Feynman Integrals and Mellin-Barnes representations

Tord Riemann   DESY, Zeuthen, Germany

Janusz Gluza   University of Silesia, Katowice

1 **Introduction**

2 **Mellin-Barnes representations**

3 **Summary**

- **Introduction + Motivation**
- **Mathematical Reminder on Γ-function, Residues, Cauchy-theorem**
- **Few simple Feynman integrals, made conventionally**
- **Mellin-Barnes representations and their evaluation**
- **Expansions in a small parameter, e.g.** $m^2/s << 1$

Introductory remarks I

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient:

★ Tensor reduction a la Passarino/Veltmann

★ Evaluate Feynman parameter integrals by direct integration

Typically 1-loop (massless: 2-loop), typically $2 \longrightarrow 2$ scattering (plus bremsstrahlung)

Feynman parameters may be used and by direct integration over them one gets objects like:

$\frac{23}{57}$, $\zeta(3)$, $\ln(\frac{t}{s})$, $\ln(\frac{t}{s}) \cdot \ln(\frac{s}{m^2})$, $\mathrm{Li}_2(\frac{t}{s+l\epsilon})$ etc.

With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated.
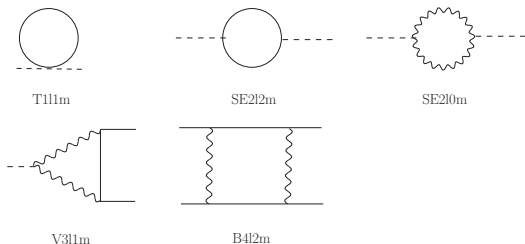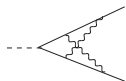
T1l1m

SE2l2m

SE2l0m

V3l1m

B4l2m

Figure shows so-called master integrals.

$$\texttt{T1l1m} = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 +$$
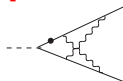
$$\texttt{B4l2m} = [-\frac{1}{\epsilon} + \ln(-s)]\frac{2y\ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots$$

with $d = 4 - 2\epsilon$ and $m = 1$ and $y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$

More loops

V6l4m1   V6l4m1d   V6l4m2
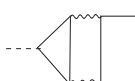
Two-loop vertex integrals with six internal lines
massless case: only fixed numbers and one scale factor

SE3l2M1m

SE3l2M1md

V4l2M2m

V4l2M2md

V4l2M1m

V4l2M1md

B5l2M2md
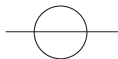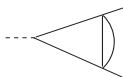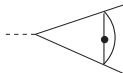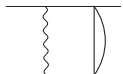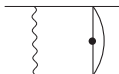
B5l2M2m

Integrals with two different mass scales m and M
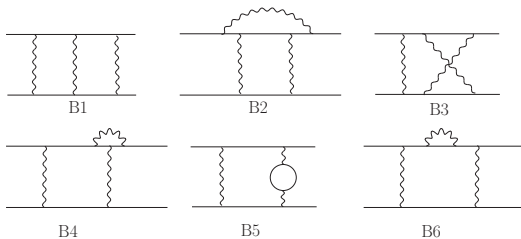
Two-loop box diagrams for massive $2 \to 2$ **scattering**
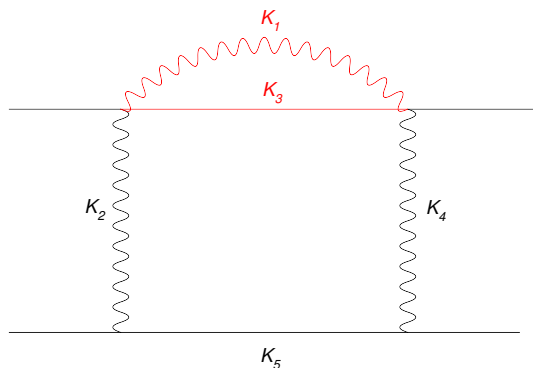
A box master integral B5l2m2, related to B2 = B7l4m2 by shrinking two lines

More legs
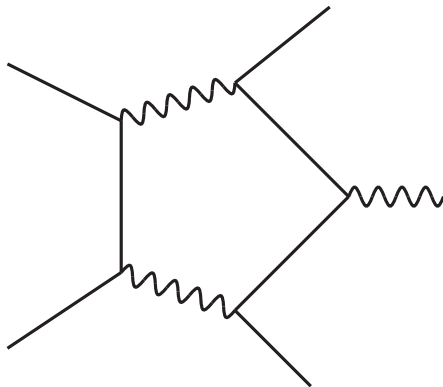
Massive pentagon: 5 kinematic variables + several masses
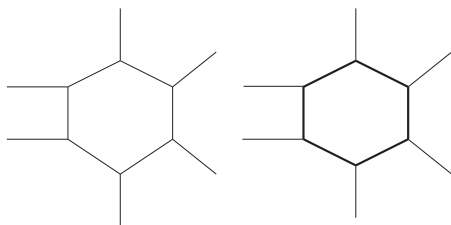
**Massless and massive hexagons: 8 kinematic variables +
several masses**

Variables for $2 \to 2$ scattering, i.e. box diagrams: s, t **or** s
and $\cos \theta$

**Variables for $2 \to 3$ scattering: 5 = 2 + 3 (three additional
momenta of a particle)**

**Variables for $2 \to 4$ scattering: 8 = 5 + 3 (another three
additional)**

Some Mathematical Preparations

We will often use, for $d = 4 - 2\epsilon$:

$$a^\epsilon \;=\; e^{\epsilon \ln(a)} = 1 + \ln(a)\,\epsilon + \frac{1}{2}\ln^2(a)\,\epsilon^2 + \ldots \quad (1)$$

The Γ-function

The Γ-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0 \qquad (2)$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad (3)$$

$$\Gamma(0) = \infty \qquad (4)$$

$$\Gamma(1) = 1 \qquad (5)$$

$$\Gamma(n) = (n-1)!, \quad n = 2, 3, \cdots \qquad (6)$$

You remember that $\Gamma(z)$ has poles at $z = -n, n = 0, 1, 2, 3, \cdots$, and it is

$$
\begin{aligned}
\Gamma[\epsilon] &= \frac{1}{\epsilon} - \gamma_E + \frac{1}{2}\left[\gamma_E{}^2 + \zeta(2)\right]\epsilon + \frac{1}{6}\left[-\gamma_E{}^3 - 3\gamma_E{}^2\zeta(2)\right. \\
&\quad \left. -2\zeta(3)\right]\epsilon^2 + \cdots \tag{7} \\
e^{\epsilon\gamma_E}\ \Gamma[\epsilon] &= \frac{1}{\epsilon} + \frac{1}{2}\zeta(2)\epsilon - \frac{1}{3}\zeta(3)\epsilon^2 + \cdots \tag{8}
\end{aligned}
$$

For definitions of Riemann's zeta-numbers $\zeta(N)$ and the Euler constant γ_E see next slides.

Look at the singularities in the complex plane.
Figure shows the real part of Γ:
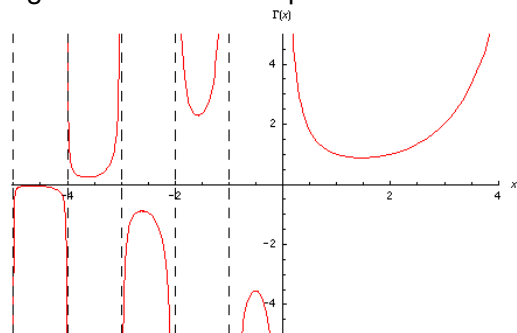
$$\text{Gamma}[-1 \pm 10i] = -4.9974 \; 10^{-9} \pm 1.07847 \; 10^{-8}i \quad (9)$$
$$\text{Gamma}[-1 \pm 100i] = 1.51438 \; 10^{-71} \pm 1.27644 \; 10^{-73}i$$
$$\text{Gamma}[\pm 100.1] \approx \pm 10^{\pm 157} \quad (10)$$

Just to remind:

$$\zeta(a) = \sum_{k=1}^{\infty} \frac{1}{k^a} \tag{11}$$

$$\text{HarmonicNumber}[N, a] = \sum_{k=1}^{N} \frac{1}{k^a} = H_{N,a} = S_a(N) \tag{12}$$

$$\text{HarmonicNumber}[N] = \sum_{k=1}^{N} \frac{1}{k^1} = H_N = S_1(N) \tag{13}$$

$$\gamma_E = \lim_{N \to \infty} [S_1(N) - \ln(N)] = 0.57721 \cdots$$

We will also need derivatives of $\Gamma(z)$:

$$\text{PolyGamma}[z] \equiv \text{PolyGamma}[0,z] = \Psi(z) = \frac{1}{\Gamma(z)} \frac{d}{dz}\Gamma(z)$$

At integer values:

$$\Psi(N+1) \;=\; \sum_{k=1}^{N} \frac{1}{k} - \gamma_E \;=\; S_1(N) - \gamma_E \tag{14}$$

The following properties hold:

$$\begin{aligned}
\Psi(z+1) &= \Psi(z) + 1/z & (15)\\
\Psi(1+\epsilon) &= -\gamma_E + \zeta_2\,\epsilon + \dots & (16)\\
\Psi(1) &= -\gamma_E & (17)\\
\Psi(2) &= 1 - \gamma_E & (18)\\
\Psi(3) &= 3/2 - \gamma_E & (19)
\end{aligned}$$

Finally:

$$\mathrm{PolyGamma}[n,z] = \frac{d^n}{dz^n}\Psi(z) \tag{20}$$

It is e.g.

$$\text{PolyGamma}[2N, 1] = -(2N)! \; \zeta(2N + 1) \tag{21}$$

Cauchy Theorem and Residues

An integral over an anti-clockwise directed closed path C is:

$$\oint F(z)dz = 2\pi i \sum_{z=z_i} \text{Res}[F(z)] \tag{22}$$

where the residues $\text{Res}[F(z)]|_{z=z_i}$ are coefficients a^i_{-1} of the Laurent series of $F(z)$ around z_i:

$$F(z) = \sum_{n=-N}^{\infty} a^i_n (z-z_i)^n = \frac{a^i_{-N}}{(z-z_i)^N} + \cdots + \frac{a^i_{-1}}{(z-z_i)} + a^i_0 + \cdots$$

$$\text{Res}[F(z)]|_{z=z_i} = a^i_{-1} \tag{23}$$

If $G(z)$ has a Taylor expansion around z_0, then it is:

$$\text{Res}[G(z)\ F(z)]|_{z=z_i} = \sum_{n=1}^{N} \frac{a^i_{-n}}{k!} \frac{d^n}{dz^n} G(z)|_{z=z_i} \tag{24}$$

Due to this property, we need for applications not only $\Gamma(z)$, but also its derivatives.

Some residues with $\Gamma(z)$

$$
\begin{aligned}
\Psi(z) &= \text{PolyGamma}[z] = \text{PolyGamma}[0, z] \\
\text{Residue}[F[z]\Gamma[z], \{z, -n\}] &= \frac{(-1)^n}{n!} F[-n] \qquad (25) \\
\text{Residue}[F[z]\Gamma[z]^2, \{z, -n\}] &= \frac{2\text{PolyGamma}[n+1]F[-n] + F'[-n]}{(n!)^2}
\end{aligned}
$$

Some further examples

derived with Mathematica:

```
Series[Gamma[z]^2, {z, -3, -1}]
```

$$\frac{1}{36(z+3)^2} + \frac{\frac{11}{108} - \frac{EulerGamma}{18}}{z+3} + O[z+3]^0$$

In[8]:= `Series[Gamma[z - 2] Gamma[z + 5]^2, {z, 2, -1}]`

Out[8]= $\frac{518\,400}{z-2} + O[z-2]^0$

In[6]:= `Series[Gamma[z + 2] Gamma[z - 1]^2, {z, -2, -1}]`

Out[6]= $\frac{1}{36(z+2)^3} + \frac{\frac{11}{108} - \frac{EulerGamma}{12}}{(z+2)^2} + \frac{97 - 132\,EulerGamma + 54\,EulerGamm}{432(z+2)}$

In[4]:= `Series[Gamma[z + 2] Gamma[z - 1]^2, {z, 1, -1}]`

Out[4]= $\frac{2}{} + \frac{3 - 6\,EulerGamma}{} + O[z-1]^0$

Integrals + Some sums `Mathematica` **can do**

$$\oint_{-1/3-9i}^{-1/3+9i} dz \Gamma[z] = (-i)\, 3.97173 \qquad (26)$$

$$2\pi i \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = (2\pi i)\frac{1-e}{e} = (-i)\, 3.97173 \qquad (27)$$

This corresonds to closing the line to the left.

While, closing to the right gives another result (of course, . . .):

$$(-1) * 2\pi i \sum_{n=0}^{0} \frac{(-1)^n}{n!} = (2\pi i) \neq (-i)\, 3.97173 \qquad (28)$$

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), n, 0, Infinity] =
.   (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])

Sum[s^(n) PolyGamma[0, n + 1], n, 0, Infinity] =
.   (EulerGamma + Log[1 - s])/(-1 + s)

The above sums were done with Mathematica 5.2.
Mathematica versions 6 and 7 are more powerful.

