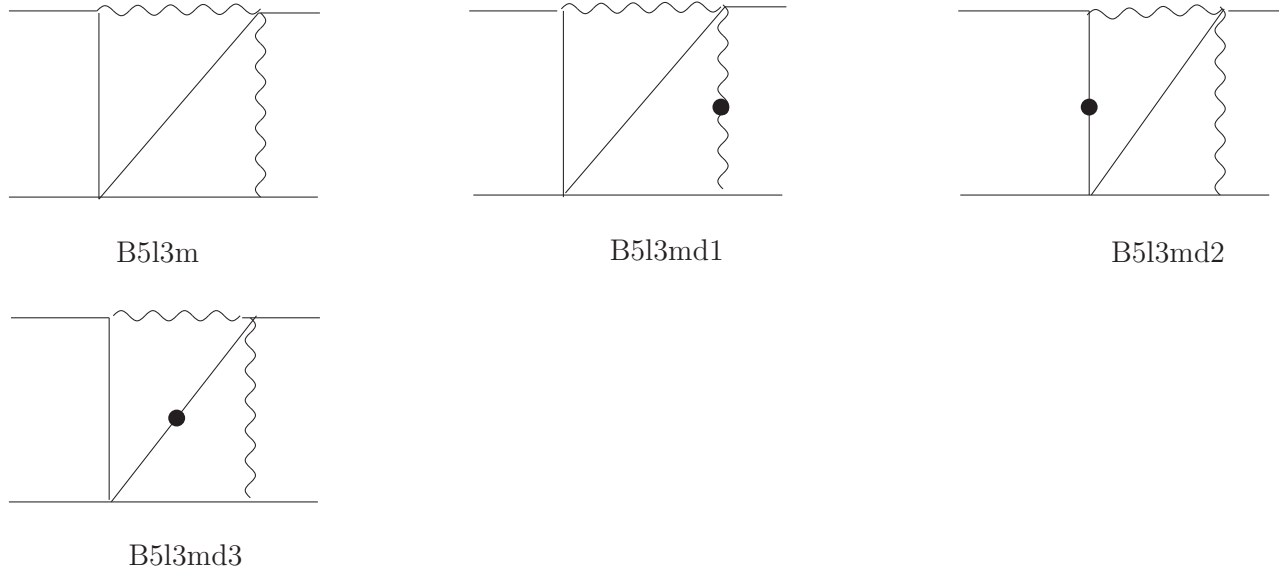


B5l2m3d3

## B5l3m: The divergences in $D - 4 = -2\epsilon$



The **B5l3m** boxes, contribute to **B2** (2nd planar 2-box) (shrink two lines...)



The **B5l3md2** topology appears twice as a master  
but the **B5l3md1** does not!

## The B5l3m topology: Gross features

$$MB5l3m[x, y] = \text{Sum}[B5l3m[k, x, y] * ep^k, k, 0, 1]; \quad (1)$$

$$MB5l3md1[x, y] = \text{Sum}[B5l3md1[k, x, y] * ep^k, k, -2, 1]; \quad (2)$$

$$MB5l3md2[x, y] = \text{Sum}[B5l3md2[k, x, y] * ep^k, k, -2, 1]; \quad (3)$$

$$MB5l3md2a[x, y] = \text{Sum}[B5l3md2a[k, x, y] * ep^k, k, -2, 1]; \quad (4)$$

$$MB5l3md3[x, y] = \text{Sum}[B5l3md3[k, x, y] * ep^k, k, -1, 1]; \quad (5)$$

Note:

- B5l3m – the basic master is finite
- B5l3md2 – use 4-dim. MB-Representation
- B5l3md2' – the same, but ( $s \leftrightarrow t$ )
- B5l3md1, B5l3md3 – system of 2 coupled differential eqns

Only BLB5l3md1 has  $1/\epsilon^2$  (so decouples), and last step is the two  $1/\epsilon$  coefficients of B5l3md1 and B5l3md3.

