http://www-zeuthen.desy.de/~riemann/Talks/capp07/

Feynman Integrals and Mellin-Barnes Representations

Janusz Gluza, U. Silesia, Katowice
Tord Riemann, DESY, Zeuthen

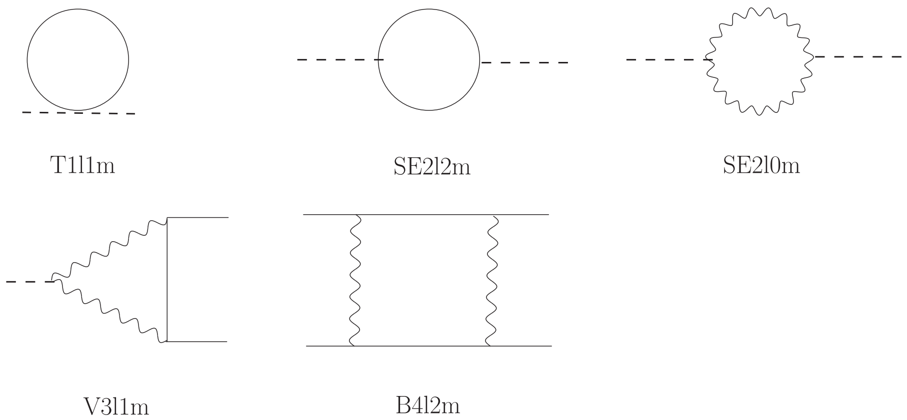
CAPP, March 2007, DESY, Zeuthen

- Introduction: Feynman integrals: M-point functions with L loops and N internal lines
- Loop momentum integrations with Feynman parameters x_i
- Doing the x_i-integrals
- Barnes' contour integrals for the hypergeometric function
- Representations by Mellin-Barnes integrals (AMBRE package (K.Kajda))
- Treatment of divergencies in $d = 4 - 2\epsilon$ (MB package (M.Czakon))
- Numerical evaluations, infinite series, approximations (XSUMMER package (S.Moch, P.Uwer))
- Summary

Introductory remarks

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient.

At most one-loop, at most $2 \longrightarrow 2$ scattering (plus bremsstrahlung)



$$T1l1m = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 +$$

$$B4l2m = [-\frac{1}{\epsilon} + \ln(-s)]\frac{2y\ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots$$

with $d = 4 - 2\epsilon$ and $m = 1$ and

$$y = \frac{\sqrt{1-4/t}-1}{\sqrt{1-4/t}+1}$$

Figure shows so-called master integrals for Bhabha scattering (see lecture by R. Harlander for algebraic methods.)

Then Feynman parameters may be used and by direct integration over them one gets things like: $\frac{23}{57},\ \ln\frac{t}{s},\ \ln\frac{t}{s}\ln\frac{m^2}{s},\ Li_2(\frac{t}{s})$ etc. With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated

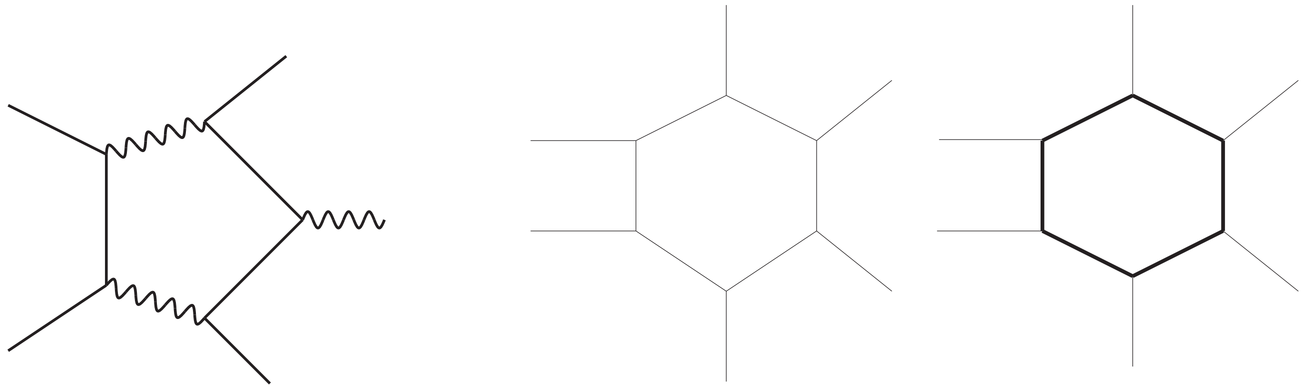


Figure 1: Massive QED pentagon (5 variables), massless and massive hexagons (8 variables