L-loop n-point Feynman Integrals of tensor rank R with N internal lines

- Internal loop momenta are $k_l, \ l = 1 \cdots L$
- Propagators have mass m_i and momentum $q_i, \ i = 1 \cdots N$ and indices ν_i – see $G(X)$
- External legs have momentum $p_e, \ e = 1 \cdots n$, with $p_e^2 = M_e^2$

The N propagators are:

$$D_i \ = \ q_i^2 - m_i^2 \ = \ [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{n} d_i^e p_e]^2 - m_i^2$$

Feynman integrals have the following general form:

$$G(X) \ = \ \frac{e^{\epsilon \gamma_E L}}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \ X(k_{l_1}, \cdots, k_{l_R})}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.$$

The numerator X may contain a tensor structure (see later for more on that):

$$X(k_{l_1}, \cdots, k_{l_R}) \ = \ (k_{l_1} P_{e_1}) \cdots (k_{l_R} P_{e_R}) \ = \ (P_{e_1}^{\alpha_1} \cdots P_{e_R}^{\alpha_R}) \, (k_{l_1}^{\alpha_1} \cdots k_{l_R}^{\alpha_R})$$

Tensor integrals

Tensor integrals appear naturally in Feynman diagrams, due to

- fermion propagators
- non-abelian triple-boson vertices
- boson propagators in R_ξ gauges and unitary gauge

Example: Fermionic vacuum polarization

$$
\begin{aligned}
\Pi^{\alpha\beta} \quad &\sim \quad \frac{1}{(i\pi^{d/2})} \int d^d k \, \mathrm{Tr} \left[\frac{[\gamma k + m_1]}{D_1} \gamma^\beta \frac{[\gamma(k+p_1)+m_2]}{D_2} \gamma^\alpha \right] \\
&\sim \quad \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \left[(m_1 m_2 - k^2 - k p_1) g^{\alpha\beta} + 2 k^\alpha k^\beta \right. \\
&\qquad \left. + k^\alpha p_1^\beta + p_1^\alpha k^\beta \right]
\end{aligned}
\tag{29}
$$

So, one needs also efficient ways to evaluate tensor integrals –
see later

Simple examples of scalar integrals
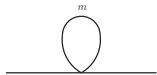
$$A_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1} \quad \to \mathrm{U}V - divergent: \ \sim \frac{d^4 k}{k^2}$$
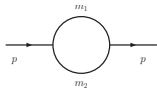
$$B_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2} \quad \to \mathrm{U}V - divergent \ \sim \frac{d^4 k}{k^4}$$
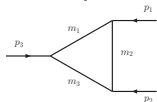
$$C_0 = \frac{1}{(i\pi^{d/2})} \int \frac{d^d k}{D_1 D_2 D_3} \quad \to \mathrm{U}V - finite \ \sim \frac{d^4 k}{k^6}$$

Dependent on conventions, where k starts to run in the loop, it is:

$$
\begin{aligned}
D_1 &= k^2 - m_1^2 \\
D_2 &= (k + p_1)^2 - m_2^2 \\
D_3 &= (k + p_1 + p_2)^2 - m_3^2
\end{aligned}
$$

Evaluate Feynman integrals

There are two strategies to solve a Feynman integral:

- Reduction
  Express the integral with the aid of recurrence relations by other, known integrals.
  These are then the Master Integrals.

- Direct evaluation

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \dots + \nu_N)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \dots x_N^{\nu_N - 1} \delta(1 - x_1 \dots - x_N)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}}$$

with $N_\nu = \nu_1 + \dots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1 + \dots + \nu_N)} = (x_1 D_1 + \dots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu} \quad (30)$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$
\begin{aligned}
k &= \bar{k} + (M^{-1})Q, \\
m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J.
\end{aligned}
\quad (31)
$$

Remember: $M_{1-loop} = 1$, in general:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M}, \quad (32)$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \rightarrow k$):

$$G(1) \quad = \quad \int \frac{Dk_1 \ldots Dk_L}{\left(kMk + J - QM^{-1}Q\right)^{N_\nu}}. \tag{33}$$

Go Euclidean: Rotate now the $k^0 \rightarrow iK_E^0$ with $k^2 \rightarrow -k_E^2$ (and again rename $k^E \rightarrow k$):

$$G(1) \rightarrow (i)^L \int \frac{Dk_1^E \ldots Dk_L^E}{\left(-k^E M k^E + J - QM^{-1}Q\right)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{\left[kMk - (J - QM^{-1}Q)\right]^{N_\nu}}.$$

Call

$$\mu^2(x) \quad = \quad -(J - QM^{-1}Q) \tag{34}$$

and get

$$G(1) \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{\left(kMk + \mu^2\right)^{N_\nu}}. \tag{35}$$

For 1-loop integrals it is $L = 1, M = 1$ - and we will use nearly only those - we are ready to do the k-integration.

Additional step for *L*-loop integrals

For L-loops go on and now diagonalize the matrix *M* by a rotation:

$$
\begin{aligned}
k \rightarrow k'(x) &= V(x)\,k, \\
kMk &= k' M_{diag} k' \\
&\rightarrow \sum \alpha_i(x) k_i^2(x), \\
M_{diag}(x) &= (V^{-1})^+ M V^{-1} = (\alpha_1, \ldots, \alpha_L).
\end{aligned}
$$

This leaves both the integration measure and the integral invariant:

$$
G(1) = (-1)^{N_\nu}(i)^L \int \frac{Dk_1 \ldots Dk_L}{\left(\sum_i \alpha_i k_i^2 + \mu^2\right)^{N_\nu}}. \tag{36}
$$

Rescale now the k_i,

$$\bar{k}_i = \sqrt{\alpha_i} k_i, \tag{37}$$

with

$$d^d k_i = (\alpha_i)^{-d/2} d^d \bar{k}_i, \tag{38}$$

$$\prod_{i=1}^{L} \alpha_i = \det M, \tag{39}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \rightarrow k$):

$$G(1) = (-1)^{N_\nu} (i)^L (\det M)^{-d/2} \int \frac{Dk_1 \ldots Dk_L}{\left(k_1^2 + \ldots + k_L^2 + \mu^2\right)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$i^L \int \frac{Dk_1 \dots Dk_L}{\left(k_1^2 + \dots + k_L^2 + \mu^2\right)^{N_\nu}} = \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2}}, \qquad (40)$$

$$i^L \int \frac{Dk_1 \dots Dk_L \, k_1^2}{\left(k_1^2 + \dots + k_L^2 + \mu^2\right)^{N_\nu}} = \frac{d}{2} \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - dL/2 - 1}}.$$

These formulae follow for $L = 1$ immediately from any textbook.
See 'Mathematical Interlude'.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Mathematical interlude: *d*-dimensional integrals (I)

After the Wick rotation, the integrand of the momentum integration is positive definite.

Further it is independent of the angular variables.

The integral is understood as symmetric limit the infinity boundaries.

$$
\begin{aligned}
\int d^d k \, k_\mu \, F(k^2) &= 0 \\
\int d^d k \, F(k + C) &= \int d^d k \, F(k).
\end{aligned}
$$

Introduce d-dim. spherical coordinates. The vector k has d components:

$$
\begin{aligned}
k_d &= r\cos\theta_d \equiv \rho_d\cos\theta_d \\
k_{d-1} &= \rho_{d-1}\cos\theta_{d-1} \\
&\cdots \\
k_3 &= \rho_3\cos\theta_3 \\
k_2 &= \rho_2\sin\phi \\
k_1 &= \rho_2\cos\phi \\
\rho_{d-1} &= \rho_d\sin\theta_d
\end{aligned}
$$

Mathematical interlude (II)

The above is the direct generalization of the 3- or
4-dimensional phase space parametrization.
With these variables, the integral over the complete
d-dimensional phase space gets the following form:

$$
\begin{aligned}
\int_{-\infty}^{\infty} d^d k \, F(k) \;=\; &\lim_{R \to \infty} \int_0^R dr r^{d-1} \int_0^\pi d\theta_{d-1} \sin^{d-2} \theta_{d-1} \\
&\int_0^\pi d\theta_{d-2} \sin^{d-3} \theta_{d-2} \dots \int_0^{2\pi} d\theta_1 F(k)
\end{aligned}
$$

The integrations met in the loop calculations may be performed using the following two integrals:

$$\int_0^\pi d\theta \sin^m \theta = \sqrt{\pi} \frac{\Gamma\left[\frac{1}{2}(m+1)\right]}{\Gamma\left[\frac{1}{2}(m+2)\right]}, \tag{41}$$

$$\int_0^\infty dr \frac{r^\beta}{(r^2 + M^2)^\alpha} = \frac{1}{2} \frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\alpha - \frac{\beta+1}{2}\right)}{\Gamma(\alpha)} \frac{1}{(M^2)^{\alpha - (\beta+1)/2}}.$$

In general, the angular integrations are influenced by the integrand too. (Remember phase space integrals of bremsstrahlung!)

Mathematical interlude (III)

If $F(k) \to F(r),\ r = |k|$, the angular integrations yield the surface of the d-dimensional sphere with radius r:

$$\omega_d(r) = \frac{2\pi^{d/2}}{\Gamma\left[\frac{d}{2}\right]} r^{d-1}. \tag{42}$$

The remaining integration, over r, yields for $F(r) = 1$ the volume of the sphere with radius R:

$$V_d(R) = \frac{\pi^{d/2}}{\Gamma\left[1 + \frac{d}{2}\right]} R^d, \tag{43}$$

$$G(1) = \int d^d k \frac{1}{(k^2 + M^2)^{N_\nu}}$$

$$= \int_0^\infty dr \frac{\omega_d(r)}{(r^2 + M^2)^{N_\nu}}$$

and we get immediately, with $M^2 \equiv M^2(x_1, x_2, \ldots)$:

$$G(1) = \left[\frac{i\pi^{d/2}\Gamma(N_\nu - d/2)}{\Gamma(N_\nu)} \frac{1}{(M^2)^{N_\nu - d/2}} \right]. \qquad (44)$$

Finally, one gets for scalar integrals:

$$
G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{(\det M)^{-d/2}}{(\mu^2)^{N_\nu - dL/2}},
$$

or

$$
G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j-1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}
$$

with

$$
U(x) = (\det M) \; (\to 1 \text{ for } L = 1) \tag{45}
$$

$$
F(x) = (\det M) \, \mu^2 = -(\det M) \, J + Q \, \tilde{M} \, Q \; (\to -J + Q^2 \text{ for } L = 1) \tag{46}
$$

Trick for one-loop functions:

$$U = \det M = 1 = \sum x_i \tag{47}$$

and so U 'disappears' and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j. \tag{48}$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand
$k \to \bar{k} + U(x)^{-1} \tilde{M} Q$
the $\int d^d \bar{k} \ \bar{k}/(\bar{k}^2 + \mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j - 1} \delta[1 - \sum_{i=1}^N x_i] \frac{U(x)^{N_\nu - d(L+1)/2 - 1}}{F(x)^{N_\nu - dL/2}}$$

$$\left[\sum_l \tilde{M}_{1l} Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$G(k_{1\alpha} k_{2\beta}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - 2 - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

$$\times \sum_l \left[[\tilde{M}_{1l} Q_l]_\alpha [\tilde{M}_{2l} Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{d}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x) F(x) \frac{(V_{1l}^{-1})^+ (V_{2l}^{-1})}{\alpha_l} \right]$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) \;=\; (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j - 1} \, \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu - d/2}}$$

$$(49)$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1 x_2$$

one-loop box:

$$F(s, t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1 x_2 + [-s]x_3 x_4$$

one-loop pentagon:

$$F(s, t, t', v_1, v_2, m^2) = m^2(x_1 + x_3 + x_4)^2 + [-t]x_1 x_3 + [-t']x_1 x_4 + [-s]x_2 x_5 + [-v_1]x_3 x_5 + [-$$

2-loop example: B7l4m2 = B2 (page 8), has a box-type sub-loop with 2 off-shell legs:
page):

$$
\begin{aligned}
F^{-(a_{4567}-d/2)} &= \Big\{ m^2(x_5 + x_6)^2 + [-t]x_4 x_7 + [-s]x_5 x_6 \\
&\quad + (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7 x_5 \Big\}^{-(a_{4567}-d/2)}
\end{aligned}
$$

2-loop: B5l2m2, sub-loop with 2 off-shell legs (diagram see p.7):

$$F_{2lines}(k_1^2, m^2) = m^2(x_3)^2 + [-k_1^2 + m^2]x_1 x_3$$

The Tadpole $A_0(m)$

$$\underset{m}{\bigcirc}\quad T1l1m[a] = A_0 \;\;=\;\; \frac{e^{\epsilon\gamma_E}}{(i\pi^{d/2})}\int \frac{d^d k}{(k^2-m^2)^a} \to \mathrm{UV-div.}$$

With our general formulae we get, in the 1-dimensional Feynman parameter integral, for the numerator
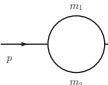
$$
\begin{aligned}
N &=& (k^2-m^2)x_1 \equiv k^2 + J \\
F &=& m^2 x_1 \equiv m^2 x_1^2
\end{aligned}
$$

and thus

$$
\begin{aligned}
T1l1m[a] &=& (-1)^a e^{\epsilon\gamma_E}\frac{\Gamma[a-d/2]}{\Gamma[a]}\int_0^1 dx x^{a-1}\delta[1-x]\frac{1}{F^{a-d/2}} \\
&=& (-1)^a e^{\epsilon\gamma_E}(m^2)^{2-a-\epsilon}\frac{\Gamma[a-2+\epsilon]}{\Gamma[a]} \\
&\to& -e^{\epsilon\gamma_E}\Gamma[-1+\epsilon] \text{ for a} = 1,\ \text{m} = 1 \\
&=& \frac{1}{\epsilon} + 1 + \left(1+\frac{\zeta_2}{2}\right)\epsilon + \left(1+\frac{\zeta_2}{2}-\frac{\zeta_3}{3}\right)\epsilon^2 + \cdots
\end{aligned}
$$