The first one is found by algebraic manipulations (see Czakon et al. LCWS Paris 2004), the second then fulfils a diff.eqn

## Differential equations

$$\frac{\partial B5l3md3[-1]}{\partial x} = \frac{1+x^2}{x(1-x^2)} B5l3md3[-1] - \frac{yH[0,y]}{(1-x^2)(1-y^2)} \quad (6)$$

with  $s = -(1-x)^2/x$ ,  $t = -(1-y)^2/y$

Solution:

$$B5l3md3[-1] = -\frac{xy}{(-1+x^2)(-1+y^2)} H[0,x]H[0,y] \quad (7)$$

with

$$H[0,x] = \ln(x) \quad (8)$$

The coefficients in the equation are of the form

$$\frac{A_1}{x-B_1} + \frac{A_2}{x-B_2} + \dots \quad (9)$$

One may derive (systems of ) differential equations for the masters, with inhomogeneity composed of simpler masters (Kotikov, Laporta, Remiddi)



$$\frac{\partial M_n}{\partial x} = A(x, y) M_n + I(x, y) \quad (10)$$

$$I(x, y) = \sum_{k=0, n-1} c_k M_k \quad (11)$$

Expand in  $\epsilon$  ( $D = 4 - 2\epsilon$ ):

$$M_n = \sum_{i=-2, i_m} M_{n,i} \epsilon^i \quad \text{etc.} \quad (12)$$

General solution for homogeneous eqn. ( $M'_h = A M_h$ ):

$$M'_h / M_h = A \quad (13)$$

$$\int (M'_h / M_h) = \ln M_h = \int A \quad (14)$$

$$= \int \sum \frac{a_i}{x - x_i} \sim \ln(x - x_i) \quad (15)$$

so:

$$M_h \sim \text{Polynomials} \quad (16)$$

Then the inhomogeneous solution is:

$$M(x, y) = M_h(x, y) \left( \text{const}(y) + \int \frac{I(x', y)}{M_h(x', y)} \right) \quad (17)$$

Result:

nested integrals over 'simple' iterated integrands

The method leads to the **HPLs**  $H(\{a\}, x)$  and **GPLs**  $G(\{a(y)\}, x)$

**Harmonic Polylogarithms**  $H(x)$

$$H[-1, 1, x] = \int_0^x \frac{dx''}{(1+x'')} \int_0^{x''} \frac{dx'}{(1-x')} \quad (18)$$

$$= Li_2\left(\frac{1+x}{2}\right) + \dots \quad (19)$$

**Generalized Harmonic Polylogarithms**  $G(x, y) \dots$

but it works only if the **polynomial structure is simple** enough for a solution with this class of functions

Method is absolutely 'super' if it works.

**But:**

one needs complete chains of masters of lower complexity, and there are **systems of up to 6 (!) potentially coupled 1st order equations**

## Mellin-Barnes representations

Boos, Davydychev 1991, Smirnov 1999, Tausk 1999, Smirnov book 2004

$$\frac{1}{(A+B)^\nu} = \frac{B^{-\nu}}{(1 - (-A/B))^{-\nu}} = \frac{B^{-\nu}}{2\pi i \Gamma(\nu)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma B^{-\sigma} \Gamma(-\sigma) \Gamma(\nu + \sigma) \quad (20)$$

Is special case of a well-known Mellin-Barnes integral for hypergeometric functions

$$\frac{1}{(1-z)^\nu} = {}_2F_1(\nu, b, b', z)|_{b=b'} \quad (21)$$

$$= \frac{1}{2\pi i \Gamma(\nu)} \frac{\Gamma(b')}{\Gamma(b)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \Gamma(\nu + \sigma) \Gamma(-\sigma) \frac{\Gamma(b + \sigma)}{\Gamma(b' + \sigma)} \quad (22)$$

with  $-z = A/B$ .

How can this be made useful here?

## Introduce Feynman parameters

The momentum integrals of a Feynman diagram may be performed with Feynman parameters, one for each line.