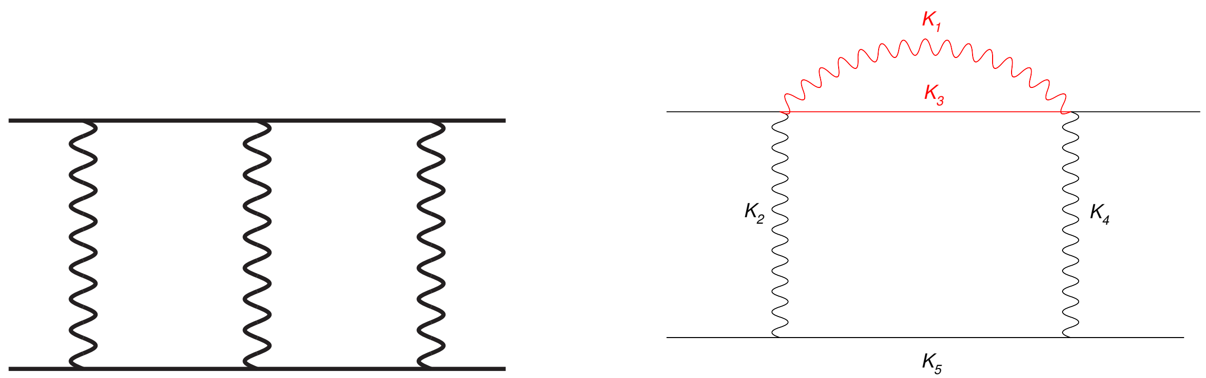


Figure 2: The two-loop planar QED box B1 = B7l4m1, another box master integral B5l2m2 (from B2 = B7l2m2)

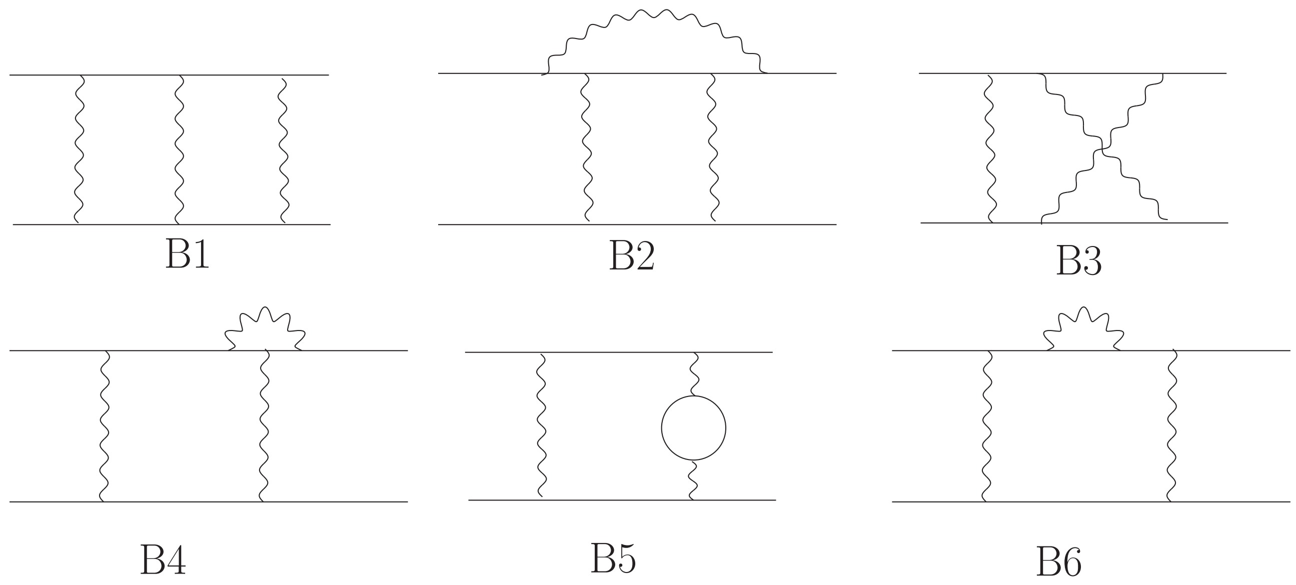


Figure 3: Two-loop box diagrams for massive Bhabha scattering

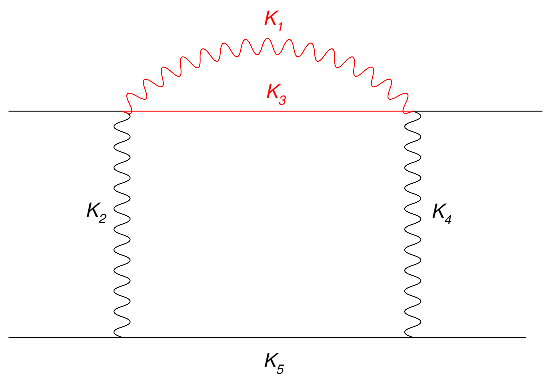


Figure 4: A box master integral B5l2m2, related to B2 = B7l2m2 by shrinking two lines

Feynman integrals: scalar and tensor integrals, shrinked and dotted ones

Just to mention what kind of integrals may appear:

- **diagrams with numerators:**
  tensors arise from internal fermion lines: $\int d^d k_i \cdots \frac{\gamma_\nu (k_i^\nu - p_n^\nu) - m_n}{(k_i - p_n)^2 - m_n^2} \cdots$
- **diagrams with shrinked and/or with dotted lines: a sample relation see next page**
- **relations to simpler diagrams may shift the complexity:** $\frac{1}{d-4} = -\frac{1}{2\epsilon}$

$$
\begin{aligned}
C_0(m,0,m;m^2,m^2,s) &= \frac{1}{s-4m^2}\left[\frac{d-2}{d-4}\;\frac{A_0(m^2)}{m^2} + \frac{2d-3}{d-4}\;B_0(m,m;s)\right] \\
V3l2m &= \frac{-1}{s-4}\left[\frac{1-\epsilon}{\epsilon}\;T1l1m + \frac{5-4\epsilon}{2\epsilon}\;SE2l2m\right] \qquad (1)
\end{aligned}
$$

$$\texttt{SE3l2m}(a,b,c,d) \quad = \quad -\frac{e^{2\epsilon\gamma_E}}{\pi^D}\int \frac{d^Dk_1 d^Dk_2 \ (k_1k_2)^{-d}}{[(k_1+k_2-p)^2-m^2]^b \ [k_1^2]^a \ [k_2^2-m^2]^c}.$$

$$\begin{aligned} \texttt{SE3l2m} &= \texttt{SE3l2m}(1,1,1,0) \\ \texttt{SE3l2md} &= \texttt{SE3l2m}(1,1,2,0) \\ \texttt{SE3l2mN} &= \texttt{SE3l2m}(1,1,1,-1) \end{aligned} \qquad (2)$$

$$\begin{aligned} \texttt{SE3l2md} &= \frac{-(1+s)+\epsilon(2+s)}{s-4}\ \texttt{SE3l2m} + \frac{2(1-\epsilon)}{s-4}\left(\texttt{T1l1m}^2 + 3\ \texttt{SE3l2mN}\right), \\ &= \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left(\frac{1-\zeta_2}{2} + \frac{1+x}{1-x}\ln(x) + \frac{1+x^2}{(1-x)^2}\frac{1}{2}\ln^2(x)\right) + \mathcal{O}(\epsilon) \end{aligned}$$

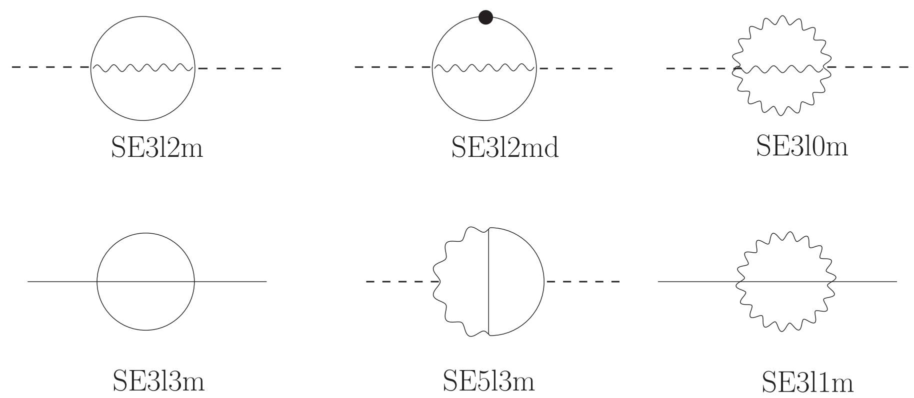


Figure 5: The six two-loop 2-point masters of Bhabha scattering.

More legs, more loops

Seek methods for an (approximated) analytical or numerical evaluation of more involved diagrams

Remember: UltraViolet (UV) und InfraRed (IR) divergencies appear

Might try:

- **same as for simple one-loop: Feynman parameters, direct integration**
- **use algebraic relations between integrals and find a (minimal) basis of master integrals – is a preparation of the final evaluation**
- **derive and solve (system of) differential equations**
- **derive and solve (system of) difference equations**
- **do something else for a direct evaluation of single integrals – of all of them or of the masters only**

Into the last category falls what we present here:
Use Feynman parameters and transform the problem

Loop momentum integrations with Feynman parameters for L-loop n-point functions

Consider an arbitrary L-loop integral $G(X)$ with loop momenta k_l, with E external legs with momenta p_e, and with N internal lines with masses m_i and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \ldots d^D k_L \; X(k_1, \ldots, k_L)}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.$$

$$D_i = q_i^2 - m_i^2 = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e]^2 - m_i^2$$

The numerator may contain a tensor structure

$$X = (k_1 p_{e_1}) \cdots (k_L p_{e_L}) = (k_1^{\alpha_1} \cdots k_L^{\beta_L}) \, (p_{e_1}^{\alpha_1} \cdots p_{e_L}^{\beta_L})$$