The Self-energy $B_0(s, m_1, m_2)$

$$SE2l = B_0[s, m_1, m_2] = (2\sqrt{\pi}\mu)^{4-d} \frac{e^{\epsilon \gamma_E}}{(i\pi^{d/2})} \int \frac{d^d k}{[k^2 - m^2][(k+p)^2 - m_2^2]}$$

The $SE2l$ is UV-divergent and the corresponding F-function is:

$$F[s, m_1, m_2] = m_1^2 x_1^2 + m_2^2 x_2^2 - [s - m_1^2 - m_2^2] x_1 x_2 \tag{50}$$

and for special cases:

$$
\begin{aligned}
F[s, m_1, 0] &= m_1^2 x_1^2 - [s - m_1^2] x_1 x_2 \\
F[s, m_1, m_1] &= m_1^2 (x_1 + x_2)^2 - [s] x_1 x_2 \\
F[s, 0, 0] &= -[s] x_1 x_2
\end{aligned}
$$

The 'conventional' Feynman parameter integral is 1-dimensional because
$x_2 \equiv 1 - x_1$**:**

$$F(x) = -sx(1-x) + m_2^2(1-x) + m_1^2 x \equiv -s(x - x_a)(x - x_b) \tag{51}$$

The result is of logarithmic type for the constant term in ϵ:

$$
\begin{aligned}
B_0[s, m_1, m_2] &= (4\pi\mu^2)^\epsilon \; e^{\epsilon\gamma_E} \; \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 \frac{dx}{F(x)^\epsilon} \\
&= \frac{1}{\epsilon} - \int_0^1 dx \ln\left(\frac{F(x)}{4\pi\mu^2}\right) \\
&\quad + \epsilon \left\{ \frac{\zeta_2}{2} + \frac{1}{2} \int_0^1 dx \ln^2\left(\frac{F(x)}{4\pi\mu^2}\right) \right\} + \mathcal{O}(\epsilon^2).
\end{aligned}
$$

Here we used the expansion:

$$
e^{\epsilon\gamma_E}\Gamma(1+\epsilon) = 1 + \frac{\zeta_2}{2}\epsilon^2 - \frac{\zeta_3}{3}\epsilon^3 \cdots \tag{52}
$$

When using `LoopTools`**, the corresponding call returns exactly the constant term of B_0 in ϵ (with use of $e^{\epsilon\gamma_E} = 1 + \epsilon\gamma_E + \cdots \to 1$):**

$$
B_0^{(0)}(s, m_1^2, m_2^2) = \texttt{b0(s,am12,am22)} \tag{53}
$$

For $4\pi\mu^2 \to 1$ B_0 looks quite compact:

$$B_0(s, m_1, m_2) = \frac{1}{\epsilon} - \int_0^1 dx \, \ln[F(x)] + \frac{\epsilon}{2}\left[\zeta_2 + \int_0^1 dx \ln^2[F(x)]\right] + \cdots \quad (54)$$

Explicitly, one has to integrate

$$\ln[F(x)] = \ln[-s(x - x_a)(x - x_b)]$$
$$\ln^2[F(x)] = \ln^2[-s(x - x_a)(x - x_b)]$$

So we will need the integrals:

$$\int dx_0^1 \{\ln(x - x_a), \, \ln(x - x_a)\ln(x - x_b)\} \quad (55)$$

which is trivial, together with some complex algebra rules how to handle complex arguments of logarithms with

$$s \to s + i\epsilon \quad (56)$$

wherever needed.

For the case $m_1 = m_2 = 1$, one gets for the first terms in ϵ:

$$
\begin{aligned}
B_0[s, 1, 1] &= \frac{1}{\epsilon} + 2 + \frac{1+y}{1-y} H(0, y), \quad (57) \\
H(0, y) &= \ln(y). \quad (58)
\end{aligned}
$$

The $H(0, y)$ is a harmonic polylogarithmic function, and
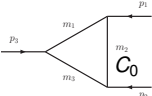
$$
\begin{aligned}
y &= \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \\
s &= -\frac{(1-x)^2}{x}
\end{aligned}
$$

The other case treated later again is $m_1 = 0, m_2 = m$:

$$
B_0[s, m^2, 0] = \frac{1}{\epsilon} + 2 + \frac{1 - s/m^2}{s/m^2} \ln(1 - s/m^2) \quad (59)
$$

The massive one-loop vertex $C_0(s, m_1, m_2)$

$$= \frac{e^{\epsilon \gamma_E}}{(i \pi^{d/2})} \int \frac{d^d k}{[(k + p_1)^2 - m^2][k^2][(k - p_2)^2 - m^2]} \sim |_{k \to \infty} \frac{d^4 k}{k^6} \to UV -$$

The massive vertex (all $m_1, m_2, m_3 \neq 0$) is a finite quantity.
We assume immediately here: $m_2 = 0, m_1 \neq 0 \neq m_3$**.**
A problem now is IR-divergence.
Appears when a massive internal line is between two external on-shell lines.
Incoming $p_1^2 = m^2$ and $p_2^2 = m^2$, look at $k \to 0$:

$$
\begin{aligned}
&d^4 k \ \frac{1}{(k - p_2)^2 - m^2} \ \frac{1}{(k)^2} \ \frac{1}{(k + p_1)^2 - m^2} \\
=\ &d^4 k \ \frac{1}{k^2 - 2kp_2} \ \frac{1}{(k)^2} \ \frac{1}{k^2 + 2kp_1} \\
\to\ &\frac{d^4 k}{k^{1+2+1}} \sim \frac{k^3 dk}{k^4} \sim \frac{dk}{k}|_{k \to 0} \longrightarrow div
\end{aligned}
$$

An IR-regularization is needed, must take $d > 4$.
Both UV-div (with $d < 4$) and IR-div together: must allow for a complex
$d = 4 - 2\epsilon$**, and take limit at the end.**

First we have a look, for later use, at the F-function:

$$
\begin{aligned}
N &= D_1 x + D_2 y + D_3 z &\quad(61)\\
&= k^2 x + (k^2 + 2kp_1)y + (k^2 - 2kp_2)z &\quad(62)\\
&= k^2(x + y + z) + 2k(p_1 y - p_2 z) &\quad(63)\\
&= (k + Q)^2 - Q^2 &\quad(64)
\end{aligned}
$$

**We used $1 = x + y + z$ here. And the F-function is
$F = Q^2 - J = Q^2$ (there is no constant term in N here), as
was shown before:**

$$
F = m^2(y + z)^2 + [-s]yz \quad(65)
$$

**This F-function does not factorize in y and z. But now
back to the direct Feynman parameter integration.**

Start with change $y \to y' = (1-x)y$, **then** $y' \to y$:

$$
\begin{aligned}
\frac{1}{D_1 D_2 D_3} &= \int_0^1 dx\,dy\,dz \frac{\delta(1-x-y-z)}{(D_2 x + D_1 y + D_3 z)^3} \quad (66) \\
&= \int_0^1 dx \int_0^{1-x} \frac{dy}{(D_2 x + D_1 y + D_3 z)^3} \quad (67) \\
&= \int_0^1 dx \int_0^1 \frac{x\,dy}{(D_2 x + D_1 y + D_3 z)^3} \quad (68)
\end{aligned}
$$

After this change of variables, the integrand factorizes in x **and** y:

$$
\begin{aligned}
N &= (k + x p_y)^2 - x^2 p_y^2 \quad (69) \\
&= (k + Q)^2 - Q^2 \quad (70)
\end{aligned}
$$

resulting into

$$
\begin{aligned}
F = Q^2 &= x^2 p_y^2 \quad (71) \\
p_y^2 &= -sy(1-y) + m^2 \quad (72)
\end{aligned}
$$

For C_0 we obtain (with $N_\nu = 3$ and $N_\nu - d/2 = 1 + \epsilon$):

$$C_0[s, m, m, 0] \quad = \quad (-1)e^{\epsilon \gamma_E} \Gamma[1 + \epsilon] \int_0^1 \frac{dx}{x^{1+2\epsilon}} \int_0^1 \frac{dy}{(p_y^2)^{1-\epsilon}} \tag{73}$$

This is integrable for $\epsilon < 0$, or $d > 4$, or more general: $d \not< 4$.
The x-integral made simple here:

$$\int_0^1 \frac{dx}{x^{1+2\epsilon}} \quad = \quad \frac{x^{-2\epsilon}|_0^1}{-2\epsilon} = -\frac{1^{-2\epsilon} - 0^{-2\epsilon}}{2\epsilon} \tag{74}$$

$$= \quad -\frac{1}{2\epsilon} \tag{75}$$

We see that the IR-singularity is an end-point-singularity in Feynman parameter space.
Further:

$$-\frac{1}{2\epsilon} \frac{dy}{(p_y^2)^{1-\epsilon}} \quad = \quad -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} (p_y^2)^\epsilon \tag{76}$$

$$= \quad -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} e^{\epsilon \ln(p_y^2)} \tag{77}$$

$$= \quad -\frac{1}{2\epsilon} \frac{dy}{(p_y^2)} \left[1 + \epsilon \ln(p_y^2) + \epsilon^2 \ln^2(p_y^2) + \cdots \right] \tag{78}$$

Here I stop this study.

We see that the further integrations proceed quite similar as for the 2-point function, in fact the $p_y^2 = -sy(1-y) + m^2$ is the same building block.
The integrals to be solved now are more general, they include also denominators $1/p_y^2$:

Some integrals

$$\int dy \ln(y - y_0) = (y - y_0)\ln(y - y_0) - y + C \tag{79}$$

$$\int dy \frac{1}{y - y_0} = \ln(y - y_0) + C \tag{80}$$

$$\int dy \frac{\ln(y - y_0)}{y - y_0} = \frac{1}{2}\ln^2(y - y_0) + C \tag{81}$$

Here, often y is real and y_0 is complex. Then no special care about phases is necessary.

$$\int_0^1 \frac{dx}{x - x_0}\left[\ln(x - x_A) - \ln(x_0 - x_A)\right] = Li_2\left(\frac{x_0}{x_0 - x_A}\right) - Li_2\left(\frac{x_0 - 1}{x_0 - x_A}\right) \tag{82}$$

This formula is valid if x_0 is real.

C_0 with a small photon mass λ

In [**1**, **2**], the C_0-integral is treated with a finite photon mass:

$$\int \frac{d^4 k}{(k^2 - \lambda^2)(k^2 + 2kp_1)(k^2 - 2kp_2)} \tag{83}$$

$$= -i\pi^2 \int_0^1 dy dx \, \frac{y}{x^2 p_y^2 + (1-x)\lambda^2} \tag{84}$$

$$= i\pi^2 \int_0^1 dy \left[\frac{1}{2p_y^2} \ln \frac{\lambda^2}{p_y^2} + \mathcal{O}\left(\lambda / \sqrt{p_y^2} \right) \right], \tag{85}$$

It is easy to see from the term $1/(2p_y^2) \, \ln(\lambda^2)$ the **correspondence of** $(d - 4)$ **and** λ^2, which is a universal relation in all 1-loop cases.

Now using Mellin-Barnes Representations

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation
Computercodes:

- Ambre.m - Derive Mellin-Barnes representations for Feynman integrals [3]

- MB.m - Find an ϵ-expansion and evaluate numerically in Euclidean region [4]

Integrating the Feynman parameters – get MB-Integrals

We derived:

$$SE2l1m = B_0(s, m, 0) \quad = \quad e^{\epsilon\gamma_E}\,\Gamma(\epsilon)\int_0^1 dx_1\,dx_2\frac{\delta(1 - x_1 - x_2)}{F(x)^\epsilon} \tag{86}$$

$$V3l2m = C_0(s, m, m, 0) \quad = \quad e^{\epsilon\gamma_E}\,\Gamma(1 + \epsilon)\int_0^1 dx_1\,dx_2\,dx_3\frac{\delta(1 - x_1 - x_2 - x_3)}{F(x)^{1+\epsilon}} \tag{87}$$

and

$$F_{SE2l1m} \quad = \quad m^2x_1^2 - (s - m^2)x_1x_2 \tag{88}$$

$$F_{V3l2m} \quad = \quad m^2(x_1 + x_2)^2 - (s)x_1x_2 \tag{89}$$

We want to apply now:

$$\int_0^1 \prod_{j=1}^N dx_j\,x_j^{\alpha_j - 1}\,\delta\left(1 - \sum x_i\right) \quad = \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_7 N)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_N)} \tag{90}$$

with coefficients α_i dependent on ν_i and on the structure of the F

See in a minute:
For this, we have to apply one or several
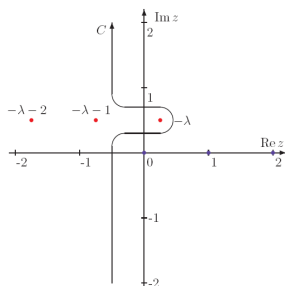## **MB-integrals here.**

$$\int_0^1 \prod_{j=1}^N dx_j \, x_j^{\alpha_j-1} \, \delta\left(1 - \sum_{i=1}^N x_i\right) \;=\; \frac{\prod_{i=1}^N \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^N \alpha_i\right)} \tag{91}$$

Simplest cases:

$$\int_0^1 dx_1 \, x_1^{\alpha_1-1} \, \delta\left(1 - x_1\right) \;=\; 1$$

$$\int_0^1 \prod_{j=1}^2 dx_j \, x_j^{\alpha_j-1} \, \delta\left(1 - \sum_{i=1}^N x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1-1}(1-x_1)^{\alpha_2-1} \;=\; B(\alpha_1, \alpha_2)$$

$$=\; \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma\left(\alpha_1 + \alpha_2\right)}$$

Here we want to go:
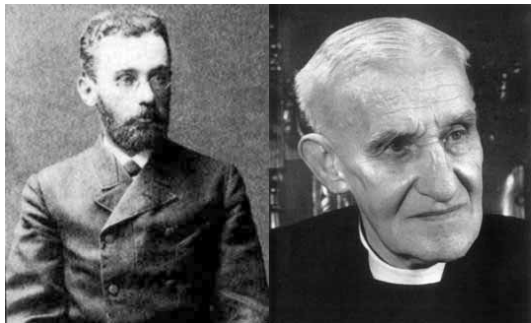
$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}} \quad (92)$$

The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$.
The formula looks a bit unusual to loop people, but for persons
with a mathematical background it is common knowledge.