In 2-loops, consider **two subsequent sub-loops** (the first: **off-shell 1-loop**, second **on-shell 1-loop**) and get e.g. for **B7l4m2**, the planar 2nd type 2-box:

allow for propagators with indices,  $1/(k_1^2 - m_1^2)^{a_1}$  etc.

$$K_{1\text{-loop Box,off}} = \frac{(-1)^{a_{4567}} \Gamma(a_{4567} - d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^1 \prod_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1 - x_4 - x_5 - x_6 - x_7)}{F^{a_{4567}-d/2}} \quad (23)$$

where  $a_{4567} = a_4 + a_5 + a_6 + a_7$  and the function  $F$  is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F = [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2 \quad (24)$$

$$+(m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5 \quad (25)$$

We want to apply now:

$$\int_0^1 \prod_i^4 dx_i x_i^{\alpha_i-1} \delta(1 - x_1 - x_2 - x_3 - x_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)} \quad (26)$$

with coefficients  $\alpha_i$  dependent on  $a_i$  and on  $F$

For this, we have to apply several MB-integrals here.

And repeat the procedure for the 2nd subloop.

For the 2nd planar 2-box, B7l4m2, one gets (Smirnov book 4.73):

$$B_{\text{pl},2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[ \frac{m^2}{-s} \right]^{z_5+z_6} \left[ \frac{-t}{-s} \right]^{z_1} \prod_{j=1}^6 [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})} \quad (27)$$

with  $a = a_1 + \dots + a_7$  and

$$z_i = \text{const} + i \Im m(z_i) \quad (28)$$

$$d = 4 - 2\epsilon \quad (29)$$

$$\text{const} = \frac{(i\pi^{d/2})^2 (-1)^a (-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d - a_{4567})} \quad (30)$$

The integrand includes e.g.:

$$\Gamma_2 = \Gamma(-z_2) \quad (31)$$

$$\Gamma_4 = \Gamma(-z_4) \quad (32)$$

$$\Gamma_7 = \Gamma(a_4 + z_2 + z_4) \quad (33)$$

$$\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \quad (34)$$

$$\dots \quad (35)$$

We now derive from B7l4m2 the MB-integral for B5l3m by setting  $a_1 = 0$  (trivial, gives B6l3m2) and  $a_4 = 0$ .

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \rightarrow \frac{1}{\Gamma(0)} = 0 \quad (36)$$

See by inspection that we will get factor  $\Gamma(a_4)$  if  $z_2, z_4 \rightarrow 0$ .

→ Start with the  $z_2, z_4$  integrations by

taking the residues for closing the integration contours to the right:

$$I_{2,4} = \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \quad (37)$$

$$= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,\dots} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) \quad (38)$$

$$= \sum_{n,m=0,1,\dots} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \rightarrow_{a=0} 1 \quad (39)$$

So, setting  $a_1 = a_4 = 0$  and eliminating  $\int dz_2 dz_4$  with setting  $z_2 = z_4 = 0$

we got a 4-fold Mellin-Barnes integral for B5l3m

with  $24 - 3 = 21$   $z_i$ -dependent  $\Gamma$ -functions which may yield residua within four-fold sums.

As mentioned:

This formula has to be calculated now explicitly for the case

$$B_{5|3md2} = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \quad (40)$$

( $B_{5|3md2}$  is a dotted master, with index  $a_2 = 2$ , all others are one)

Next tasks:

- Find a **region of definiteness** of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \quad (41)$$

- Then go to the physical region where  $\epsilon \ll 1$  by distorting the integration path step by step (adding each crossed residuum – **per residue this means one integral less!!!**)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