Some numerators are reducible, i.e. one may divide them out against the numerators a la:

$$\begin{aligned}
\frac{2kp_e}{[(k+p_e)^2 - m_1^2] D_2 \ldots D_N} &\equiv \frac{[(k+p_e)^2 - m_1^2] - [k^2 - m_2^2] + (m_1^2 + m_2^2 - m_e^2)]}{[(k+p_e)^2 - m_1^2] D_2 \ldots D_N} \\
&= \frac{1}{D_2 \ldots D_N} - \frac{1}{[(k+p_e)^2 - m^2] D_3 \ldots D_N} + \frac{m_1^2 + m_2^2 - m_e^2}{D_1 D_2 \ldots D_N}
\end{aligned}$$

Irreducible numerators

For a two-loop QED box diagram, it is e.g. $L = 2$, $E = 4$, and we have as potential simplest numerators:

$$k_1^2, k_2^2, k_1 k_2 \text{ and } 2(E-1) \text{ products } k_1 p_e, k_2 p_e$$

compared to N internal lines, $N = 5, 6, 7$. This gives

$$I = L + L(L-1)/2 + L(E-1) - N \text{ irreducible numerators}$$

of this type. Here:

$$I(N) = 9 - N = 4, 3, 2$$

This observation is of practical importance:
Imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals.
Which momenta combinations are irreducible is dependent on the choice of momenta flows.
Message:
When evaluating all Feynman integrals by Mellin-Barnes-integrals, one should also learn to handle numerator integrals
. . . and it is - in some cases - not too complicated compared to scalar ones

The one-loop case: $L = 1, E = N$, so

$$I(N) = 1 + (E-1) - N = 0$$

irreducible numerators

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \dots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \dots + \nu_N)}{\Gamma(\nu_1) \dots \Gamma(\nu_N)} \int_0^1 dx_1 \dots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \dots x_N^{\nu_N - 1} \delta(1 - x_1 \dots - x_N)}{(x_1 D_1 + \dots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \dots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1 + \dots + \nu_N)} = (x_1 D_1 + \dots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1}) Q, \\ m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J. \end{aligned}$$

Remember:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to M. The shift leaves the integral unchanged.

The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$I_k(1) \quad = \quad \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$I_k(1) \to (i)^L \int \frac{Dk_1^E \ldots Dk_L^E}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} \; = \; (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) \quad = \quad -(J - QM^{-1}Q)$$

and get

$$I_k(1) \quad = \quad (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals - will use only those - we are ready. For L-loops go on and now diagonalize the matrix M by a rotation:

$$\begin{aligned}
k \to k'(x) &= V(x)\; k, && (3)\\
kMk &= k' M_{diag} k' && (4)\\
&\to \sum \alpha_i(x) k_i^2(x), && (5)\\
M_{diag}(x) &= (V^{-1})^+ M V^{-1} \; = \; (\alpha_1, \ldots, \alpha_L).
\end{aligned}$$

This leaves both the integration measure and the integral invariant:

$$I_k(1) \quad = \quad (-1)^{N_\nu}(i)^L \int \frac{Dk_1 \dots Dk_L}{(\sum_i \alpha_i k_i^2 + \mu^2)^{N_\nu}}.$$

Rescale now the k_i,

$$\bar{k}_i \quad = \quad \sqrt{\alpha_i} k_i,$$

with

$$\begin{aligned} d^D k_i &= (\alpha_i)^{-D/2} d^D \bar{k}_i, \\ \prod_{i=1}^{L} \alpha_i &= \det M, \end{aligned} \tag{6}$$

and get the Euclidean integral to be calculated (and rename $\bar{k} \to k$):

$$I_k(1) \quad = \quad (-1)^{N_\nu}(i)^L (\det M)^{-D/2} \int \frac{Dk_1 \dots Dk_L}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}}.$$

Use now (remembering that $Dk = dk/(i\pi^{d/2})$):

$$\begin{aligned} i^L \int \frac{Dk_1 \dots Dk_L}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - DL/2}}, \\ i^L \int \frac{Dk_1 \dots Dk_L \; k_1^2}{(k_1^2 + \dots + k_L^2 + \mu^2)^{N_\nu}} &= \frac{d}{2} \; \frac{\Gamma\left(N_\nu - \frac{D}{2}L - 1\right)}{\Gamma\left(N_\nu\right)} \frac{1}{(\mu^2)^{N_\nu - DL/2 - 1}}. \end{aligned} \tag{7}$$

These formulae follow for $L = 1$ immediately from any textbook.

For $L > 1$, get it iteratively, with setting $(k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N$, $M^2 = k_2^2 + m^2$, etc.

