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Fermionic NNLO contributions to Bhabha Scattering

Tord Riemann, DESY, Zeuthen
based on work with:
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and J. Gluza (Silesian U. Katowice)

Project B1 – SFB/TRR 9, HUB, Berlin, 30 Oct. 2007

See also: • http://www-zeuthen.desy.de/theory/research/bhabha/
and • hep-ph/0412164,0604101, NPB 2006 (Massive Bhabha 2-loop masters)
and • hep-ph/0609051,0704.2400, NPB 2007 (Fermionic 2-loop corrections)

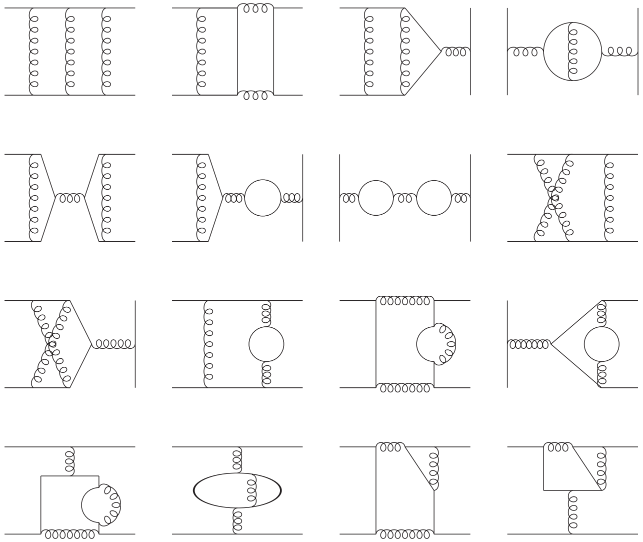
- **Introduction: Two-loop corrections to Bhabha Scattering**
- $n_f = 2$ **contributions with** $m_e^2 << M^2 << s, t$ **[arXiv:0704.2400]**
- **New: dispersive approach to** $n_f = 2$ **at arbitrary** M^2 **and hadronic corr's. [arXiv:0710.5111], hadrons: nearly final**
- **Summary**

Adapted from Z.Bern, LoopFest 2002

Two Loop Bhabha Scattering

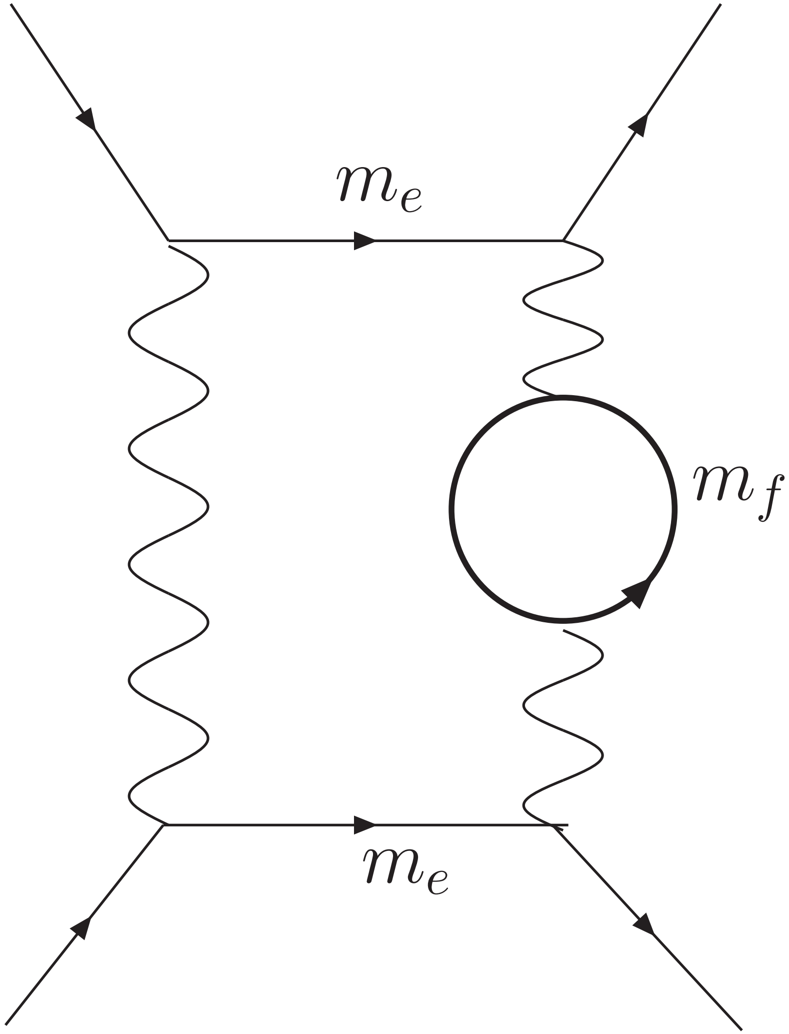
To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