Here this means:

$$B5l3md2 \rightarrow MB(4\text{-dim,fin}) + MB_3(3\text{-dim,fin}) \quad (42)$$

$$+ MB_{36}(2\text{-dim}, \epsilon^{-1}, fin) + MB_{365}(1\text{-dim}, \epsilon^{-2}, \epsilon^{-1}, fin) \quad (43)$$

$$+ MB_5(3\text{-dim,fin}) \quad (44)$$

After these preparations e.g.:

$$MB_{365}(1\text{-dim}, \epsilon^{-2}) \sim \frac{1}{\epsilon^2} \int dz_6 \frac{(-s)^{(z_6-1)} \Gamma(-z_6)^3 \Gamma(1+z_6)}{8\Gamma(-2z_6)} \quad (45)$$

$$\sim \frac{1}{\epsilon^2} \sum_{n=0, \infty} - \frac{(-1)^n (-s)^n \Gamma(1+n)^3}{8n! \Gamma(-2(-1-n))} \quad (46)$$

$$= - \frac{1}{\epsilon^2} \frac{\text{ArcSin}(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \quad (47)$$

$$= \frac{1}{\epsilon^2} \frac{x}{4(1-x^2)} H[0, x] \quad (48)$$

Here were residua at  $z_6 = -n - 1, n = 0, 1, ..$  taken



## The divergent parts of the masters B5l3m are:

$$B5l3m[-2,x_,y_] = B5l3m[-1,x_,y_] = 0;$$

$$B5l3md1[-2,x_,y_] = (((-1 + x)^2*y*(-1 + y^2 + 2*y*H[0, y]))/(8*x*(-1 + y)*(1 + y)^3));$$

$$B5l3md1[-1,x_,y_] = ((y*(6*(-1 + x - x^2 + x^3)*H[0, x]*(-1 + y^2 + 2*y*H[0, y]) - 6*(1 + x)*(-2 - 2*x^2 + 2*y^2 + 2*x^2*y^2 + y*z^2 - 2*x*y*z^2 + x^2*y*z^2 + 2*(-2*x - y + 2*x*y - x^2*y - 2*x*y^2 + (-1 + x)^2*y*H[-1, -y] + 3*(-1 + x)^2*y*H[-1, y]))*H[0, y] - 6*(-1 + x)^2*y*H[0, -1, y] - 4*y*H[0, 0, y] + 8*x*y*H[0, 0, y] - 4*x^2*y*H[0, 0, y] + 2*y*H[0, 1, y] - 4*x*y*H[0, 1, y] + 2*x^2*y*H[0, 1, y])))/(24*x*(1 + x)*(-1 + y)*(1 + y)^3));$$

$$B5l3md2[-2,x_,y_] = -x/(1 - x^2)/4 H[0, x];$$

$$B5l3md2[-1,x_,y_] = ((x*(2*(1 + y^2)*H[0, x]*H[0, y] - (-1 + y^2)*(z^2 + 6*H[-1, 0, x] - 4*H[0, 0, x] - 2*H[1, 0, x])))/(4*(-1 + x^2)*(-1 + y^2)));$$

$$B5l3md2a[-a,x_,y_] = B5l3md2[-a,y,x], \quad a=-2,-1;$$

$$B5l3md3[-2,x_,y_] = 0;$$

$$B5l3md3[-1,x_,y_] = -((x*y*H[0, x]*H[0, y])/((-1 + x^2)*(-1 + y^2)));$$

## Summary

A calculation of the constant 2-loop term for Bhabha scattering is derived from massless calculations Penin, Bonciani et al.

In parallel:

- A complete list of MASSIVE masters was derived (2004)
- Huge files with algebraic relations for all the reducible Feynman integrals needed for the interferences of boxes with Born (not complete, but fully understood)
- Essential progress for the massive 2-box master integral determination. Underway: Determination of all 2-box masters in a systematic approach use Generalized Harmonic Polylogarithms Remiddi, Vermaseren plus potentially ... )
- An unsolved problem is the systematic summation of the massive multiple sums after the MB-integral evaluation

It is also possible to do the massive 2-loop calculation with present computers.

Improve the existing MC-codes with that.

Care about the radiative 1-loops (with 5-point functions).