Finally, one gets for **Scalar integrals:**

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{(\text{det } M)^{-D/2}}{(\mu^2)^{N_\nu - DL/2}},$$

or

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2}}{F(x)^{N_\nu - DL/2}}$$

with

$$\begin{aligned} U(x) \quad &= \quad (\text{det } M) \; (\to 1 \text{ for } L = 1) \\ F(x) \quad &= \quad (\text{det } M) \; \mu^2 \; = \; -(\text{det } M) \; J + Q \; \tilde{M} \; Q \; (\to -J + Q^2 \text{ for } L = 1) \end{aligned}$$

Trick for one-loop functions:
$U = \text{det } M = 1 = \sum x_i$ and so U 'disaapears' and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j.$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1}\tilde{M}Q$$

the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2-1}}{F(x)^{N_\nu - DL/2}} \left[\sum_l \tilde{M}_{1l}Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha}k_{2\beta}) &= (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{U(x)^{N_\nu-2-D(L+1)/2}}{F(x)^{N_\nu-DL/2}} \\ &\times \sum_l \left[[\tilde{M}_{1l}Q_l]_\alpha [\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu - \frac{D}{2}L - 1\right)}{\Gamma\left(N_\nu - \frac{D}{2}L\right)} \frac{g_{\alpha\beta}}{2} U(x)F(x) \frac{(V_{1l}^{-1})^+ (V_{2l}^{-1})}{\alpha_l} \right]. \end{aligned}$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j\ x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^N x_i\right) \frac{[1, Qp_e]}{F(x)^{N_\nu - D/2}}$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s, t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s, t, t', v_1, v_2, m^2) = m^2(x_1 + x_3 + x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

2-loop: B7l4m2, sub-loop with 2 off-shell legs (diagram see next page):

$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2 + (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

2-loop: B5l2m2, sub-loop with 2 off-shell legs (diagram see p.4):

$$F_{2lines}(k_1^2, m^2) = m^2(x_3)^2 + [-k_1^2 + m^2]x_1x_3$$

For both first subloops of one-loop type:

$$K_{\text{up,1-loop}} \quad = \quad \frac{(-1)^{N_\nu}\Gamma(N_\nu - d/2)}{\prod \Gamma(\nu_i)} \int\limits_0^1 \Pi_j dx_j x_j^{\nu_j - 1} \frac{\delta(1-\sum x_i)}{F^{N_\nu - d/2}}$$

where $N_\nu = \sum \nu_i$, and the second k-integral has to be done yet.

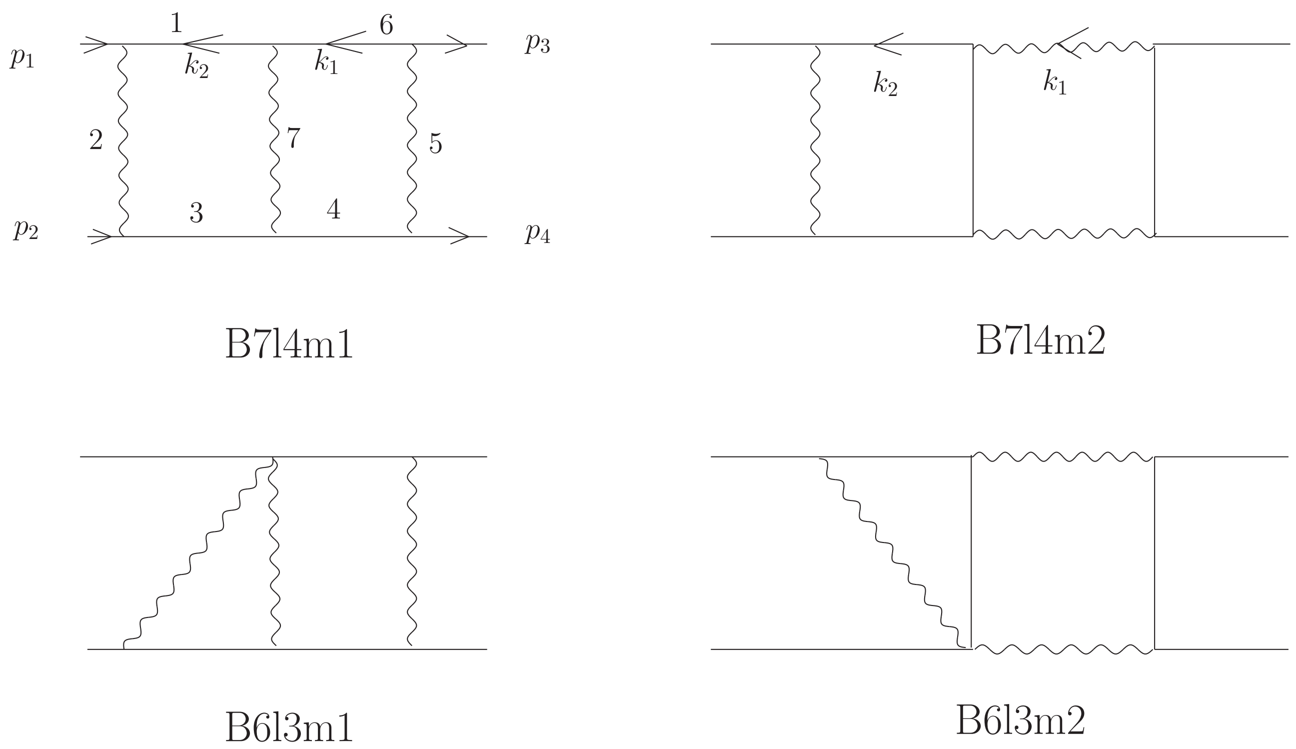


Figure 6: The planar 6- and 7-line topologies.