One might well assume that these two gentlemen did not dream of so heavy use of their results in basic research · · ·

Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953

Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290
Consider

$$F(z) \quad = \quad \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} \tag{93}$$

where $|\arg(-z)| < \pi$ (i.e. $(-z)$ is not on the neg. real axis) and the path is such that it separates the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ from the poles of $\Gamma(-\sigma)$.
$1/\Gamma(c+\sigma)$ **has no pole.**
Assume $a \neq -n$ and $b \neq -n, n = 0, 1, 2, \cdots$ so that the contour can be drawn.
The poles of $\Gamma(\sigma)$ are at $\sigma = -n, n = 1, 2, \cdots$, and it is:

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ the hypergeometric function $_2F_1(a, b, c, z)$ (for proof see textbook):

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma(-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N\to\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \quad _2F_1(a, b, c, z)$$

The continuation of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n}$$

$$+ \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \ _2F_1(a,b,c,z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a} \ _2F_1(a,1-c+a,1-b+ac,z^{-1})$$

$$+ \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b} \ _2F_1(b,1-c+b,1-a+b,z^{-1})$$

Corollary I

Putting $b = c$, we see that

$$
\begin{aligned}
{}_2F_1(a, b, b, z) &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!} \\
&= \frac{1}{(1-z)^a} \quad = \quad \frac{1}{2\pi i\, \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma \, (-z)^\sigma \, \Gamma(a+\sigma)\Gamma(-\sigma)
\end{aligned}
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{1}{B^a[1-(-A/B)]^a} \quad = \quad \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma \, B^{-\sigma-a} \, \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$ lie on the right of the path and the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$ lie on the left, then

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma \Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(c - \sigma)\Gamma(d - \sigma) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

It is supposed that a, b, c, d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $(a + b + c + d - 1) < 0$. Take sum of residues of the integrand at poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of *sin*, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $(a + b + c + d - 1) < 0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a, b, c, d for which none of the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$.

Corollary II Any real shift k: $\sigma + k, a - k, b - k, c + k, d + k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can the Mellin-Barnes formula be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i \, \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \, \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which transforms a massive propagator to a massless one (with index a of the line changed to $(a + \sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get a single monomial in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \, \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral

- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral

- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'

- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined

- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

A self-energy: SE2l1m

This is a nice example, being simple but showing [nearly] all essentials in a nutshell.
We get for this $F(x) = m^2 x_1^2 - (s - m^2)x_1 x_2$ the following representation:

$$SE2l1m = \frac{e^{\epsilon \gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s + m^2}{m^2}\right]^{-\epsilon - z} \tag{94}$$

$$\times \Gamma_1[1 - \epsilon - z]\Gamma_2[-z]\Gamma_3[1 - \epsilon + z]\Gamma_4[\epsilon + z] \tag{95}$$

Tausk approach:
Seek a configuration where all arguments of Γ-functions have positive real part. Then the $SE2l1m$ is well-defined and finite.
For small ϵ this is - here - evidently impossible; set $\epsilon \to 0$ and look at $\Gamma_2[-z]\Gamma_4[+z]$:

$$\Gamma_1[1 - z]\Gamma_2[-z]\Gamma_3[1 + z]\Gamma_4[+z] \tag{96}$$

What to do ????
Tausk: Set ϵ such that all arguments of Γ-functions get positive real parts, e.g. with the choice:

$$\epsilon = 3/8 \tag{97}$$

To make physics we have now to deform the integrand or the path such that $\epsilon \to 0$; when crossing a residue, take it and add it up.
Varying $\epsilon \to 0$ from 3/8 makes crossing in $\Gamma_4[\epsilon + z]$ a pole at $\epsilon = -z = +1/8$; there is $\epsilon + z = 0$:

$$\text{\textbf{Residue}[\textbf{SE2l1m}}, \{z, -\epsilon\}] = e^{\epsilon \gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \Gamma_3[1 - 2\epsilon] \Gamma_2[\epsilon] \tag{98}$$

Here we 'loose' one integration (easier term!) and catch the IR-singularity in $\Gamma_2[\epsilon] \sim 1/\epsilon$!
The function becomes now, for small ϵ:

$$\textbf{SE2l1m} = \frac{e^{\epsilon \gamma_E}}{2\pi i} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \int_{\Re z = -1/8} dz \left[\frac{-s + m^2}{m^2} \right]^{-\epsilon - z} \Gamma_1[1 - \epsilon - z] \Gamma_2[$$

$$\times \Gamma_3[1 - \epsilon + z] \Gamma_4[\epsilon + z] + e^{\epsilon \gamma_E} \frac{(m^2)^{-\epsilon}}{\Gamma[2 - 2\epsilon]} \Gamma_1[1 - 2\epsilon] \Gamma_2[\epsilon]$$

Now we may take safely the limit of small ϵ:

$$
\textbf{SE2l1m} \quad = \quad \frac{e^{\epsilon \gamma_E}}{2\pi i} \int_{\Re z = -1/8} dz \left[\frac{-s + m^2}{m^2} \right]^{-z} \Gamma_3[1 - z]\Gamma_4[-z]\Gamma[z]\Gamma[1 + z] \quad (99)
$$

$$
+ e^{\epsilon \gamma_E} \left(2 + \frac{1}{\epsilon} - \ln[m^2] \right) \ + \ )O(\epsilon^1) \quad (100)
$$

Now we close the integration path to the left, catch all residues from $\Gamma_3[1 - z]$ for $z < -1/8$, i.e. at $z = -n$, $n = 1, 2, \ldots$:

$$
\textbf{Res} \left\{ \left[\frac{-s + m^2}{m^2} \right]^{-z} \Gamma_1[1 - \epsilon - z]\Gamma_2[-z]\Gamma_3[1 - \epsilon + z]\Gamma_4[\epsilon + z], \{z, -n\} \right\}
$$

$$
= (-s + m^2)^n \ln(-s + m^2) \quad (101)
$$

The sum to be done is trivial (in this trivial case!!):

$$
\sum_{n=1}^{\infty} \left[\frac{-s + m^2}{m^2} \right]^n = \frac{1}{1 - \frac{-s + m^2}{m^2}} - 1 \quad (102)
$$

and we end up with:

$$\textbf{SE2l1m} \quad = \quad \frac{1}{\epsilon} + 2 + \left[\frac{1 - s/m^2}{s/m^2} \ln(1 - s/m^2) \right] \tag{103}$$

This is what we had also from the direct Feynman parameter integration above, page 53.

A vertex: V3l2m

The Feynman integral V3l2m is the QED one-loop vertex function, which is no master. It is infrared-divergent (see this by counting of powers of loop integration momentum k or know it from: massless line between two external on-shell lines)

$$F = m^2(x_1 + x_2)^2 + [-s]x_1 x_2 \tag{104}$$

We will also use the variable

$$y = \frac{\sqrt{-s+4} - \sqrt{-s}}{\sqrt{-s+4} + \sqrt{-s}} \tag{105}$$

$$\begin{aligned}
\text{V3l2m} &= \frac{e^{\epsilon \gamma_E} \Gamma(-2\epsilon)}{2\pi i} \int dz_{\Re z = -1/2} (-s)^{-\epsilon-1-z} \frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)} \\
&= \frac{\text{V3l2m}[-1]}{\epsilon} + \text{V3l2m}[0] + \epsilon \, \text{V3l2m}[1] + \cdots.
\end{aligned}$$

One may slightly shift the contour by $(-\epsilon)$ and then close the path to the left and get residua from (and only from) $\Gamma(1 + z)$: [5]:

$$
\begin{aligned}
V(s) &= \frac{1}{2s\epsilon} \frac{e^{\epsilon\gamma_E}}{2\pi i} \int\limits_{-i\infty-1/2}^{-i\infty-1/2} dz \; (-s)^{-z} \frac{\Gamma^2(-z)\Gamma(-z+\epsilon)\Gamma(1+z)}{\Gamma(-2z)} \\
&= -\frac{e^{\epsilon\gamma_E}}{2\epsilon} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} \frac{\Gamma(n+1+\epsilon)}{\Gamma(n+1)}.
\end{aligned}
$$

This series may be summed directly with Mathematica![1], and the vertex becomes:

$$
V(s) = -\frac{e^{\epsilon\gamma_E}}{2\epsilon} \Gamma(1+\epsilon) \; _2F_1 \left[1, 1+\epsilon; 3/2; s/4 \right]. \tag{106}
$$

Alternatively, one may derive the ϵ-expansion by exploiting the well-known relation with harmonic numbers $S_k(n) = \sum_{i=1}^{n} 1/i^k$:

$$
\frac{\Gamma(n+a\epsilon)}{\Gamma(n)} = \Gamma(1+a\epsilon) \exp\left[-\sum_{k=1}^{\infty} \frac{(-a\epsilon)^k}{k} S_k(n-1) \right]. \tag{107}
$$

The product $\exp(\epsilon\gamma_E)\Gamma(1+\epsilon) = 1 + \frac{1}{2}\zeta[2]\epsilon^2 + O(\epsilon^3)$ yields expressions with zeta numbers $\zeta[n]$, and, taking all terms together, one gets a collection of inverse binomial sums[2]; the first of them is the IR divergent part:

$$V(s) = \frac{V_{-1}(s)}{\epsilon} + V_0(s) + \cdots \tag{108}$$

$$V_{-1}(s) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{1}{2}\frac{4\arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}} = \frac{y}{y^2-1}\ln(y). \tag{109}$$

---

[1]The expression for $V(s)$ was also derived in [6]; see additionally [7].

[2]For the first four terms of the ϵ-expansion in terms of inverse binomial sums or of polylogarithmic functions, see [5].

The constant term:

$$
\begin{aligned}
\texttt{V3l2m[0]} \;=\; & \frac{1}{2\pi i} \int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r} \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]} \\
& \frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1+r]\right] \\
\\
=\; & \frac{1}{2} \sum_{n=0}^{\infty} \frac{s^n}{\dbinom{2n}{n}(2n+1)} S_1(n),
\end{aligned}
$$

There is also the opportunity to evaluate the MB-integrals numerically by following with e.g. a Fortran routine the straight contour.
This applies after the ϵ-expansion.

$\int_{-5i+z}^{+5i+z}$ is usually sufficient.