New: treatment of the 2-boxes with two different fermions involved



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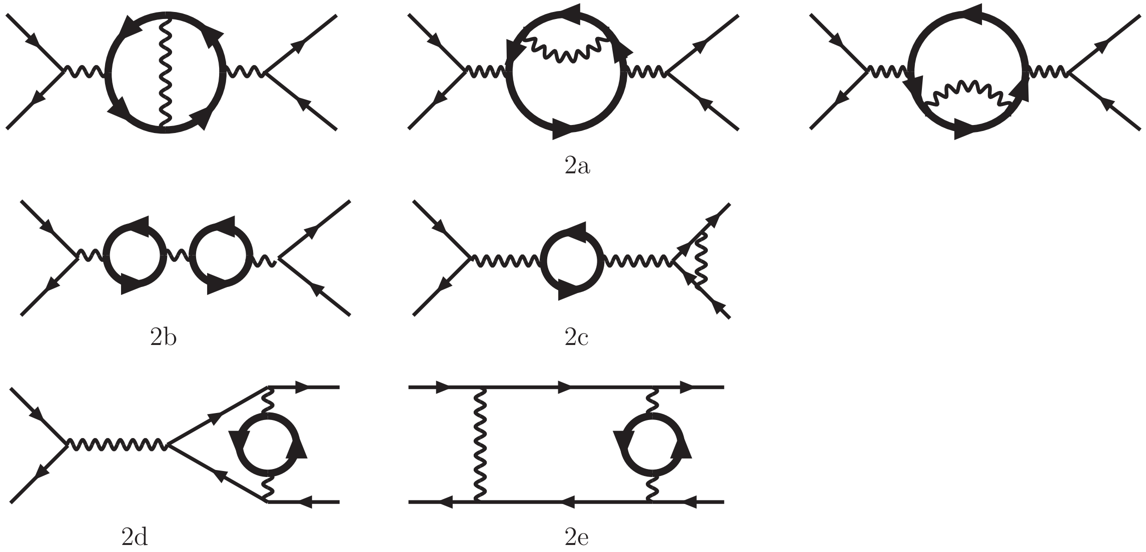
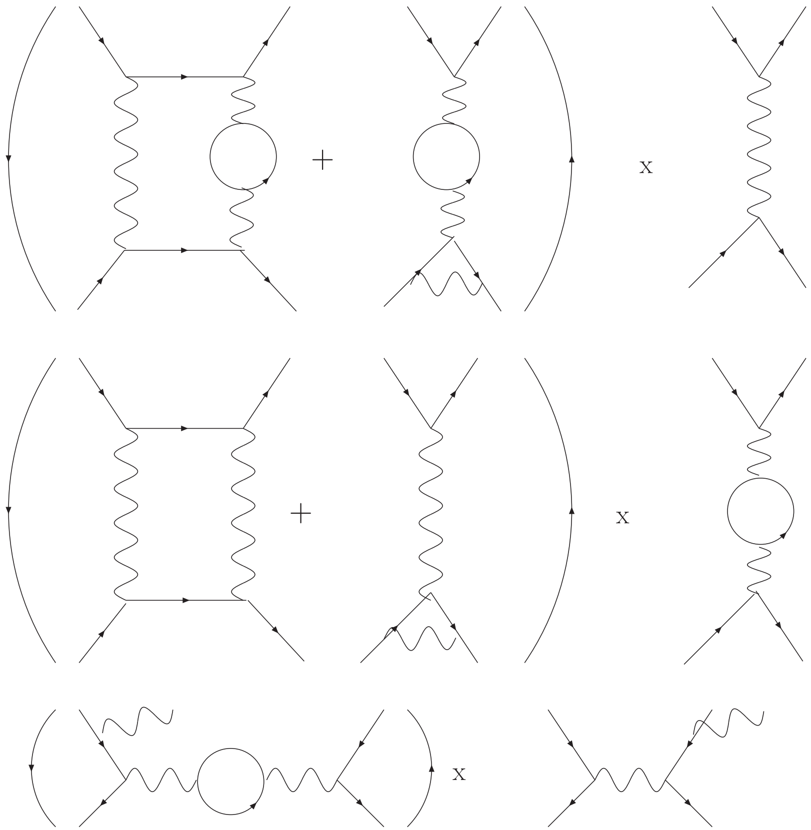
Virtual 2-loop corrections to Bhabha scattering – recent developments

- 1998 Arbuzov,Kuraev,Shaikhatdenov: $(\alpha^2 L)$ cross-section formula
- 1999,2004 Smirnov; Tausk; Heinrich,Smirnov: Few massive planar and non-planar two-loop 7-line box diagrams
- 2001 Bern,Dixon,Ghinculov: Massless photonic two-loop corrections
- 2001 Glover,Tausk,v.d.Bij: $(\alpha^2 L)$ cross-section from 2001 BDG
- 2004 Bonciani,Ferroglia,Mastrolia,Remiddi,v.d.Bij: The $n_f = 1$ SE and vertex masters and the $n_f = 1$ fermionic 2-box
- 2005 Czakon,Gluza,TR: List of all masters (33 2-boxes, 9 of them with seven lines), some 2-box masters evaluated
- 2005 Penin: $(\alpha^2 L^0)$ photonic cross-section ($n_f = 1$ fermionic box from Bonciani et al.)
- 2006 Czakon,Gluza,TR: All massive, planar $n_f = 1$ 2-boxes for $m_e^2 << s, t, u$
- 2006 Actis,Czakon,Gluza,TR: All masters for $n_f = 2$ at $m_e^2 << m_f^2 << s, t, u$
- 2007 Becher,Melnikov; Actis,Czakon,Gluza,TR: cross-section from $n_f = 2$ at $m_e^2 << m_f^2 << s, t, u$
- 2007 Actis,Gluza,TR: $n_f = 2$ cross-section with dispersion relation, $m_e^2 << m_f^2, s, t, u$, inclusion of hadronic insertions (preliminary)



Classes of Bhabha-scattering 2-loop diagrams containing at least one fermion loop.

The eight (i.e. 4 direct and 4 crossed) fermionic 2-loop box diagrams have to be combined with other diagrams for an IR-finite contribution:



After combining the 2-loop terms with the loop-by-loop terms and with soft real corrections:

$$\frac{d\sigma^{\rm NNLO}}{d\Omega} + \frac{d\sigma_{\gamma}^{\rm NLO}}{d\Omega} = \frac{d\sigma^{\rm NNLO,e}}{d\Omega} + \sum_{f\neq e} Q_f^2 \frac{d\sigma^{\rm NNLO,f^2}}{d\Omega} + \sum_{f\neq e} Q_f^4 \frac{d\sigma^{\rm NNLO,f^4}}{d\Omega}$$
$$+ \sum_{f_1,f_2\neq e} Q_{f_1}^2 Q_{f_2}^2 \frac{d\sigma^{\rm NNLO,2f}}{d\Omega}.$$

The Box Corrections

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2\mathrm{e}\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\mathrm{I,f}}(x,y)$, where $\mathrm{i} = \mathrm{A},\mathrm{B},\mathrm{C}$,

$$\begin{aligned} A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\, \mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right], \\ A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\, \mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right]. \end{aligned}$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

.

How to evaluate the $N_f = 2$ diagrams?

We did it in 2 ways

- **Decompose the 2-loop integrals to master integrals, solve them.**
  Here: In the limit $m_e^2 << m_f^2 << s, t, u$
  This is finished, hep-ph/07042400v2 $\longrightarrow$ NPB, to appear

- **Alternatively, rewrite the 2-loop integrals as dispersion integrals.**
  Decompose afterwards into master integrals
  They are simpler, of one-loop type, but a numerical integration remains then.
  Advantages:
  – $m_e^2 << m_f^2, s, t, u$
  – **apply also to hadronic insertions**

The Box Corrections (repeated here from above)

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2\mathrm{e}\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t)\right]$$

with

$$A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) = F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right],$$

$$A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t) = F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right].$$

Look e.g. at $B^{(2)}_{\mathrm{C},f}(t,s)$ for hadrons:

$$B^{(2)}_{\mathrm{C},had}(t,s) = \int_{4M_\pi^2}^{\infty} \frac{dz}{z} R_{had}(z) K_C(s,t,z)$$

And similarly for leptons:

$$4M_\pi^2 \longrightarrow 4m_l^2$$

$$R_{had}(z) \longrightarrow R_{lep}(z) \sim \sqrt{1-\frac{4m_l^2}{z}}\left(1+\frac{2m_l^2}{z}\right)+\epsilon R_{lep}^{\epsilon}(z)$$