What to be done now?

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Integrating the Feynman parameters – get MB-Integrals

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int\limits_0^\infty \Pi_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F^{-(a_{4567}-d/2)} = \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \\ +(m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)}$$

We want to apply now:

$$\int_0^1 \prod_{j=4}^7 dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-x_4-x_5-x_6-x_7\right) = \frac{\Gamma(\alpha_4)\Gamma(\alpha_5)\Gamma(\alpha_6)\Gamma(\alpha_7)}{\Gamma\left(\alpha_4+\alpha_5+\alpha_6++\alpha_7\right)}$$

with coefficients α_i dependent on a_i and on F

See in a minute:
For this, we have to apply several MB-integrals here.

And do this, if needed, several times; here: repeat the procedure for the 2nd subloop.

$$\int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\alpha_j - 1} \; \delta\left(1 - \sum_{i=1}^{N} x_i\right) \quad = \quad \frac{\prod_{i=1}^{N} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{N} \alpha_i\right)}$$

Simplest cases:

$$\begin{aligned}
\int_0^1 dx_1 \; x_1^{\alpha_1 - 1} \; \delta\left(1 - x_1\right) \quad &= \quad 1 \\
\int_0^1 \prod_{j=1}^{2} dx_j \; x_j^{\alpha_j - 1} \; \delta\left(1 - \sum_{i=1}^{N} x_i\right) = \int_0^1 dx_1 x_1^{\alpha_1 - 1} (1 - x_1)^{\alpha_2 - 1} \quad &= \quad B(\alpha_1, \alpha_2) \\
&= \quad \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma\left(\alpha_1 + \alpha_2\right)}
\end{aligned}$$

Mathematical interlude

what we need is a sequential use of a formula like:

$\frac{1}{[A(x)+B(x)]^z} => A(x)^{z_A} B(x)^{z_B}$

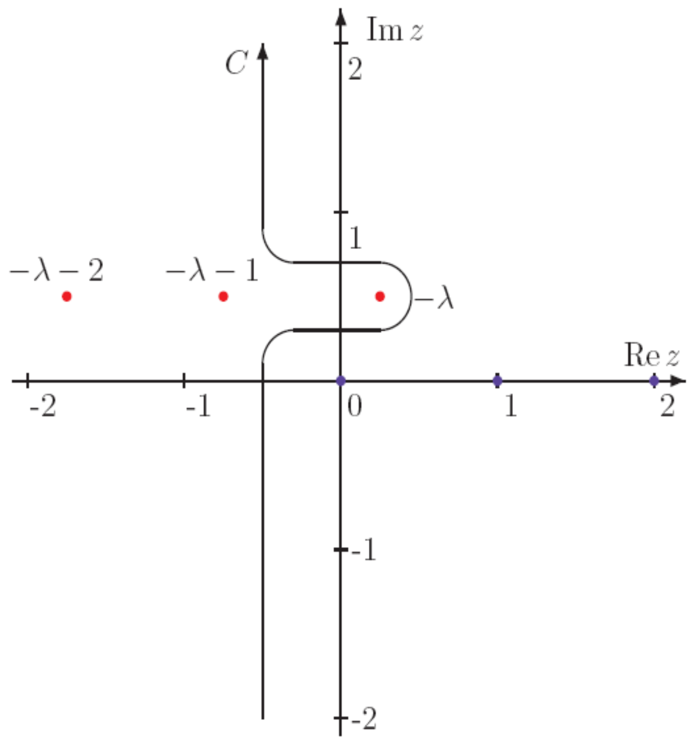
Remark:

Such a formula would also be useful in a completely different ideology:

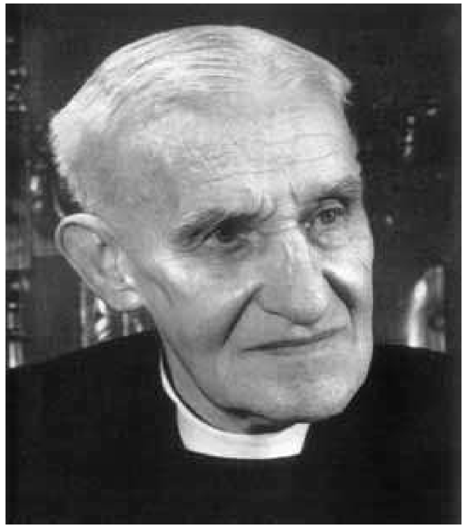
$\frac{1}{(p^2-m^2)} => \frac{\left(m^2\right)^{z_m}}{(p^2)^{z_p}}$

transforms a massive propagator into a massless propagator - with different exponent (index)

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}}$$



Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953

.