But: This works fast and stable for Euclidean kinematics where $-s > 0$.

and the ϵ-term:

$$
\begin{aligned}
\text{V3l2m[1]} \quad = \quad & \frac{1/4}{2\pi i} \int_{-i\infty+u}^{+i\infty+u} dr(-s)^{-1-r} \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]} \\
& \Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\
& + \gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\
& - 4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\
& + 4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\
& - \Psi[-z]\Psi[1+z]) + \Psi[1+z]^2 \Big] \\
= \quad & [const = 1?] \times \frac{1}{4} \sum_{n=0}^{\infty} \frac{(t)^n}{\binom{2n}{n}(2n+1)} \Big[S_1(n)^2 + \zeta_2 - S_2(n) \Big] .
\end{aligned}
$$

Here, $\Psi[r] = ...$ and $\Psi[1,r] = ...$, and the harmonic numbers $S_k(n)$ are

$$
S_k(n) \quad = \quad \sum_{i=1}^{n} \frac{1}{i^k}, \tag{110}
$$

The sums appearing above may be obtained from sums listed in Table 1 of Appendix D in [5, 8]:

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{y}{y^2-1} 2\ln(y),$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n) = \frac{y}{y^2-1} \left[-4\text{Li}_2(-y) - 4\ln(y)\ln(1+y) + \ln^2(y) - 2\zeta_2 \right],$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n)^2 = \frac{y}{y^2-1} \left[16 S_{1,2}(-y) - 8\text{Li}_3(-y) + 16\text{Li}_2(-y)\ln(1+y) \right.$$

$$+ 8\ln^2(1+y)\ln(y) - 4\ln(1+y)\ln^2(y) + \frac{1}{3}\ln^3(y) \quad (111)$$

$$\left. + 8\zeta_2\ln(1+y) - 4\zeta_2\ln(y) - 8\zeta_3 \right],$$

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_2(n) = -\frac{y}{3(y^2-1)} \ln^3(y),$$

Expansion in a small parameter: vertex V3l2m for m^2/s

Use as an example for determining the small mass expansion:

$$\text{V3coefm1} = \text{Coefficient}[\text{V3l2m}[[1, 1]], \epsilon, -1] \tag{112}$$

$$= -\frac{1}{2s}\frac{1}{2\pi i}\int_{-i\infty-1/2}^{+i\infty-1/2} dz \left(-\frac{m^2}{s}\right)^z \frac{\Gamma_1[-z]^3\Gamma_2[1+z]}{\Gamma_3[-2z]} \tag{113}$$

If $|m^2/s| << 1$, then the smallest [positive] power of it gives the biggest contribution: its exponent has to be positive and small.
So, close the contour to the right (positive z), and leading terms come from the residua expansion of $\Gamma_1[-z]^3/\Gamma_3[-2z]$ at $z = -1, -2, \cdots$. The residues are terms of a binomial sum:

$$\text{Residue} = -\frac{1}{s}\left(\frac{m^2}{s}\right)^n \frac{(2n)!}{(n!)^2}\left[2\text{HarmonicNumber}[n] - 2\text{HarmonicNumber}[2n]\right.$$

$$\left. - \ln\left(-\frac{m^2}{s}\right)\right]$$

with first terms equal to (-1)*Residua:

$$\text{V3l2m} = \frac{1}{s}\ln\left(-\frac{m^2}{s}\right) + \frac{m^2}{s^2}\left[\ln\left(2 + 2\frac{m^2}{s}\right)\right] + \frac{m^4}{s^3}\left[\ln\left(7 + 6\frac{m^2}{s}\right)\right] + O(m^6/s^3) \tag{114}$$

End of first print file

Another nice box with numerator, B5l3m($p_e.k_1$).

We used it for the determination if the small mass expansion.

$$\text{B513m}(p_e \cdot k_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\, e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}$$

$$(-s)^{(4-2\epsilon)-a_{12345}-\alpha-\beta-\delta}\,(-t)^{\delta}$$

$$\frac{\Gamma[-4+2\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\epsilon-a_{12345}-\alpha]}\quad\frac{\Gamma[-\alpha]\,\Gamma[-\beta]}{\Gamma[7-3\epsilon-a_{12345}-\alpha]\,\Gamma[5-2\epsilon-a_{123}]\,\Gamma[4-2\epsilon-a}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma]}\quad\frac{\Gamma[4-2\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-}$$

$$\Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta]\,\Gamma[3-\epsilon-a_{12}-\alpha]\,\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma]$$

$$\Gamma[5-2\epsilon-a_{1123}-\gamma]\,\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma]\,\Gamma[a_1+\gamma]\,\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+}$$

$$\Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma]\Gamma[9-4\epsilon-a_{112233}}$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\,\Gamma[5-2\epsilon-a_{1123}-\gamma]\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma]\,\Gamma[a_1+\gamma]+\Gamma[2-$$

$$\Gamma[5-2\epsilon-a_{1123}-2\alpha-\gamma]\,\Gamma[1+a_1+\gamma]\Big]\,\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\epsilon-$$

$$\Gamma[5-2\epsilon-a_{1123}-\gamma]\,\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\,\Gamma[5-2\epsilon-$$

$$\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma]$$

$$\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\,\Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta]$$

Example beyond Harmonic Polylogs: QED Box B4l2m

[9]

```
F[x]      = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7
```

B4l2m, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite ϵ:

$$\text{B4l2m} = \text{Box}(t, s) = \frac{e^{\epsilon \gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \qquad (116)$$

$$\frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2+\epsilon+z_1+z_2]\Gamma^2[1+z_1]\Gamma[-z_1]\Gamma[-z_2]$$

$$\Gamma^2[-1-\epsilon-z_1-z_2]\frac{\Gamma[-2-2\epsilon-2z_1]}{\Gamma[-2-2\epsilon-2z_1-2z_2]}$$

Mathematica package MB used for analytical expansion $\epsilon \to 0$:[4]

$$\texttt{B4l2m} = -\frac{1}{\epsilon}\mathtt{J}_1 + \ln(-s)\mathtt{J}_1 + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]\mathtt{J}_1 - 2\mathtt{J}_2\right). \tag{117}$$

with J_1 being also the divergent part of the vertex function $C_0(t; m, 0, m)/s = \texttt{V3l2m}/s$ (as is well-known):

$$\mathtt{J}_1 = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1} \frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y) \tag{118}$$

with

$$y = \frac{\sqrt{1-4m^2/t}-1)}{(\sqrt{1-4m^2/t}+1)} \tag{119}$$

The J_2 is more complicated:

$$\mathtt{J}_2 = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2} \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{s}{t}\right)^{z_1} \Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1] \tag{120}$$

$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(-\frac{m^2}{t}\right)^{z_2} \Gamma[-z_2]\frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]}\Gamma[2+z_1+z_2].$$

The expansion of `B4l2m` at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \tag{121}$$

$$r = \frac{s}{t}, \tag{122}$$

Look, under the integral, at $(-m^2/t)^{z_2}$,
and close the path to the right.
Seek the residua from the poles of Γ-functions with the smallest powers in m^2 and sum the resulting series.
we have obtained a compact answer for J_2 with the additional aid of XSUMMER [10] The box contribution of order ϵ in this limit becomes:

$$
\begin{aligned}
\texttt{B4l2m}[t, s, m^2; +1] = \frac{1}{st}\Big\{ & 4\zeta_3 - 9\zeta_2\ln(m_t) + \frac{2}{3}\ln^3(m_t) + 6\zeta_2\ln(r) - \ln^2(m_t)\ln(r) \\
& + \frac{1}{3}\ln^3(r) - 6\zeta_2\ln(1+r) + 2\ln(-r)\ln(r)\ln(1+r) - \ln^2(r)\ln(1+r) \\
& + 2\ln(r)\text{Li}_2(1+r) + 2\text{Li}_3(-r) \Big\} + \mathcal{O}(m_t).
\end{aligned} \tag{123}
$$

Remark:

The exact Box function is NOT expressible by Harmonic Polylogs, one may introduce a generalization of them: Generalized HPLs.
Automatized tools for this might be developed.
A scetch of the small mass expansion may be made as follows.
First the 1-dim. integral J_1.
The leading term comes from the first residue:

```
J1 =  Residue[m_t^z1 Gamma[-z1]^3 Gamma[1 + z1]/Gamma[-2 z1], {z1, 0}]
    = 2 Log[m_t]
```

We get a logarithmic mass dependence.
The second integral: Start with z_2, first residue is:

```
I2 = Residue[ m_t^z2 Gamma[-z2] Gamma[-1 - z1 - z2]^2
          Gamma[2 + z1 + z2]/Gamma[-2 - 2 z1 - 2 z2], {z2, 0}]
```

$$= -\frac{\Gamma[-1 - z_1]^2 \Gamma[2 + z_1]}{\Gamma[-2 - 2z_1]} \tag{124}$$

The residue is independent of m^2/t.

It has to be integrated over z_1 yet, together with the terms which were independent of z_2:

$$I_2 \sim \int dz_1 \, r^{z_1+1} \Gamma[-z_1]\Gamma[-2-2z_1]\Gamma[1+z_1]^2 \frac{\Gamma[-1-z_1]^2\Gamma[2+z_1]}{\Gamma[-2-2z_1]} \tag{125}$$

Sum over residues, close path to the left:

$$
\begin{aligned}
Residue[z_1 = -n] &= \frac{(-1)^n r^{1-n}}{2(-1+n)^3}\left[2 + (-1+n)^2\pi^2 + (-1+n)\ln[r](2 + (-1+n)\ln[r])\right] \\
Residue[z_1 = -1] &= \frac{1}{6}(3\pi^2\ln[r] + \ln[r]^3) \tag{126}
\end{aligned}
$$

and finally:

$$I_2 \sim Residue[z_1 = -1] + \sum_{n=2}^{\infty} Residue[z_1 = -n] \tag{127}$$

The sum can be done also without using XSUMMER (here at least), e.g.

$$\ln[r]\sum_{n=2}^{Infty} \frac{(-1)^n(2+\pi^2-2n\pi^2+n^2\pi^2)r^{1-n}}{2(-1+n)^3} = \frac{1}{2}\left[\pi^2\ln(1+1/r) - 2Li_3\left(-1/r\right)\right]$$

etc

Sector decomposition

For Euclidean kinematics, the integrand for the multi-dimensional x-integrations is positive semi-definite.

In numerical integrations, one has to separate the poles in $(d - 4)$, and in doing so one has to avoid overlapping singularities.

A method for that is sector decomposition.

There are quite a few recent papers on that, and also nice reviews are given [11, 12, 13, 14, 15] The intention is to separate singular regions in different variables from each other, as is nicely demonstrated by an example borrowed from [14]:

$$
\begin{aligned}
I &= \int_0^1 dx \int_0^1 dy \frac{1}{x^{1+a\epsilon} y^{b\epsilon} [x + (1-x)y]} \\
&= \int_0^1 \frac{dx}{x^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{b\epsilon} [1 + (1-x)t]} + \int_0^1 \frac{dy}{y^{1+(a+b)\epsilon}} \int_0^1 \frac{dt}{t^{1+a\epsilon} [1 + (1-y)t]}
\end{aligned}
\tag{128}
$$
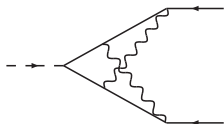
The master integral `V6l4m1`

At several occasions, we used for cross checks the package `sector_decomposition` [13] built on the C++ library GINAC [16] For that reason, the interface `CSectors` was written [**?**, **?**].

The syntax is similar to that of `AMBRE`.

Example:

The program input for the evaluation of the integral `V6l4m1` is simple; we choose $m = 1, s = -11$, and the topology may be read from the arguments of propagator functions `PR`:

```
<< CSectors.m

Options[DoSectors]
SetOptions[DoSectors, TempFileDelete -> False, SetStrategy -> C]

n1 = n2 = n3 = n4 = n5 = n6 = n7 = 1;
m = 1; s = -11;
invariants = {p1^2 -> m^2, p2^2 -> m^2, p1 p2 -> (s - 2 m^2)/2};

DoSectors[{1},
   {PR[k1,0,n1]      PR[k2,0,n2]      PR[k1+p1,m,n3]
    PR[k1+k2+p1,m,n5] PR[k1+k2-p2,m,n6] PR[k2-p2,m,n7]},
    {k2, k1}, invariants][-4, 2]
```

Here, the numerator is 1 (see the first argument $\{1\}$ of DoSectors), and the output contains the functions U_2 and F_2:

```
Using strategy C
U = x3 x4+x3 x5+x4 x5+x3 x6+x5 x6+x2 (x3+x4+x6)+x1 (x2+x4+x5+x6)

F = x1 x4^2+13 x1 x4 x5+x4^2 x5+x1 x5^2+x4 x5^2+13 x1 x4 x6
+2 x1 x5 x6+13 x4 x5 x6+x5^2 x6+x1 x6^2+x5 x6^2+x3^2 (x4+x5+x6)
+x2(x3^2+x4^2+13 x4 x6+x6^2+x3 (2 x4+13 x6))+x3 (x4^2+(x5+x6)^2
+x4 (2x5+13 x6))
```

Notice the presence of a *U*-function and the complexity of the *F*-function (compared to $U = 1$ and `f1` and `f2` in the loop-by-loop MB-approach) due to the non-sequential, direct performance of both momentum integrals at once. Both *U* and *F* are evidently positive semi-definite. The numerical result for the Feynman integral is:

$$\texttt{V6l4m1}(-s)^{2\epsilon} = -0.052210 \frac{1}{\epsilon} - 0.17004 + 0.24634\,\epsilon + 4.8773\,\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (129)$$

The numbers may be compared to (132). We obtained a third numerical result, also by sector decomposition, with the Mathematica package FIESTA [17]

$$\texttt{V6l4m1}(-s)^{2\epsilon} = -0.052208 \frac{1}{\epsilon} - 0.17002 + 0.24622\,\epsilon + 4.8746\,\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (130)$$

Most accurate result: obtained with an analytical representation based on harmonic polylogarithmic functions obtained by solving a system of differential equations (Gluza, Riemann, unpublished, [18, 19])

$$\texttt{V6l4m1}(-s)^{2\epsilon} = -0.0522082\,\frac{1}{\epsilon} - 0.170013 + 0.246253\,\epsilon + 4.87500\,\epsilon^2 + \mathcal{O}(\epsilon^3). \quad (131)$$

All displayed digits are accurate here.

V6l4m: Compare to MB-integrals

In a loop-by-loop approach, after the first momentum integration one gets here $U = 1$ and a first F-function (**??**), which depends yet on one internal momentum k_1:

```
f1 = m^2 [X[2]+X[3]+X[4]]^2 - s X[2]X[4] - PR[k1+p1,m] X[1]X[2]
           - PR[k1+p1+p2,0] X[2]X[3] - PR[k1-p2,m] X[1]X[4]
              - PR[k1,0] X[3]X[4] ,
```

leading to a 7-dimensional MB-representation; after the second momentum integration, one has:

```
f2 = m^2 [X[2]+X[3]]^2 - s X[2]X[3] - s X[1]X[4] - 2s X[3]X[4],
```

leading to another 4-dimensional integral.