Get:

$$B^{(2)}_{\mathrm{C},lep}(t,s) = \int_{4m_l^2}^{\infty} \frac{dz}{z} R_{lep}(z) K_C(s,t,z)$$

$$\begin{aligned}
K_C(x,y;z) &= F_\epsilon \sum_{i=1}^{8} c_{Ci} M_i(s,t,z) \\
&= \frac{1}{3\,m_e^2\,(y-z)} \Big\{ 2\,\frac{F_\epsilon}{\epsilon}\,x^2\,L_x + 4\,\zeta_2\,x^2\left(\frac{z}{y}-2\right) - 2\,\left(x^2+y^2+x\,y\right)\,L_x \\
&+ x^2\left(\frac{z}{y}-1\right)L_y + 2\,x^2\left(\frac{z}{y}-1\right)L_y^2 + 4\,x^2\,L_x\,L_y + x^2\left(\frac{z}{y}-1\right)\,\ln\!\left(\frac{z}{m_e^2}\right) \\
&- 2\,x^2\left(\frac{z}{y}-\frac{1}{2}\right)\,\ln^2\!\left(\frac{z}{m_e^2}\right) + 4\,x^2\left(\frac{z}{y}-1\right)\,\ln\!\left(\frac{z}{m_e^2}\right)\,\ln\!\left(1-\frac{z}{y}\right) \\
&+ 2\,x^2\,\ln\!\left(\frac{z}{m_e^2}\right)L_x - x^2\left(\frac{z}{y}+\frac{y}{z}-2\right)\,\ln\!\left(1-\frac{z}{y}\right) - 4\,x^2\,\ln\!\left(1-\frac{z}{y}\right)L_x \\
&+ 4\,x^2\left(\frac{z}{y}-1\right)\mathrm{Li}_2\!\left(\frac{z}{y}\right) - 2\,x^2\,\mathrm{Li}_2\!\left(1+\frac{z}{x}\right)\Big\}.
\end{aligned}$$

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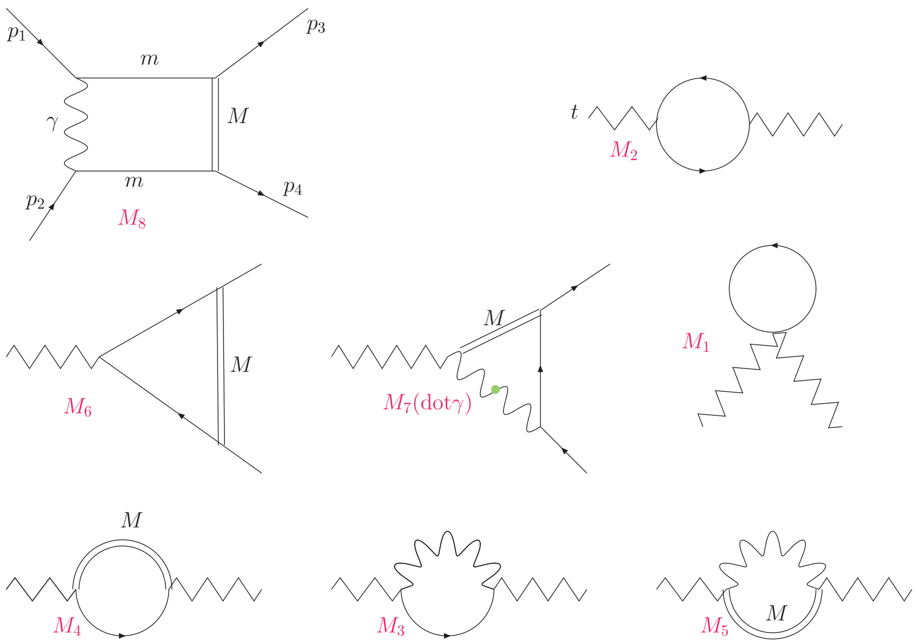
The dispersion master integrals for the $N_f = 2$ contributions

There are three box kernel functions, depending on m_e, m_f, s, t**.with** $m_e^2 << z = m_f^2, s, t$**.**
Some of them are IR-divergent.
The eight master integrals for the 2-loop boxes are:



The contributing masters are:

$$
\begin{aligned}
M_1 &= N \int d^D k \frac{1}{k^2 - m^2}, && (1)\\
M_2 &= N \int d^D k \frac{1}{(k^2 - m^2)[(k - p_1 - p_2)^2 - m^2]}, && (2)\\
M_3 &= N \int d^D k \frac{1}{k^2 (k - p_1 + p_3)^2}, && (3)\\
M_4 &= N \int d^D k \frac{1}{(k^2 - m^2)[(k - p_3)^2 - z]}, && (4)\\
M_5 &= N \int d^D k \frac{1}{(k^2 - z)(k - p_1 + p_3)^2}, && (5)\\
M_6 &= N \int d^D k \frac{1}{(k^2 - z)[(k + p_3)^2 - m^2][(k + p_3 - p_1 - p_2)^2 - m^2]}, && (6)\\
M_7 &= N \int d^D k \frac{1}{(k^2 - z)[(k + p_3)^2 - m^2](k + p_3 - p_1)^2}, && (7)\\
M_8 &= N \int d^D k \frac{1}{(k^2 - z)[(k + p_3)^2 - m^2](k + p_3 - p_1)^2[(k + p_3 - p_1 - p_2)^2 - m^2]}, && (8)
\end{aligned}
$$

where

$$F_\epsilon = N = m^{2\epsilon} \frac{e^{\gamma} \epsilon}{i\pi^{2-\epsilon}}. \tag{9}$$

and e.g. the box integral $M_8 = Bo1$ is:

```
Bo1 = (4*ep*z2 + 2*Log[-(m^2/t)] - 4*ep*Log[me]*Log[-(m^2/t)] -
      4*ep*Log[1 - m^2/t]*Log[-(m^2/t)] + 3*ep*Log[-(m^2/t)]^2 -
      2*Log[-(me^2/t)] + 4*ep*Log[me]*Log[-(me^2/t)] +
      4*ep*Log[1 - m^2/t]*Log[-(me^2/t)] - 2*ep*Log[-(m^2/t)]*
       Log[-(me^2/t)] - ep*Log[-(me^2/t)]^2 + Log[-(m^2/s)]*
       (4*ep*Log[me] + 4*ep*Log[1 - m^2/t] - 2*(1 + ep*Log[-(m^2/t)] +
          ep*Log[-(me^2/t)])) + 2*ep*PolyLog[2, (m^2 + s)/s])/
     (2*ep*s*(m^2 - t))
```