Barnes' contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

$$F(z) \;=\; \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma}\frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)}$$

where $|\mathrm{arg(-z)}| < \pi$ **(i.e.** $(-z)$ **is not on the neg. real axis) and the path is such that it separates the poles of** $\Gamma(a+\sigma)\Gamma(b+\sigma)$ **from the poles of** $\Gamma(-\sigma)$**.**
$1/\Gamma(c+\sigma)$ **has no pole.**
Assume $a \neq -n$ **and** $b \neq -n, n = 0, 1, 2, \cdots$ **so that the contour can be drawn.**

The poles of $\Gamma(\sigma)$ **are at** $\sigma = -n, n = 1, 2, \cdots$**, and it is:**

```
Residue[ F[s] Gamma[-s] , {s,n} ]  =  (-1)^n/n!  F(n)
```

Closing the path to the right gives then, by Cauchy's theorem, for $|z| < 1$ **the**

hypergeometric function $_2F_1(a,b,c,z)$ (for proof see textbook):

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N\to\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$$
$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {_2F_1}(a,b,c,z)$$

The continuation of the hypergeometric series for $|z|>1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]}(-z)^{-a-n}$$
$$+\sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]}(-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\ {_2F_1}(a,b,c,z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)}(-z)^{-a}\ {_2F_1}(a,1-c+a,1-b+ac,z^{-1})$$
$$+\frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)}(-z)^{-b}\ {_2F_1}(b,1-c+b,1-a+b,z^{-1})$$

Corollary I

Putting $b = c$, we see that

$$
{}_2F_1(a,b,b,z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{z^n}{n!}
$$
$$
= \frac{1}{(1-z)^a} = \frac{1}{2\pi i\ \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma\ (-z)^\sigma\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

This allows to replace sum by product:

$$
\frac{1}{(A+B)^a} = \frac{1}{B^a[1-(-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma\ B^{-\sigma-a}\ \Gamma(a+\sigma)\Gamma(-\sigma)
$$

Barnes' lemma

If the path of integration is curved so that the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$ lie on the right of the path and the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$ lie on the left, then

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} d\sigma \Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(c-\sigma)\Gamma(d-\sigma) \quad = \quad \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

It is supposed that a,b,c,d are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle C to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a+b+c+d-1)<0$. Take sum of residues of the integrand at poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of Γ-functions, this in turn by combinations of sin, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a+b+c+d-1)<0$. Both sides are analytical functions of e.g. a. So the relation remains true for all values of a,b,c,d for which none of the poles of $\Gamma(a+\sigma)\Gamma(b+\sigma)$, as a function of σ, coincide with any of the poles of $\Gamma(c-\sigma)\Gamma(d-\sigma)$.

Corollary II Any real shift k: $\sigma+k, a-k, b-k, c+k, d+k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.

How can this be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

$$\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i \ \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \ \Gamma(a+\sigma)\Gamma(-\sigma)$$

  which may allow to perform the (massless) momentum integral (with index a of the line changed to $(a+\sigma)$).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting F- and U-forms, in order to get a single monomial in the x_i, which allows the integration over the x_i:

$$\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i \ \Gamma(a)} \int\limits_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \ \Gamma(a+\sigma)\Gamma(-\sigma)$$

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.

A short remark on history

- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use

The Γ-function

The Γ-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z+1) = 0$$

$$\Gamma(z) \quad = \quad \int_0^\infty t^{z-1} e^{-t} dt$$

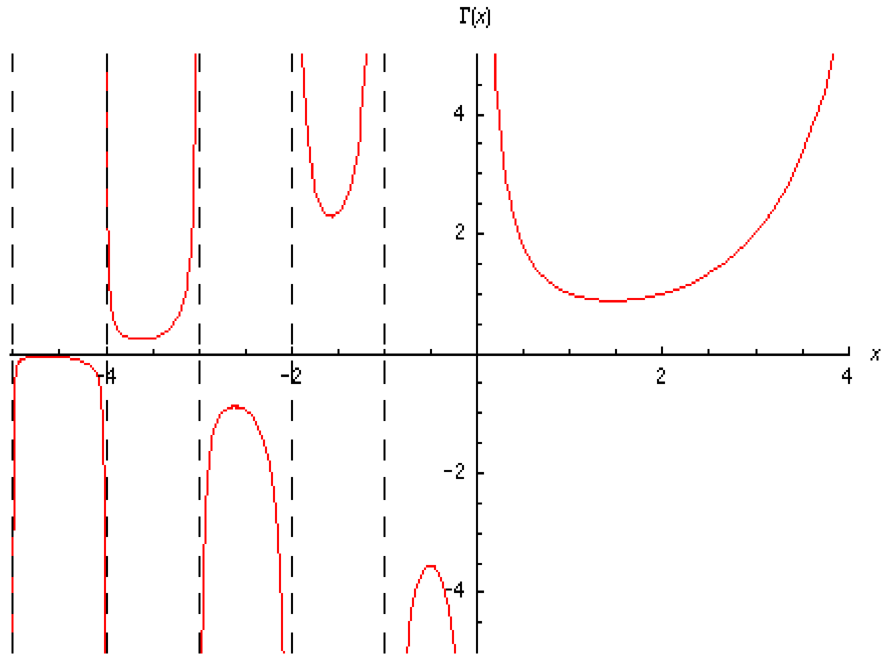
Series[Gamma[ep], {ep, 0, 2}] =

$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \frac{1}{12}(6{\gamma_E}^2 + \pi^2)\epsilon + \frac{1}{12}(-2{\gamma_E}^3 - {\gamma_E}^2\pi + 2\Psi(2,1))\epsilon^2 + \cdots$$

exp(ep EulerGamma)Series[Gamma[ep], {ep, 0, 2}] =

$$e^{\epsilon\gamma_E}\Gamma[\epsilon] = \frac{1}{\epsilon} + \frac{1}{12}(\pi^2)\epsilon + \frac{1}{6}(\Psi(2,1))\epsilon^2 + \cdots$$