After several applications of Barnes' first lemma, an 8-dimensional integral has to be treated.

We made no attempt here to simplify the situation by any of the numerous tricks and reformulations etc. known to experts.

The package `AMBRE.m` is designed for a semi-automatic derivation of Mellin-Barnes (MB) representations for Feynman diagrams; for details and examples of use see the webpage http://prac.us.edu.pl/~gluza/ambre/. The package is also available from http://projects.hepforge.org/mbtools/.

Version 1.0 is described in [3, **?**], the last released version is version 2.0 [**?**, **?**], which allows to construct MB-representations for two-loop *tensor* integrals.

The package is yet restricted to the so-called loop-by-loop approach, which yields compact representations, but is known to potentially fail for non-planar topologies with several scales. An instructive example has been discussed in [20].

For one-scale problems, one may safely apply `AMBRE.m` to non-planar diagrams. For our example `V6l4m1`, one gets e.g. with the 8-dimensional MB-representation scetched above the following numerical output after running also `MB.m` [4] (see also the webpage `http://projects.hepforge.org/mbtools/`),

at $s = -11$:

$$\texttt{V6l4m} \, (-s)^{2\epsilon} = -0.0522082 \, \frac{1}{\epsilon} - 0.17002 + 0.25606 \, \epsilon + 4.67 \, \epsilon^2 + \mathcal{O}(\epsilon^3). \quad (132)$$

Compare this to an MB-integral for V6l0m
Was solved first in [21]

```
fupc1 = m^2 FX[X[3] + X[4]]^2 - PR[k1 + p1, 0] X[1] X[2] -
 PR[k1 + p1 + p2, m] X[2] X[3] - PR[k1 - p2, m] X[1] X[4] -
 S X[2] X[4] - PR[k1, 0] X[3] X[4]

fupc2 = -s X[2] X[3]-s X[1] X[4]-2 s X[3] X[4]
```

The function is a scale factor times a pure number

$$V6l0m = const(-s)^{-2-2\epsilon} \left[\frac{A_4}{\epsilon^4} + \frac{A_3}{\epsilon^3} + \frac{A_2}{\epsilon^2} + \frac{A_1}{\epsilon} + A_0 + \dots \right] \qquad (133)$$

```
V6l0m = (-s)^{-2 - 2 eps}
  *
Gamma[-z1] Gamma[-1 - eps - z1 - z2] Gamma[
    1 + z1 + z3] Gamma[-1 - eps - z1 - z3 - z4] Gamma[-z4] Gamma[
    1 + z1 + z2 + z4] Gamma[
    2 + eps + z1 + z2 + z3 + z4] Gamma[-2 - 2 eps - z2 - z4 -
     z5] Gamma[-z5] Gamma[
    1 - z1 + z5] Gamma[-2 - 2 eps - z3 - z4 - z6] Gamma[-z6] Gamma[
    3 + eps + z1 + z2 + z3 + z4 + z6] Gamma[
    2 + 2 eps + z4 + z5 + z6])/(Gamma[-2 eps] Gamma[
    1 - z1] Gamma[-3 eps - z4] Gamma[3 + eps + z1 + z2 + z3 + z4])


integrals = MBcontinue[(-s)^(2 + 2 eps) fin, eps -> 0, rules];

26 integral(s) found
```

From $1/\epsilon^4$ until ϵ^0: at most 4-dimensional
AMBRE output:

```
{-47.8911 + 1/eps^4 - 9.8696/eps^2 - 33.2569/eps}

1

0

-9.869604401660688     -> -Pi^2

-33.25689147976733     -> -(83/3*Zeta[3])

-47.89108304541082    -> (-59 Pi^4/120)
```

Try PSQL for a systematic study

Tensor integrals

At the LHC, we need also massive 5-point and 6-point functions at least, for NLO corrections.

In general, the treatment of tensor integrals is a non-trivial task.

- One might think that all tensor numerators may be reduced to (even simpler) scalar integrands. Important notice: For $L > 1$, this is not true, we have *irreducible numerators*, see next slide.

- For many problems, it is preferable to evaluate the tensors without knowing the scalar products. The reasons are different.

Simple tensor integrals

$$B^{\mu} \;\equiv\; p_1^{\mu} B_1 \;=\; \frac{1}{(i\pi^{d/2})} \int \frac{d^d k \; k^{\mu}}{D_1 D_2}$$

$$B^{\mu\nu} \;\equiv\; p_1^{\mu} p_1^{\nu} B_{22} + g^{\mu\nu} B_{20} \;=\; \frac{1}{(i\pi^{d/2})} \int \frac{d^d k \; k^{\mu} k^{\nu}}{D_1 D_2}$$

and B_1 and B_{22}, B_{20} have to be determined.

Reducible numerators

Some numerators are reducible – one may divide them out against the denominators:

$$\frac{2kp_1}{D_1\,[(k+p_1)^2 - m_2^2]\,\ldots D_N} \equiv \frac{[(k+p_1)^2 - m_2^2] - [k^2 - m_1^2] + (-m_1^2 + m_2^2 - p_1^2)]}{D_1[(k+p_1)^2 - m_2^2]\ldots D_N}$$

$$= \frac{1}{D_1 D_3 \ldots D_N} - \frac{1}{D_2 D_3 \ldots D_N} + \frac{-m_1^2 + m_2^2 - p_1^2}{D_1 D_2 D_3 \ldots D_N}$$

This way one derives:

$$p_1^\mu B^\mu = p^2 B_1 = \frac{1}{(i\pi^{d/2})} \int d^d k \frac{p_1 k}{D_1 D_2}$$

$$= \frac{1}{(i\pi^{d/2})} \frac{1}{2} \int d^d k \left[\frac{1}{D_1} - \frac{1}{D_2} + \frac{-m_1^2 + m_2^2 - p_1^2}{D_1 D_2} \right]$$

and finally:

$$B_1 = \frac{1}{2p_1^2} \left[A_0(m_1) - A_0(m_2) + (-m_1^2 + m_2^2 - p_1^2)B_0(m_1, m_2, p^2) \right] \quad (134)$$

Known to everybody: The Passarino-Veltman reduction scheme for 1-loop tensors

worked out in [22] until 4-point functions.

Irreducible numerators

For a two-loop QED box diagram, it is e.g. $L = 2$, $E = 4$, and we have as potential simplest numerators:

$k_1^2, k_2^2, k_1 k_2$ **and** $2(E - 1)$ **products** $k_1 p_e, k_2 p_e$
compared to N **internal lines,** $N = 5, 6, 7$**. This gives**

$I = L + L(L - 1)/2 + L(E - 1) - N$ **irreducible numerators**

of this type. Here:

$$I(N) = 9 - N = 4, 3, 2$$

This observation is of practical importance:

Imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals. Which momenta combinations are irreducible is dependent on the choice of momenta flows.

Message:

When evaluating all Feynman integrals by Mellin-Barnes-integrals, one should also learn to handle numerator integrals

. . . and it is - in some cases - not too complicated compared to scalar ones

The one-loop case: $L = 1$, $E = N$, so

$$I(N) = 1 + (E - 1) - N = 0$$

irreducible numerators

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1} \tilde{M} Q$$

the $\int d^d \bar{k} \, \bar{k}/(\bar{k}^2 + \mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - d(L+1)/2 - 1}}{F(x)^{N_\nu - dL/2}} \left[\sum_l \tilde{M}\right]$$

Here also a tensor integral:

$$\begin{aligned}
G(k_{1\alpha} k_{2\beta}) &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\Gamma(\nu_1)\dots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \, x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - 2 - d(L+1)/2}}{F(x)^{N_\nu - dL/2}} \\
&\quad \times \sum_l \left[[\tilde{M}_{1l} Q_l]_\alpha [\tilde{M}_{2l} Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{d}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{d}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x) F(x) \frac{(V_{1l}^{-1})^+ (V_{2l}^{-1})}{\alpha_l} \right]
\end{aligned}$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{d}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \, x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu - d/2}} \quad (135)$$

$$(136)$$

Another nice box with numerator, **B5l3m(**$p_e.k_1$**)**

We used it for the determination if the small mass expansion.

$$B513m(p_e \cdot k_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}} e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_j]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}$$

$$(-s)^{(4-2\epsilon)-a_{12345}-\alpha-\beta-\delta}(-t)^{\delta}$$

$$\frac{\Gamma[-4+2\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\Gamma[-\beta]}{\Gamma[7-3\epsilon-a_{12345}-\alpha]\Gamma[5-2\epsilon-a_{123}]\Gamma[4-2\epsilon-a}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma]}\frac{\Gamma[4-2\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-}$$

$$\Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[3-\epsilon-a_{12}-\alpha]\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma$$

$$\Gamma[5-2\epsilon-a_{1123}-\gamma]\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma]\Gamma[a_1+\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+$$

$$\Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\delta-\gamma]\Gamma[9-4\epsilon-a_{112233}$$

$$\left[\Gamma[3-\epsilon-a_{12}-\alpha]\Gamma[5-2\epsilon-a_{1123}-\gamma]\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma]\Gamma[a_1+\gamma]+\Gamma[2-\right.$$

$$\Gamma[5-2\epsilon-a_{1123}-2\alpha-\gamma]\Gamma[1+a_1+\gamma]\Big]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\epsilon-$$

$$\Gamma[5-2\epsilon-a_{1123}-\gamma]\Gamma[4-2\epsilon-a_{1123}-2\alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\Gamma[5-2\epsilon-$$

$$\Gamma[8-4\epsilon-a_{112233445}-2\alpha-2\beta-2\delta-\gamma]$$

$$\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\Gamma[4-2\epsilon-a_{1234}-\alpha-\beta-\delta]$$

Summary

- We have introduced to the representation of *L*-loop *N*-point Feynman integrals of general type
- The determination of the ϵ-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, . . . so you have something to do if you like to . . .

Problem: Determine the small mass limit of B5l2m2 or of any other of the 2-loop boxes for Bhabha scattering.
Prof. Gluza may check your solution.
He leaves soon.

On-shell example: `B4l2m`, the 1-loop on-shell box

```
den = (x4 d4 + x5 d5 + x6 d6 + x7 d7 // Expand) /. kinBha /. m^2 -> 1

Q = -Coefficient[den, k]/2 // Simplify
  =  p3 x4 + p2 x5 - p1 (x4 + x6)

M = Coefficient[den, k^2] // Simplify
  = x4 + x5 + x6 + x7 -> 1

J = den /. k -> 0 // Simplify
  = t x4

F[x] = (Q^2 - J  M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4
     = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7

B4l2ma = mb[(x5+x6)^2, -tx7x4 - sx5x6, nu, ga]

B4l2mb = B4l2ma /. (-sx5x6 - tx4x7)^(-ga - nu) ->
             mb[(-s)x5x6, (-t)x7x4, nu+ga, de]
             /.((-s)x5x6)^de_ -> (-s)^de x5^de x6^de
```

```
             /.((x56^2)^ga      -> (x5 + x6)^(2ga))
   =

       (inv2piI^2(-s)^de x5^de x6^de ((x5 + x6)^(2ga)((-t)x4x7)^(-de-ga-nu
             Gamma[-de] Gamma[-ga] Gamma[de + ga + nu] /Gamma[nu]


   B4l2mc = B4l2mb /. (x5 + x6)^(2ga) ->
              mb[x5, x6, -2ga, si]
              /. ((-t)x4x7)^si_ -> (-t)^si x4^si x7^si // ExpandAll
   =
       1/(Gamma[-2ga] Gamma[nu])
   inv2piI^3  (-s)^de (-t)^(-de - ga - nu)
              x4^(-de - ga - nu) x5^(de + si) x6^(de + 2 ga - si) x7^(-d
              Gamma[-de] Gamma[-ga] Gamma[ de + ga + nu] Gamma[-si] Gamm


   B4l2md =  xfactor4[a4, x4, a5, x5, a6, x6, a7, x7] B4l2mc
   =
   .... (-s)^de (-t)^(-de-ga-nu)
   x4^(-1+a4-de-ga-nu) x5^(-1+a5+de+si) x6^(-1+a6+de+2ga-si) x7^(-1+a7- d


   B4l2me =
     B4l2md /.
         x4^B4_  x5^B5_  x6^B6_  x7^B7_ -> xint4[x4^B4 x5^B5 x6^B6 x7^B7]
   = ...
```

```
B4l2mf =  B4l2me /.
     Gamma[a6 + de + 2 ga - si]Gamma[-si]Gamma[ a5 + de + si] Gamma[-
   -> barne1[si, a5 + de, -2 ga, a6 + de + 2 ga, 0]
```

This finishes the evaluation of the MB-representation for `B4l2m`.