dσ / dΩ [nb] | $\sqrt{s}$ [GeV]	1	10	91	500
LO QED	46.6409	0.466409	0.00563228	0.000186564
LO `Zfitter`	46.643	0.468499	0.127292	0.0000854731
NNLO (e)	-0.230927	-0.00453987	-0.0000919387	-4.28105 $\cdot 10^{-6}$
NNLO ($e + \mu$) "	-0.256679	-0.00570942	-0.000122796	-5.90469 $\cdot 10^{-6}$
NNLO ($e + \mu + \tau$) "		-0.00586082	-0.000135449	-6.7059 $\cdot 10^{-6}$
NNLO ($e + \mu + \tau + t$) "				-6.6927 $\cdot 10^{-6}$
NNLO photonic	2.07476	0.0358755	0.000655126	0.0000284063
NNLO IR e	-0.19927	-0.00359349	-0.0000672264	-2.95317$\cdot 10^{-6}$
NNLO IR μ (analytic)	-0.0314292	-0.00134635	-0.0000335037	-1.66781 $\cdot 10^{-6}$
NNLO IR μ (dispersion)	-0.0333538	-0.00134663	-0.0000335037	-1.66781 $\cdot 10^{-6}$
NNLO IR τ (analytic)		-0.00021027	-0.0000162977	-1.00877 $\cdot 10^{-6}$
NNLO IR τ (dispersion)		-0.000272634	-0.0000163119	-1.00878 $\cdot 10^{-6}$

Table 1: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 90^\circ$. Empty entries are related to cases where the high-energy approximation cannot be applied.

Using R_{had}

This is a topic by itself, because R_{had} is basically unpublished.

N.N.1:
Fuer R(s) mit Fehlern, Kontinuum + Resonanzen haben wir nur unsere interne Arbeitsversion.

N.N.2:
This procedure is a follow up of complicated programs, which unfortunately do not exist in a really user-friendly form.

N.N.3:
I understand that for your problem it is probably too cumbersome (and time-consuming) to use the data.

N.N.4:
es hat etwas gedauert, bis ich in meinen alten Verzeichnissen auf einer 1994er Vax am MPI fuendig geworden bin.

So, finally, we could reproduce the old estimates given for the vertex dispersion relation in Kniehl, Krawczyk, Kühn, Stuart (1988) $\longrightarrow$ soon we have final numerics

$\sqrt{s}$ **[GeV]**	**1**	**10**	M_Z	**500**
Born	**440994**	**4409.94**	**53.0348**	**1.76398**
Born run.				
vertices				
rest e	**193**	**5.73**	**0.1357**	**0.00673**
μ	$<$ **1**	**0.42**	**0.0408**	**0.00288**
	$\times$	**0.08**	**0.0407**	**0.00288**
τ	$<$ **1**	$<$ **10**$^{-2}$	**0.0027**	**0.00088**
	$\times$	$\times$	**-0.0096**	**0.00084**
t	$<$ **1**	$<$ **10**$^{-2}$	$<$ **10**$^{-4}$	$<$ **10**$^{-5}$
	$\times$	$\times$	$\times$	$\times$
had.	$<$ **1**	**0.39**	**0.0877**	**0.00811**

Table 2: Numerical values for the differential cross section in nanobarns at a scattering angle $\theta = 3^\circ$, in units of 10^2. For each fermion flavour, we show the result obtained through the dispersion-based approach (first line) and the one coming from the analytical expansion (second line), neglecting $\mathcal{O}(m_f^2/x)$, where $x = s, |t|, |u|$. When $m_f^2 > x$, the entry is suppressed.

$\sqrt{s}$ **[GeV]**		**1**	**10**	M_Z	**500**
Born		**466537**	**4665.37**	**56.1067**	**1.86615**
Born run.					
vertices					
rest	e	**807**	**14.53**	**0.2706**	**0.01193**
	μ	**160**	**6.08**	**0.1470**	**0.00726**
		153	**6.08**	**0.1470**	**0.00726**
	τ	**2**	**1.33**	**0.0752**	**0.00457**
		$\times$	**1.07**	**0.0752**	**0.00457**
	t	$<$ **1**	$<$ **10**$^{-2}$	**0.0005**	**0.00043**
		$\times$	$\times$	$\times$	**-0.00013**
had.		**234**	**16.07**	**0.4701**	**0.02461**

Table 3: Numerical values for the differential cross section in nanobarns at a scattering angle $\theta = 90^\circ$, in units of 10^{-4}. See the caption of Table 2 for further details.

Summary

- **We determined the $N_f = 2$ contributions to 2-loop Bhabha scattering**
- **The contribution is small, but non-negligible at the scale 10^{-4} ($\rightarrow$ LEP influencing ??)**
- **Agreement for $m_e^2 << m_l^2 << s, t, u$ with:**
  "Two-loop QED corrections to Bhabha scattering"
  Thomas Becher, Kirill Melnikov, arXiv:0704.3582 [hep-ph], JHEP
- **Status:**
  $\longrightarrow$ Leptonic case where $m_l^2 \sim s, t, u$ (with dispersion integrals) final;
  also: Bonciani, Ferroglia, Penin
  To be finalized numerically yet:
  $\longrightarrow$ Hadronic $N_f = 2$ contributions (non-perturbative)

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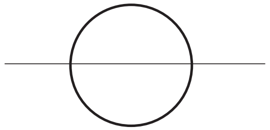
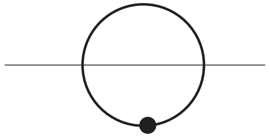
The 2-loop master integrals for the $N_f = 2$ contributions

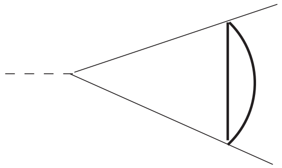
S. Actis, M. Czakon, J. Gluza, TR, 2006(publ.) / 2007(box master expansion corrected)



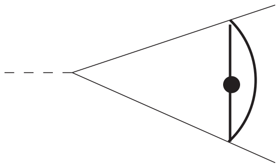
SE3l2M1m



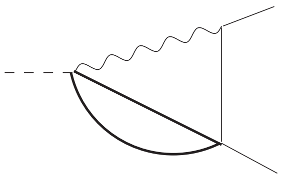
SE3l2M1md



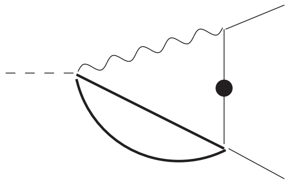
V4l2M2m



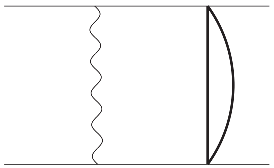
V4l2M2md



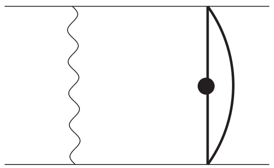
V4l2M1m



V4l2M1md



B5l2M2md



B5l2M2m

There are eight additional master integrals with two different mass scales.

The 2-box-diagrams represent a three-scale problem: $s/m_e^2, t/m_e^2, M^2/m_e^2$

There are several opportunities to evaluate the master integrals.

We used here the following:

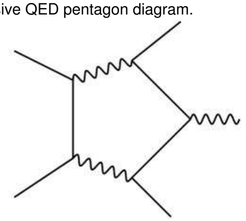
- **Feynman parameter representation**
- **derive Mellin-Barnes-representation**
  ($\longrightarrow$ with package AMBRE (public, Gluza,Kajda,TR))
- **The ϵ-expansion in $d = (4 - 2\epsilon$**
  ($\longrightarrow$ with package MB (public, Czakon))
- **Perform 2 subsequent small mass expansions**

Next slide: propaganda for AMBRE

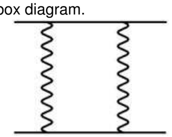
AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

To download 'right click' and 'save target as'.

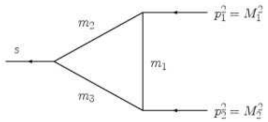
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



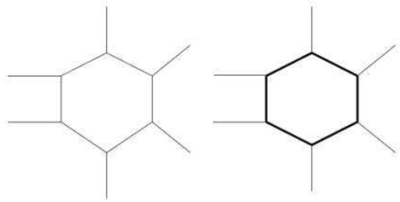
  - example3.nb - Massive QED one-loop box diagram.



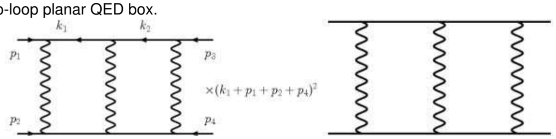
  - example4.nb - General one-loop vertex.



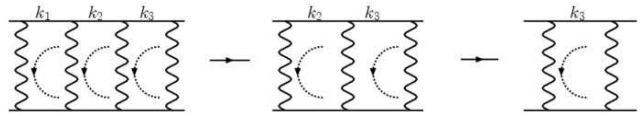
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.



  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



  - example8.nb - The loop-by-loop iterative procedure.



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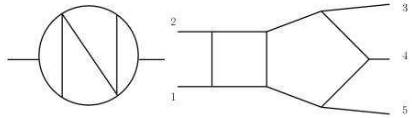

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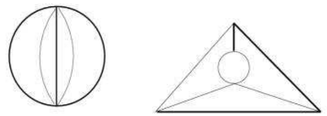

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  !" . – (

Self-energy master integrals:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051v2

$$L(R) = \ln\left(\frac{m_e^2}{M^2}\right)$$

$$\begin{aligned}
\texttt{SE3l2M1m[on shell]} &= M^2\, m^{-4\epsilon}\Big\{R\Big[\frac{1}{2\epsilon^2}+\frac{5}{4\epsilon}-\frac{3}{8}+\frac{\zeta_2}{2}+\frac{3}{2}L(R)-\frac{1}{2}L^2(R)\Big] \\
&+ R^2\Big[\frac{11}{18}-\frac{1}{3}L(R)\Big]+\epsilon\Big[R\Big(\frac{45}{16}+\frac{5}{4}\zeta_2-\frac{\zeta_3}{3}-\frac{7}{4}L(R)+L^2(R) \\
&- \frac{1}{2}L^3(R)\Big)+R^2\Big(-\frac{3}{4}+\frac{8}{9}L(R)-\frac{1}{2}L^2(R)\Big)\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
\texttt{SE3l2M1md[on shell]} &= m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\Big[1+2L\,(R)\Big]+\frac{1}{2}\,(1+\zeta_2)+L\,(R)+L^2\,(R) \\
&+ \epsilon\Big[\frac{1}{6}\,(3+3\zeta_2-2\zeta_3)+(1+\zeta_2)\,L\,(R)+L^2\,(R)+\frac{2}{3}L^3\,(R)\Big] \\
&+ R\Big[-\frac{3}{4}+\frac{1}{2}L(R)+\epsilon\left(\frac{7}{8}-L(R)+\frac{3}{4}L^2(R)\right)\Big] \\
&+ R^2\Big[-\frac{5}{36}+\frac{1}{6}L(R)+\epsilon\Big(-\frac{5}{72}+\frac{1}{18}L(R)+\frac{1}{4}L^2(R)\Big)\Big]\Big\}.
\end{aligned}$$

Vertex master integrals:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051v2

$L_m(x) = \ln(-m^2/x)$ and $L_M(x) = \ln(-M^2/x)$,

$$
\begin{aligned}
\texttt{V412M1m[x]} &= m^{-4\epsilon}\Bigg\{\frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \frac{1}{2}\Big[19 - 3\zeta_2 - L_m^2(x)\Big] \\
&+ \frac{M^2}{x}\Big[-2 + 4\zeta_2 - 4\zeta_3 - 2L_m(x) + 2L_M(x) - 4\zeta_2 L_M(x) \\
&+ 2L_m(x)L_M(x) - L_M^2(x) - L_m(x)L_M^2(x) + \frac{1}{3}L_M^3(x)\Big]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\texttt{V412M1md[x]} &= \frac{m^{-4\epsilon}}{m^2}\Bigg\{\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big[1 + \frac{1}{2}L_m(x)\Big] + 2 - \zeta_2 + L_m(x) + \frac{1}{4}L_m^2(x) \\
&+ \frac{M^2}{x}\Big[\frac{1}{\epsilon} - \frac{1}{\epsilon}L_M(x) - 1 + 3\zeta_2 + L_m(x) + L_M(x) \\
&- L_m(x)L_M(x) - \frac{1}{2}L_M^2(x)\Big]\Bigg\},
\end{aligned}
$$

$$
\texttt{V412M2m[x]} = m^{-4\epsilon}\Bigg\{\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big[\frac{5}{2} + L_m(x)\Big] + \frac{1}{2}(19 + \zeta_2) + 5\ L_m(x) + L_m^2(x)\Bigg\},
$$

$$
\texttt{V412M2md[x]} = \frac{m^{-4\epsilon}}{6x}\Big[12\zeta_3 - 6\zeta_2 L_M(x) - L_M^3(x)\Big],
$$

Box master integrals:

Correct Mellin-Barnes representations in Actis et al., NPB(PS) 160 (2006) 91, hep-ph/0609051v2

$$
\begin{aligned}
\texttt{B512M2m[x,y]} &= \frac{m^{-4\epsilon}}{x}\Bigg\{\frac{1}{\epsilon^2}L_m(x) + \frac{1}{\epsilon}\Big(-\zeta_2 + 2L_m(x) + \frac{1}{2}L_m^2(x) + L_m(x)L_m(y)\Big) \\
&- 2\zeta_2 - 2\zeta_3 + 4L_m(x) + L_m^2(x) + \frac{1}{3}L_m^3(x) - 4\zeta_2 L_m(y) \\
&+ 2L_m(x)L_m(y) + L_m(x)L_m^2(y) - \frac{1}{6}L_m^3(y) \\
&- \Big(3\zeta_2 + \frac{1}{2}L_m^2(x) - L_m(x)L_m(y) + \frac{1}{2}L_m^2(y)\Big)\ln\Big(1+\frac{y}{x}\Big) \\
&- \Big(L_m(x) - L_m(y)\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big) + \mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B512M2md[x,y]} &= \frac{m^{-4\epsilon}}{xy}\Bigg\{\frac{1}{\epsilon}\Big[-L_m(x)L_m(y) + L_m(x)L(R)\Big] - 2\zeta_3 + \zeta_2 L_m(x) + 4\zeta_2 L_m(y) \\
&- 2L_m(x)L_m^2(y) + \frac{1}{6}L_m^3(y) - 2\zeta_2 L(R) + 2L_m(x)L_m(y)L(R) - \frac{1}{6}L^3(R) \\
&+ \Big(3\zeta_2 + \frac{1}{2}L_m^2(x) - L_m(x)L_m(y) + \frac{1}{2}L_m^2(y)\Big)\ln\Big(1+\frac{y}{x}\Big) \\
&+ \Big(L_m(x) - L_m(y)\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big) - \mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
B^{(2)}_{\mathrm{A},f}(x,y) \;=\;& \frac{1}{\epsilon}\,\frac{2}{3}\left(\frac{x^2}{y}+2\,x+y\right)\left[\frac{5}{3}-L(R_f)+L_e(y)\right]L_e(x) \\
+\;& \frac{1}{3}\,\frac{x^2}{y}\Big\{2\Big(\frac{131}{27}-10\,\zeta_2-2\,\zeta_3\Big)-2\Big(\frac{25}{9}-6\,\zeta_2\Big)L(R_f)+\frac{7}{6}\,L^2(R_f) \\
-\;& \frac{1}{3}\,L^3(R_f)+\Big[\frac{82}{9}-2\,\zeta_2-\frac{4}{3}\,L(R_f)\Big]L_e(x)-2\Big[\frac{1}{3}+8\,\zeta_2-\frac{1}{2}\,L(R_f)\Big]L_e(y) \\
-\;& \Big[\frac{23}{6}-2\,L(R_f)\Big]L_e^2(y)+4\Big[2-L(R_f)\Big]L_e(x)\,L_e(y)-4\Big[\frac{5}{12}\,L_e^3(y) \\
-\;& L_e(x)\,L_e^2(y)\Big]-\Big[6\,\zeta_2+\ln^2\Big(\frac{y}{x}\Big)\Big]\ln\Big(1+\frac{y}{x}\Big)-2\,\ln\Big(\frac{y}{x}\Big)\,\mathrm{Li}_2\Big(-\frac{y}{x}\Big) \\
+\;& 2\,\mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}+\frac{x}{3}\Big\{2\Big(\frac{262}{27}-9\,\zeta_2-4\,\zeta_3\Big)-4\Big(\frac{25}{9}-3\,\zeta_2\Big)L(R_f) \\
+\;& \frac{7}{3}\,L^2(R_f)-\frac{2}{3}\,L^3(R_f)+2\Big[\frac{121}{9}-\frac{10}{3}\,L(R_f)\Big]L_e(x)-2\Big[\frac{10}{3}+12\,\zeta_2 \\
-\;& 2\,L(R_f)\Big]L_e(y)+\Big[\frac{13}{3}-2\,L(R_f)\Big]L_e^2(x)-\Big[\frac{16}{3}-2\,L(R_f)\Big]L_e^2(y) \\
+\;& 2\Big[\frac{17}{3}-2\,L(R_f)\Big]L_e(x)\,L_e(y)+\frac{2}{3}\,L_e^3(x) \\
+\;& 6\,L_e(x)\,L_e^2(y)-2\,L_e^3(y)-2\Big[6\,\zeta_2+\ln^2\Big(\frac{y}{x}\Big)\Big]\ln\Big(1+\frac{y}{x}\Big) \\
-\;& 4\,\ln\Big(\frac{y}{x}\Big)\,\mathrm{Li}_2\Big(-\frac{y}{x}\Big)+4\,\mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}+\frac{y}{3}\Big\{2\Big(\frac{131}{27}-7\zeta_2-2\zeta_3\Big) \\
-\;& 2\Big(\frac{25}{9}-3\zeta_2\Big)L(R_f)+\frac{7}{6}L^2(R_f)-\frac{1}{3}L^3(R_f)+\Big[\frac{130}{9}-\frac{10}{3}L(R_f)\Big]L_e(x) \\
-\;& \Big[6+12\zeta_2-3L(R_f)\Big]L_e(y)+\Big[\frac{5}{3}-L(R_f)\Big]L_e^2(x)-\Big[\frac{25}{6}-L(R_f)\Big]L_e^2(y) \\
+\;& 2\Big[\frac{10}{3}-L(R_f)\Big]L_e(x)L_e(y)+\frac{1}{3}L_e^3(x)-L_e^3(y)+3L_e(x)L_e^2(y) \\
-\;& \Big[6\,\zeta_2+\ln^2\Big(\frac{y}{x}\Big)\Big]\ln\Big(1+\frac{y}{x}\Big)-2\ln\Big(\frac{y}{x}\Big)\,\mathrm{Li}_2\Big(-\frac{y}{x}\Big)+2\,\mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}
\end{aligned}
$$

$B_{2e,f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	188758	5200.08	284.711
μ	1635.62	1686.88	130.579
τ			39.5554

Table 4: Finite part of $B_{2e,f}$ in nanobarns at a scattering angle $\theta = 3°$.

$B_{2e,f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	143.162	3.23102	0.160582
μ	61.3875	1.79381	0.0995184
τ	10.0105	0.935319	0.0639576
t			-0.00256757

Table 5: Finite part of $B_{2e,f}$ in nanobarns at a scattering angle $\theta = 90°$.

$\sqrt{s}$ [GeV]	10	91	500
e	-124.237	-254.293	-400.574
μ	-4.8036	-29.1057	-70.1032
τ		-2.08719	-13.4901

Table 6: Real part for the vertex form factor.

$$\frac{d\sigma^{\mathrm{NNLO,f^2}}}{d\Omega} = \frac{\alpha^2}{s}\left\{\sigma_1^{\mathrm{NNLO,f^2}} + \sigma_2^{\mathrm{NNLO,f^2}} \ln\left(\frac{2\omega}{\sqrt{s}}\right)\right\}$$