Package: AMBRE.m (K. Kajda, with support by J. Gluza and TR)

Example beyond Harmonic Polylogs: QED Box B4l2m

[9]

```
F[x]        = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7
```

B4l2m, the 1-loop QED box, with two photons in the *s*-channel; the Mellin-Barnes representation reads for finite ϵ:

$$\texttt{B4l2m} = \text{Box}(t, s) = \frac{e^{\epsilon \gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \qquad (138)$$

$$\frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2 + \epsilon + z_1 + z_2] \Gamma^2[1 + z_1] \Gamma[-z_1] \Gamma[-z_2]$$

$$\Gamma^2[-1 - \epsilon - z_1 - z_2] \frac{\Gamma[-2 - 2\epsilon - 2z_1]}{\Gamma[-2 - 2\epsilon - 2z_1 - 2z_2]}$$

Mathematica package MB used for analytical expansion $\epsilon \to 0$:[4]

$$\text{B412m} = -\frac{1}{\epsilon}J_1 + \ln(-s)J_1 + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]J_1 - 2J_2\right). \tag{139}$$

with J_1 being also the divergent part of the vertex function $C_0(t; m, 0, m)/s = \text{V312m}/s$ (as is well-known):

$$J_1 = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1}\frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y) \tag{140}$$

with

$$y = \frac{\sqrt{1 - 4m^2/t} - 1)}{(\sqrt{1 - 4m^2/t} + 1)} \tag{141}$$

The J_2 is more complicated:

$$J_2 = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2}\int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1\left(\frac{s}{t}\right)^{z_1}\Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1] \tag{142}$$

$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2\left(-\frac{m^2}{t}\right)^{z_2}\Gamma[-z_2]\frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]}\Gamma[2+z_1+z_2].$$

The expansion of `B4l2m` at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \tag{143}$$

$$r = \frac{s}{t}, \tag{144}$$

Look, under the integral, at $(-m^2/t)^{z_2}$,
and close the path to the right.
Seek the residua from the poles of Γ-functions with the smallest powers in m^2 and sum the resulting series.
we have obtained a compact answer for J_2 with the additional aid of XSUMMER [10] The box contribution of order ϵ in this limit becomes:

$$
\begin{aligned}
\text{B4l2m}[t, s, m^2; +1] = \frac{1}{st} \Big\{ & 4\zeta_3 - 9\zeta_2\ln(m_t) + \frac{2}{3}\ln^3(m_t) + 6\zeta_2\ln(r) - \ln^2(m_t)\ln(r) \quad (145) \\
& + \frac{1}{3}\ln^3(r) - 6\zeta_2\ln(1+r) + 2\ln(-r)\ln(r)\ln(1+r) - \ln^2(r)\ln(1+r) \\
& + 2\ln(r)\text{Li}_2(1+r) + 2\text{Li}_3(-r) \Big\} + \mathcal{O}(m_t).
\end{aligned}
$$

Remark:

The exact Box function is NOT expressible by Harmonic Polylogs, one may introduce a generalization of them: Generalized HPLs.

Automatized tools for this might be developed.

A scetch of the small mass expansion may be made as follows.

First the 1-dim. integral J_1.

The leading term comes from the first residue:

```
J1 =  Residue[m_t^z1 Gamma[-z1]^3 Gamma[1 + z1]/Gamma[-2 z1], {z1, 0}]

    = 2 Log[m_t]
```

We get a logarithmic mass dependence.

The second integral: Start with z_2, first residue is:

```
I2 = Residue[ m_t^z2 Gamma[-z2] Gamma[-1 - z1 - z2]^2
                  Gamma[2 + z1 + z2]/Gamma[-2 - 2 z1 - 2 z2], {z2, 0
```

$$= -\frac{\Gamma[-1 - z_1]^2 \Gamma[2 + z_1]}{\Gamma[-2 - 2z_1]} \tag{146}$$

qqq

The residue is independent of m^2/t.

It has to be integrated over z_1 yet, together with the terms which were independent of z_2:

$$I_2 \sim \int dz_1 r^{z_1+1} \Gamma[-z_1] \Gamma[-2 - 2z_1] \Gamma[1 + z_1]^2 \frac{\Gamma[-1 - z_1]^2 \Gamma[2 + z_1]}{\Gamma[-2 - 2z_1]} \tag{147}$$

Sum over residues, close path to the left:

$$Residue[z_1 = -n] = \frac{(-1)^n r^{1-n}}{2(-1 + n)^3} \left[2 + (-1 + n)^2 \pi^2 + (-1 + n) \ln[r](2 + (-1 + n) \ln[r]) \right] \tag{148}$$

$$Residue[z_1 = -1] = \frac{1}{6}(3\pi^2 \ln[r] + \ln[r]^3) \tag{149}$$

and finally:

$$I_2 \sim Residue[z_1 = -1] + \sum_{n=2}^{\infty} Residue[z_1 = -n] \tag{150}$$

The sum can be done also withoput using XSUMMER (here at least), e.g.

$$\ln[r] \sum_{n=2}^{Infty} \frac{(-1)^n (2 + \pi^2 - 2n\pi^2 + n^2\pi^2) r^{1-n}}{2(-1 + n)^3} = \frac{1}{2} \left[\pi^2 \ln(1 + 1/r) - 2 Li_3(-1/r) \right] \tag{151}$$

etc

Some routines in mathematica which may be used:

```
(* Barnes' first lemma: \int d(si) Gamma(si1p+si)Gamma(si2p+si)Gamma(s
                        with 1/inv2piI = 2 Pi I *)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
  1/inv2piI  Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
      si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m

  (* Mellin-Barnes integral: (A+B)^(-nu) = 1/(2 Pi I) \int d(si) a^si b
                             Gamma[-si]Gamma[nu+si]/Gamma[nu]  *)

mb[a_, b_, nu_, si_]:=inv2piI a^si b^(-nu-si)Gamma[-si]Gamma[nu+si]/Ga

  (* After the k-integration, the integrand for \int\prod(dxi xi^(ai -
              will be (L=1 loop) : xfactorn F^(-nu) Q(xi).pe with  nu

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
  I Pi^(d/2) (-1)^(a1 + a2 + a3) x1^(a1 - 1) x2^(a2 - 1)x3^(a3 - 1)Gam
        a1 + a2 + a3 - d/2]/(Gamma[a1]Gamma[a2]Gamma[a3])

  (* xinti - the i-dimensional x - integration over Feynman parameters
```

```
xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_)  ] :=
   Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1]  / Gamma[a1 + a2 + a3 + 3]
```

quote also:
[23] [17] [24] [25] [26] [13] [14] [27] [28]

Feynman integrals: scalar and tensor integrals, shrinked and dotted ones

Just to mention what kind of integrals may appear:

- **diagrams with numerators:**
  tensors arise from internal fermion lines: $\int d^d k_i \cdots \frac{\Gamma_\nu (k_i^\nu - p_n^\nu) - m_n}{(k_i - p_n)^2 - m_n^2} \cdots$

- **diagrams with shrinked and/or with dotted lines: a sample relation see next page**

- **relations to simpler diagrams may shift the complexity:** $\frac{1}{d-4} = -\frac{1}{2\epsilon}$

$$
\begin{aligned}
C_0(m, 0, m; m^2, m^2, s) &= \frac{1}{s - 4m^2} \left[\frac{d-2}{d-4} \frac{A_0(m^2)}{m^2} + \frac{2d-3}{d-4} B_0(m, m; s) \right] \\
V3l2m &= \frac{-1}{s-4} \left[\frac{1-\epsilon}{\epsilon} T1l1m + \frac{5-4\epsilon}{2\epsilon} SE2l2m \right] \quad (152)
\end{aligned}
$$

$$\texttt{SE3l2m}(a, b, c, d) = -\frac{e^{2\epsilon\Gamma_E}}{\pi^D} \int \frac{d^D k_1 d^D k_2 (k_1 k_2)^{-d}}{[(k_1 + k_2 - p)^2 - m^2]^b [k_1^2]^a [k_2^2 - m^2]^c}.$$

$$\texttt{SE3l2m} = \texttt{SE3l2m}(1, 1, 1, 0)$$
$$\texttt{SE3l2md} = \texttt{SE3l2m}(1, 1, 2, 0)$$
$$\texttt{SE3l2mN} = \texttt{SE3l2m}(1, 1, 1, -1)$$

$$\texttt{SE3l2md} = \frac{-(1 + s) + \epsilon(2 + s)}{s - 4} \texttt{SE3l2m} + \frac{2(1 - \epsilon)}{s - 4} \left(\texttt{T1l1m}^2 + 3 \texttt{SE3l2} \right.$$

$$= \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left(\frac{1 - \zeta_2}{2} + \frac{1 + x}{1 - x} \ln(x) + \frac{1 + x^2}{(1 - x)^2} \frac{1}{2} \ln^2(x) \right) + \mathcal{O}($$
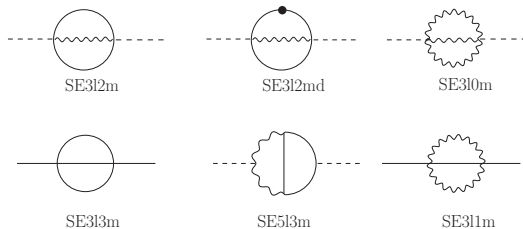
Figure: The six two-loop 2-point masters of Bhabha scattering.

More legs, more loops

Seek methods for an (approximated) analytical or numerical evaluation of more involved diagrams
Remember: UltraViolet (UV) und InfraRed (IR) divergencies appear
Might try:

- **same as for simple one-loop: Feynman parameters, direct integration**
- **use algebraic relations between integrals and find a (minimal) basis of master integrals – is a preparation of the final evaluation**
- **derive and solve (system of) differential equations**
- **derive and solve (system of) difference equations**
- **do something else for a direct evaluation of single integrals – of all of them or of the masters only**

Into the last category falls what we present here:
Use Feynman parameters and transform the problem

This is not a review ...

... and it is also not a lecture

For many references and a more balanced list of references see some older lectures located at my homepage in Zeuthen:
http://www-zeuthen.desy.de/~riemann/
e.g.

"Feynman Integrals and Mellin-Barnes representations"
Lecture, DESY CAPP School, March 2011, Zeuthen, Germany
http://www-zeuthen.desy.de/~riemann/Talks/riemann-capp-2011.pdf
"Feynman Integrals - Mellin-Barnes representations - Sums"
Two lectures, Humboldt-Universitt, Berlin, June 2010
http://www-zeuthen.desy.de/~riemann/Talks/
HUB-Berlin-SS10-MB-lectures.pdf
"Evaluation of Feynman diagrams with Mellin-Barnes representations"
Lecture at Helmholtz-JINR CALC School, July 2009, JINR, Dubna, Russia
http://www-zeuthen.desy.de/~riemann/Talks/calc2009-riemann.pdf
"Evaluation of Feynman Integrals: Advanced Methods"
Three lectures, RECAPP 2009, February 2009, Allahabad, India
http://www-zeuthen.desy.de/~riemann/Talks/riemann-recapp-09.pdf

Introduction – Feynman integrals I

The perturbative expansion in a small coupling constant may be treated with Feynman integrals.

The Feynman integrals may be visualized by Feynman diagrams.

Higher orders in perturbation theory lead to Feynman diagrams with more and more loops - closed lines, repesenting internal virtual particles.

Normally, the lowest perturbative order has no Feynman diagrams with loops and is then called Born approximation.
The higher orders of perturbation theory are also called "radiative corrections".

The one-loop corrections are, at least in principle, easy to calculate.

For non-abelian gauge field theories with spontaneous symmetry breaking - e.g. the Standard Model -
the basics for one-loop calculations with dimensional regularization have been systematically worked out more than 30 years ago.

Seminal articles are:

All the scalar Feynman integrals for 4-particle interactions ...

... are expressed in terms of a basis of
1-point functions – tadpoles
2-point functions – self-energies
3-point functions – vertices

4-point functions – box diagrams
't Hooft, Veltman, "Scalar One Loop Integrals", [29, Nucl.Phys. B153, 1978]
All the scalar integrals lead to Euler Dilogarithms or simpler functions.

The tensor Feynman integrals ...

... stem from fermion and/or boson propagators and from vector boson couplings,
and they may be traced back to the basis of scalar Feynman integrals, see
Passarino, Veltman, "One Loop Corrections for e^+e^- Annihilation Into $\mu^+\mu^-$ in the Weinberg Model", [30, Nucl.Phys. B160, 1979]
Basic idea:
Derive a system of algebraic equations for the tensor integrals and solve it.

Not solved in the old papers:

more external particles

case of more external particles, i.e. N-point functions with $N > 4$

kinematical singularities

treatment of kinematical singularities or of spurious singularities, artificially introduced by the algorithm

higher terms in ϵ

higher terms in ϵ, stemming from the so-called ϵ-expansion in $d = 4 - 2\epsilon$ due to the dimensional regularization

The $\epsilon^0 =$ constant terms are physical, singular terms have to compensate each other, and higher powers of ϵ interact with higher order corrections from perturbation theory.

Some later generalizations I

Look at 1-loop Feynman integrals, but we are interested in expressions fulfilled for ...

- *n* external particles, $n = 5, \cdots 8$ and higher, with proper treatment of singularities
- higher orders in the ϵ-expansion
- arbitrary kinematical situations, including massive particles

All this leads to Feynman integrals with **many different scales**

e.g. a **4-point function**, with 4 external momenta $p_i, i = 1, \cdots 4$ may depend in general on:
- **2 kinematical variables**, usually called $s \sim p_1 p_2$ and $t \sim p_1 p_3$, where $s + t + u = 0, u \sim p_1 p_4$, but due to $p_1 + p_2 + p_3 = -p_4$ the u is not independent
- **plus 4 internal masses** M_i
- **plus 4 external "virtualities"** $p_i^2 \equiv m_i^2$

Some later generalizations II

It would be wonderful to have an algorithm for **automatic evaluation** of ll the scalar integrals by infinite sums.