The $\sigma_1^{\mathrm{NNLO,f^2}}$ is the main result of this study:

$$
\begin{aligned}
\sigma_1^{\mathrm{NNLO,f^2}} \; = \; & \frac{\left(1-x+x^2\right)^2}{3x^2}\Bigg\{-\frac{1}{3}\left[\ln^3\left(\frac{s}{m_e^2}\right)+\ln^3\left(R_f\right)\right]+\ln^2\left(\frac{s}{m_e^2}\right)\left[\frac{55}{6}-\ln\left(R_f\right)\right.\\
+ \; & \left.\ln\left(1-x\right)-\ln\left(x\right)\right]+\ln\left(\frac{s}{m_e^2}\right)\left[-\frac{589}{18}+\frac{37}{3}\ln\left(R_f\right)-\ln^2\left(R_f\right)\right.\\
- \; & \left.2\ln\left(R_f\right)\Big(\ln\left(x\right)-\ln\left(1-x\right)\Big)-8\mathrm{Li}_2\left(x\right)\right]+\frac{4795}{108}-\frac{409}{18}\ln\left(R_f\right)+\frac{19}{6}\ln^2\left(R_f\right)\\
- \; & \ln^2\left(R_f\right)\Big(\ln\left(x\right)-\ln\left(1-x\right)\Big)-8\ln\left(R_f\right)\mathrm{Li}_2\left(x\right)+\frac{40}{3}\mathrm{Li}_2\left(x\right)\Bigg\}\\
+ \; & \ln\left(\frac{s}{m_e^2}\right)\left[\zeta_2\left(-\frac{2}{3x^2}+\frac{4}{3x}+\frac{11}{2}-\frac{23}{3}x+\frac{16}{3}x^2\right)+\ln^2\left(x\right)\left(-\frac{1}{3x^2}+\frac{17}{12x}\right.\right.\\
- \; & \left.\frac{5}{4}-\frac{x}{12}+\frac{2}{3}x^2\right)+\ln^2\left(1-x\right)\left(-\frac{2}{3x^2}+\frac{11}{6x}-\frac{5}{2}+\frac{11}{6}x-\frac{2}{3}x^2\right)\\
+ \; & \ln\left(x\right)\ln\left(1-x\right)\left(\frac{2}{3x^2}-\frac{4}{3x}-\frac{1}{2}+\frac{5}{3}x-\frac{4}{3}x^2\right)+\ln\left(x\right)\left(\frac{55}{9x^2}-\frac{83}{9x}+\frac{65}{6}\right.\\
- \; & \left.\left.\frac{85}{18}x+\frac{10}{9}x^2\right)+\frac{1}{3}\ln\left(1-x\right)\left(-\frac{10}{3x^2}+\frac{31}{6x}-10+\frac{31}{6}x-\frac{10}{3}x^2\right)\right]\\
+ \; & \frac{1}{3}\ln^3\left(x\right)\left(-\frac{1}{3x^2}+\frac{31}{12x}-\frac{11}{6}-\frac{x}{6}+\frac{x^2}{3}\right)+\frac{1}{3}\ln^3\left(1-x\right)\left(-\frac{1}{3x^2}+\frac{1}{x}\right.\\
- \; & \left.\frac{4}{3}+x-\frac{x^2}{3}\right)+\ln^2\left(x\right)\ln\left(1-x\right)\left(-\frac{1}{3x^2}+\frac{1}{3x}-\frac{4}{3}+x-\frac{x^2}{3}\right)\\
+ \; & \frac{1}{3}\ln\left(x\right)\ln^2\left(1-x\right)\left(-\frac{1}{x^2}+\frac{2}{x}-\frac{7}{4}+\frac{x}{2}\right)+\ln^2\left(x\right)\left[\frac{55}{18x^2}-\frac{46}{9x}+\cdots\cdots\cdots\right.
\end{aligned}
$$

$$
\begin{aligned}
\cdots\cdots\cdots + \quad & \frac{14}{3} - \frac{4}{9}x - \frac{10}{9}x^2 + \ln(R_f)\left(-\frac{1}{3x^2} + \frac{17}{12x} - \frac{5}{4} - \frac{x}{12} + \frac{2}{3}x^2\right)\Big] \\
+ \quad & \ln^2(1-x)\Big[\frac{10}{9x^2} - \frac{29}{9x} + \frac{9}{2} - \frac{29}{9}x + \frac{10}{9}x^2 + \ln(R_f)\left(-\frac{2}{3x^2} + \frac{11}{6x}\right. \\
- \quad & \left.\frac{5}{2} + \frac{11}{6}x - \frac{2}{3}x^2\right)\Big] + \ln(x)\ln(1-x)\Big[-\frac{10}{9x^2} + \frac{37}{18x} + \frac{1}{2} - \frac{25}{9}x \\
+ \quad & \frac{20}{9}x^2 + \ln(R_f)\left(\frac{2}{3x^2} - \frac{4}{3x} - \frac{1}{2} + \frac{5}{3}x - \frac{4}{3}x^2\right)\Big] + \ln(x)\Big[-\frac{589}{54x^2} + \frac{1753}{108x} \\
- \quad & \frac{701}{36} + \frac{925}{108}x - \frac{56}{27}x^2 + \mathrm{Li}_2(x)\left(-\frac{4}{x^2} + \frac{19}{3x} - 7 + 3x - \frac{2}{3}x^2\right) \\
+ \quad & \ln(R_f)\left(\frac{37}{9x^2} - \frac{56}{9x} + \frac{47}{6} - \frac{67}{18}x + \frac{10}{9}x^2\right) + \zeta_2\left(-\frac{2}{3x^2} + \frac{4}{x} - \frac{1}{6}\right. \\
- \quad & \left.\frac{10}{3}x + 2x^2\right)\Big] + \ln(1-x)\Big[\frac{56}{27x^2} - \frac{161}{54x} + \frac{56}{9} - \frac{161}{54}x + \frac{56}{27}x^2 \\
+ \quad & \ln(R_f)\left(-\frac{10}{9x^2} + \frac{31}{18x} - \frac{10}{3} + \frac{31}{18}x - \frac{10}{9}x^2\right) + \zeta_2\left(-\frac{2}{x^2} + \frac{20}{3x} - \frac{32}{3} + \frac{20}{3}x\right. \\
- \quad & \left.2x^2\right)\Big] + \mathrm{Li}_3(x)\left(\frac{4}{3x^2} - \frac{7}{3x} + 3 - \frac{5}{3}x + \frac{2}{3}x^2\right) + \frac{2}{3}S_{1,2}(x)\left(-\frac{1}{x^2} + \frac{1}{x}\right. \\
- \quad & \left.x + x^2\right) + \zeta_2\Big[\frac{19}{9x^2} - \frac{13}{18x} - \frac{43}{3} + \frac{311}{18}x - \frac{98}{9}x^2 + \ln(R_f)\left(-\frac{2}{3x^2} + \frac{4}{3x}\right. \\
+ \quad & \left.\frac{11}{2} - \frac{23}{3}x + \frac{16}{3}x^2\right)\Big] + \zeta_3\left(-\frac{4}{3x^2} + \frac{3}{x} - 5 + \frac{11}{3}x - 2x^2\right)
\end{aligned}
$$