One question:
How to systematically evaluate the 1-loop Feynman integrals

Can this be extended to *n* loop?

Rest of this talk:

Use of Mellin-Barnes integrals

Explain one approach with use of Mellin-Barnes integrals, leading to infinite sums to be evaluated somehow by somebody.

Somehow ... somebody = question to people at RISC $\cdots$

Tensor integrals: A simple example

1-loop self-energy:

$$I_2^\mu = \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]}$$

$$= p_\mu B_1$$

Solve:

$$p_\mu I_2^\mu = p^2 B_1(p, M_1, M_2)$$

$$= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1\,D_2}$$

$$= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\,D_2} \right],$$

$$B_1(p, M_1, M_2) = \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

Feynman integrals: The Bhabha box diagram

In the rest of the talk we use as example, in $d = 4 - 2\epsilon$:

$$e^-(p_1) + e^+(p_2) \quad \rightarrow \quad e^-(p_3) + e^+(p_4) \tag{153}$$

with kinematics

$$p_1 + p_2 = p_3 + p_4 \tag{154}$$
$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = m^2 \tag{155}$$
$$(p_1 + p_2)^2 = s \tag{156}$$
$$(p_1 - p_3)^2 = t \tag{157}$$

Bhabha box, with photon exchange in s channel:

$$I_4 = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{[k^2 - m^2 + i\epsilon] \times [(k + p_1)^2] + i\epsilon]} \tag{158}$$

$$\times \frac{1}{[(k + p_1 + p_2)^2 - m^2 + i\epsilon] \times [(k + p_3)^2 + i\epsilon]} \tag{159}$$
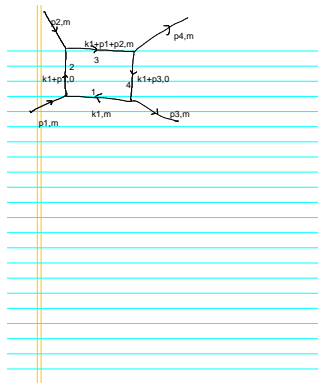
The Feynman diagram I

$$I_4 = Box(s, t) =$$

Figure: The topology of the massive QED box diagram (aka B4l2m).

There are many ways to evaluate the ϵ-expansion of l_4:

$$l_4 = \frac{1}{\epsilon}B_{-1}(s,t) + B_0(s,t) + \epsilon B_1(s,t) + \cdots \tag{160}$$

Already for $B_1(s,t)$, the direct integration of the Feynman parameter integral becomes difficult.

System of differential equations

One way, not further exemplified here, but very efficient:
Derive a system of differential equations for the original k-momentum integral
and solve it iteratively
This leads to a solution in terms of e.g.
Generalized Harmonic Polylogarithms
and allows to determine arbitrary high terms $B_n(s,t)$.
See: Fleischer, Gluza, Lorca, Riemann, [9, EPJC48, 2006] and references
therein, especially Bonciani et al., [31, NPB681, 2004].

The general analytical result in d dimensions ...

... was given already in 2003 by Fleischer, Jegerlehner, Tarasov in [32, NPB672, 2003] **in terms of two Appell hypergeometric functions F_1 and F_2**
and a Kampe de Feriet function.

The first ϵ-terms of the QED box diagram

$$I_4 = \frac{1}{\epsilon} \ B_{-1}(s,t) + B_0(s,t) + \epsilon \ B_1(s,t) + \cdots \qquad (161)$$

with

$$B_{-1} = \frac{2xy}{(1-x^2)(1-y)^2} \ H(0,x) \qquad (162)$$

where $H(0,x) \equiv \ln(x)$ has been introduced, and

$$x \ = \ \frac{\sqrt{1-4/s}-1}{\sqrt{1-4/s}+1}, \quad y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}, \qquad (163)$$

Further,

$$B_0 = \frac{2}{st\sqrt{1-4/s}} \ H(0,x) \ \left[H(0,y) + 2H(1,y)\right] \qquad (164)$$

Finally, here the ϵ-term:

$$
\begin{aligned}
B_1 = \frac{-2}{st\sqrt{1-4/s}} \Bigg\{ & G(-\frac{1}{y},0,0,x) + G(-y,0,0,x) \\
& -2\Big(G(-\frac{1}{y},-1,0,x) + G(-y,-1,0,x)\Big) \\
& -\Big(G(-\frac{1}{y},0,x) + G(-y,0,x) - 2H(-1,0,x)\Big)[H(0,y)+2H(1,y)] \\
& -\Big(G(-\frac{1}{y},x) - G(-y,x) + H(0,x)\Big)[H(0,0,y)+2H(0,1,y)] \\
& -\Big(5G(-\frac{1}{y},x) - 3G(-y,x) - \frac{1}{2}H(0,x) - 2H(-1,x) - 4H(0,y)\Big)\zeta_2 \\
& -2\Big(H(1,y)H(0,0,y) - H(0,y)H(0,1,y)\Big) \\
& -2\Big(H(-1,0,0,x) - 2H(-1,-1,0,x)\Big) - 2H(0,x)[H(1,0,y)+2H(1,1,y)] \\
& +H(0,0,0,y) + 2H(1,0,0,y) - 2\zeta_3 \Bigg\} . \tag{165}
\end{aligned}
$$

The functions G are generalized harmonic polylogarithms, see Gehrmann and Remiddi, [33, NPB601, 2001], and [31, NPB681, 2004].

The 2-dimensional Harmonic Polylogarithms
in Bonciani et al., [31, NPB681, 2004], are defined as the set of
functions generated by the repeated integrations

$$\int_0^x dz \ \{g(j; z)\} \ G(m_w; z) \tag{166}$$

with

$$g(-1; x) = \frac{1}{1+x} \tag{167}$$

$$g(0; x) = \frac{1}{x} \tag{168}$$

$$g(-y; x) = \frac{1}{y+x} \tag{169}$$

$$g(-1/y; x) = \frac{1}{1/y+x} \tag{170}$$

Summary

Although for the QED 1-loop box we know the result, it would be interesting to get it via summation techniques

This would allow an algorithmic solution with useful properties:

- only one diagram per calculation – no coupled system of equations

- derivations are relatively simple

- generalization to higher loop problems – in principle – easy

References I

F. A. Berends, G. Komen, Soft and hard photon corrections for mu pair production and BHABHA scattering in presence of a resonance, Nucl. Phys. B115 (1976) 114.

G. 't Hooft, M. Veltman, Scalar one loop integrals, Nucl. Phys. B153 (1979) 365–401.

J. Gluza, K. Kajda, T. Riemann, AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals, Comput. Phys. Commun. 177 (2007) 879–893. `arXiv:0704.2423`, `doi:10.1016/j.cpc.2007.07.001`.

M. Czakon, Automatized analytic continuation of Mellin-Barnes integrals, Comput. Phys. Commun. 175 (2006) 559–571. `arXiv:hep-ph/0511200`, `doi:10.1016/j.cpc.2006.07.002`.

J. Gluza, F. Haas, K. Kajda, T. Riemann, Automatizing the application of Mellin-Barnes representations for Feynman integrals, PoS ACAT2007 (2007) 081. `arXiv:0707.3567`.

T. Huber, D. Maitre, HypExp 2, expanding hypergeometric functions about half- integer parameters`arXiv:arXiv:0708.2443[hep-ph]`.

A. Davydychev, M. Kalmykov, New results for the ϵ-expansion of certain one-, two- and three-loop Feynman diagrams, Nucl. Phys. B605 (2001) 266–318. `arXiv:hep-th/0012189`.

References II

A. Davydychev, M. Kalmykov, Massive Feynman diagrams and inverse binomial sums, Nucl. Phys. B699 (2004) 3–64.
`arXiv:hep-th/0303162`.

J. Fleischer, J. Gluza, A. Lorca, T. Riemann, First order radiative corrections to Bhabha scattering in d dimensions, Eur. J. Phys. 48 (2006) 35–52.
`arXiv:hep-ph/0606210`, `doi:10.1140/epjc/s10052-006-0008-6`.

S. Moch, P. Uwer, Xsummer: Transcendental functions and symbolic summation in form, Comput. Phys. Commun. 174 (2006) 759–770.
`arXiv:math-ph/0508008`.

T. Binoth, G. Heinrich, An automatized algorithm to compute infrared divergent multi-loop integrals, Nucl. Phys. B585 (2000) 741–759.
`arXiv:hep-ph/0004013v.2`.

A. Denner, S. Pozzorini, An algorithm for the high-energy expansion of multi-loop diagrams to next-to-leading logarithmic accuracyNumerators, sector decomposition etc.
`arXiv:hep-ph/0408068`.

C. Bogner, S. Weinzierl, Resolution of singularities for multi-loop integrals, Comput. Phys. Commun. 178 (2008) 596–610.
`arXiv:0709.4092`, `doi:10.1016/j.cpc.2007.11.012`.

G. Heinrich, Sector Decomposition, Int.J.Mod.Phys. A23 (2008) 1457–1486.
`arXiv:0803.4177`, `doi:10.1142/S0217751X08040263`.

References III

A. Smirnov, V. Smirnov, Hepp and Speer Sectors within Modern Strategies of Sector Decomposition, JHEP 0905 (2009) 004.
arXiv:0812.4700, doi:10.1088/1126-6708/2009/05/004.

C. Bauer, A. Frink, R. Kreckel, Introduction to the GiNaC framework for symbolic computation within the C++ programming language, J. Symbolic Computation 33 (2002) 1.
arXiv:cs.sc/0004015.

A. Smirnov, M. Tentyukov, Feynman Integral Evaluation by a Sector decomposiTion Approach (FIESTA), Comput.Phys.Commun. 180 (2009) 735–746.
arXiv:0807.4129, doi:10.1016/j.cpc.2008.11.006.

E. Remiddi, J. Vermaseren, Harmonic polylogarithms, Int.J.Mod.Phys. A15 (2000) 725–754.
arXiv:hep-ph/9905237, doi:10.1142/S0217751X00000367.

D. Maitre, HPL, a Mathematica implementation of the harmonic polylogarithms, Comput. Phys. Commun. 174 (2006) 222–240.
arXiv:hep-ph/0507152, doi:10.1016/j.cpc.2005.10.008.

M. Czakon, A. Mitov, S. Moch, Heavy-quark production in gluon fusion at two loops in QCD, Nucl. Phys. B798 (2008) 210–250.
arXiv:0707.4139, doi:10.1016/j.nuclphysb.2008.02.001.

R. Gonsalves, Dimensionally regularized two loop on-shell quark form- factor, Phys. Rev. D28 (1983) 1542.

References IV

G. Passarino, M. Veltman, One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model, Nucl. Phys. B160 (1979) 151.
doi:10.1016/0550-3213(79)90234-7.

M. Argeri, P. Mastrolia, Feynman Diagrams and Differential Equations, Int.J.Mod.Phys. A22 (2007) 4375–4436.
arXiv:0707.4037, doi:10.1142/S0217751X07037147.

R. K. Ellis, G. Zanderighi, Scalar one-loop integrals for QCD, JHEP 02 (2008) 002.
arXiv:arXiv:0712.1851, doi:10.1088/1126-6708/2008/02/002.

T. Binoth, J. P. Guillet, G. Heinrich, E. Pilon, T. Reiter, Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs, Comput. Phys. Commun. 180 (2009) 2317–2330.
arXiv:0810.0992, doi:10.1016/j.cpc.2009.06.024.

T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, J. Tausk, A complete reduction of one-loop tensor 5- and 6-point integrals, Phys. Rev. D80 (2009) 036003.
arXiv:arXiv:0812.2134, doi:10.1103/PhysRevD.80.036003.

J. Fleischer, J. Gluza, K. Kajda, T. Riemann, Pentagon diagrams of Bhabha scattering, Acta Phys. Polon. B38 (2007) 3529–3536.
arXiv:0710.5100.

T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, J. Tausk, On the tensor reduction of one-loop pentagons and hexagons, Nucl. Phys. Proc. Suppl. 183 (2008) 109–115.
arXiv:arXiv:0807.2984, doi:10.1016/j.nuclphysbps.2008.09.091.

References V

G. 't Hooft, M. Veltman, Scalar One Loop Integrals, Nucl. Phys. B153 (1979) 365–401.
doi:10.1016/0550-3213(79)90605-9.

G. Passarino, M. Veltman, One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model,
Nucl. Phys. B160 (1979) 151.
doi:10.1016/0550-3213(79)90234-7.

R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi, J. van der Bij, Planar box diagram for the (N(F) = 1) two
loop QED virtual corrections to Bhabha scattering, Nucl. Phys. B681 (2004) 261–291.
arXiv:hep-ph/0310333,
doi:10.1016/j.nuclphysb.2004.08.003, 10.1016/j.nuclphysb.2004.08.003.

J. Fleischer, F. Jegerlehner, O. Tarasov, A New hypergeometric representation of one loop scalar integrals in
d dimensions, Nucl. Phys. B672 (2003) 303–328.
arXiv:hep-ph/0307113, doi:10.1016/j.nuclphysb.2003.09.004.

T. Gehrmann, E. Remiddi, Two loop master integrals for $\gamma^* \rightarrow 3$ jets: The Planar topologies, Nucl. Phys.
B601 (2001) 248–286.
arXiv:hep-ph/0008287, doi:10.1016/S0550-3213(01)00057-8.