dσ / dΩ [nb] | $\sqrt{s}$ [GeV]	10	91	500
LO QED	440873	5323.91	176.349
LO `Zfitter`	440875	5331.5	176.283
NNLO (e)	-1397.35	-35.8374	-1.88151
NNLO $(e + \mu)$	-1394.74	-43.1888	-2.41643
NNLO $(e + \mu + \tau)$			-2.55179
NNLO photonic	9564.09	251.661	12.7943

dσ / dΩ [nb] | $\sqrt{s}$ [GeV]	10	91	500
LO QED [Eq. (**??**)]	0.466409	0.00563228	0.000186564
LO `Zfitter`	0.468499	0.127292	0.0000854731
NNLO (e)	-0.00453987	-0.0000919387	-4.28105 $\cdot 10^{-6}$
NNLO $(e + \mu)$	-0.00570942	-0.000122796	-5.90469 $\cdot 10^{-6}$
NNLO $(e + \mu + \tau)$	-0.00586082	-0.000135449	-6.7059 $\cdot 10^{-6}$
NNLO $(e + \mu + \tau + t)$			-6.6927 $\cdot 10^{-6}$
NNLO photonic	0.0358755	0.000655126	0.0000284063

Table 7: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 3^\circ$ and $\theta = 90^\circ$. Empty entries are related to cases where the high-energy approximation cannot be applied.

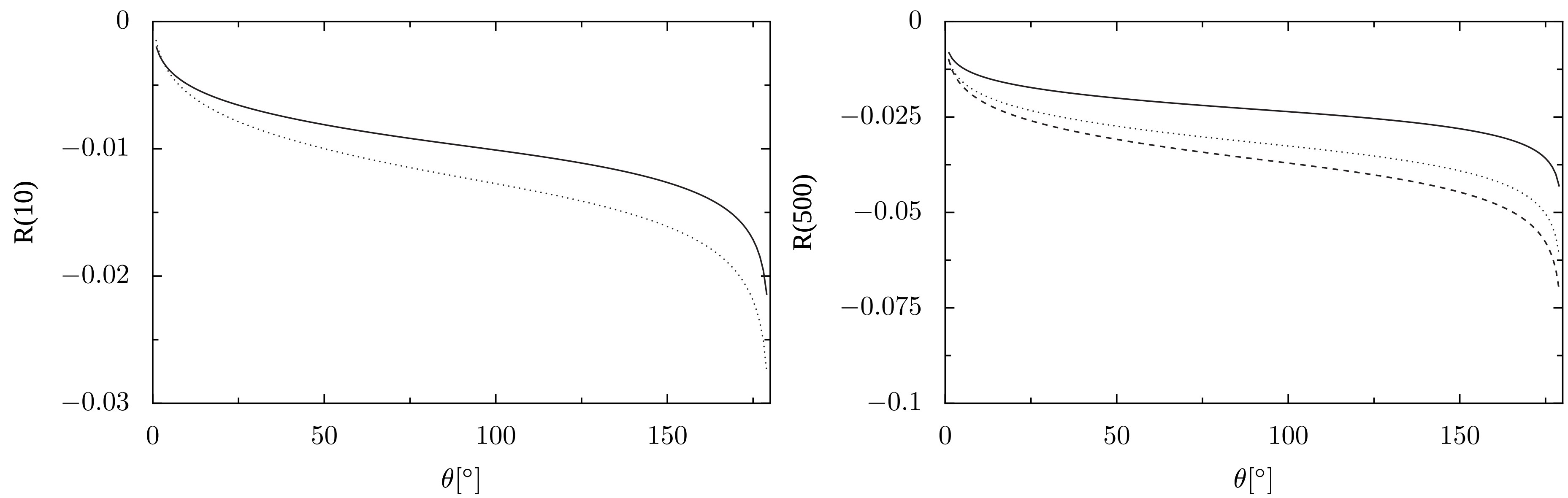

Figure 1: Ratio of the fermionic NNLO corrections to the differential cross section respect to the tree-level result for $\sqrt{s} = 10$ GeV and $\sqrt{s} = 500$ GeV. Solid line: electron-loop contributions, a dotted one the sum of electron- and muon-loop ones, and a dashed one includes also τ leptons.

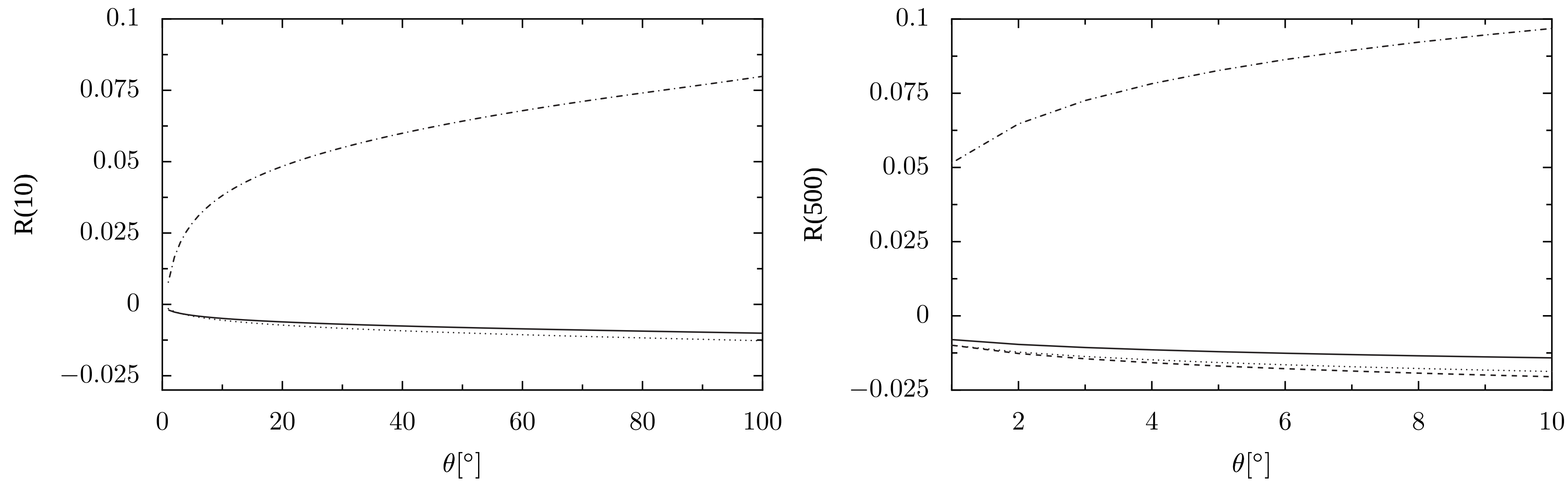

Figure 2: Here also with the photonic contributions of Arbuzov et al., Glover et al., Penin (dash-dotted lines).