Singularities in the complex plane:



Some facts on residua

The function

$$F(z) \quad = \quad \sum_{i=-N}^{\infty} \frac{a_i}{(z-z_0)^i}$$

has the residue

$$\text{Res } F(z)|_{z=z_0} = a_{-1}$$

An integral over an anti-clockwize directed closed path C around z_0 then is

$$\frac{1}{2\pi i} \int_C dz F(z) = 2\pi i a_{-1}$$

If $G(z)$ has a Taylor expansion around z_0 and $F(z)$ has a Laurent expansion beginning with $a_{-N}/(z-z_0)^N + \ldots$, then their product has the residue:

$$\text{Res}[G(z)\ F(z)]|_{z=z_0} = \sum_{k=1}^{N} \frac{a_{-k} G(z_0)^{(k)}}{k!}$$

Some residua with $\Gamma(z)$ and $\Psi(z)$

Residue[F[z]Gamma[z], {z, -n}] $= \frac{(-1)^n}{n!} F[-n]$

Residue[F[z]Gamma[z]^2, {z, -n}] $= \frac{2PolyGamma[n+1]F[-n]+F'[-n]}{(n!)^2}$

Residue[F[z]Gamma[z - 1]^2, {z, -3}] $= \frac{25F[-3]-12\gamma_E F[-3]+6F'[-3]}{3456}$

Series[F[z]Gamma[z - 1]^2, {z, -3, -1}] $= \frac{F[-3]}{576(z+3)^2} + \frac{25F[-3]-12\gamma_E F[-3]+6F'[-3]}{3456(z+3)}$

$+ a_0 + a_1(z+3) + \cdots$

Series[Gamma[z + a]Gamma[z - 1]^2, {z, -3, -1 }] $= \frac{\Gamma[-3+a]}{576(z+3)^2}$

$+ \frac{(25\Gamma[-3+a]-12\gamma_E\Gamma[-3+a]+6(\Gamma[-3+a]PolyGamma[0,-3+a])}{3456} + a_0 + a_1(z+3) + \cdots$

Where

$$\begin{aligned} \mathsf{P}olygamma[n+1] &\equiv \mathsf{P}olygamma[0,n+1] \\ &= \Psi(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)} = S_1(n) - \gamma_E = \sum_{k=1}^{n} \frac{1}{k} - \gamma_E \end{aligned}$$

The following properties hold:

$$\Psi(z+1) = \Psi(z) + 1/z \tag{8}$$

$$\Psi(1+\epsilon) = -\gamma_E + \zeta_2 \; \epsilon + \ldots \tag{9}$$

$$\Psi(1) = -\gamma_E \tag{10}$$

$$\Psi(2) = 1 - \gamma_E \tag{11}$$

$$\Psi(3) = 3/2 - \gamma_E$$

Some sums `Mathematica` can do

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), n, 0, Infinity] =

. (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])

Sum[s^(n) PolyGamma[0, n + 1], n, 0, Infinity] =

. (EulerGamma + Log[1 - s])/(-1 + s)

V3l2m

The Feynman integral V3l2m is the QED one-loop vertex function, which is no master.
It is infrared-divergent (see this by counting of powers of loop integration momentum k or know it from: massless line between two external on-shell lines)

$$F = m^2(x_1+x_2)^2 + [-s]x_1x_2$$

Here: $s \equiv t$ (sorry!!!). We will also use the variable

$$y = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$$

$$\begin{aligned} \mathtt{V3l2m[y]} &= \frac{e^{\epsilon\gamma_E}\Gamma(-2\epsilon)}{2\pi i}\int dz(-s)^{-\epsilon-1-z}\frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)} \\ &= \frac{\mathtt{V3l2m[-1,y]}}{\epsilon}+\mathtt{V3l2m[0,y]}+\epsilon\ \mathtt{V3l2m[1,y]}+\cdots . \end{aligned} \tag{12}$$

$$
\begin{aligned}
\mathtt{V3l2m}[-1,\mathtt{y}] &= \frac{1}{2}\frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(t)^n}{\begin{pmatrix}2n\\n\end{pmatrix}(2n+1)} \\
&= \frac{1}{2}\frac{4\arcsin(\sqrt{t/2})}{\sqrt{4-t}\sqrt{t}} \\
&= \frac{1}{2}\frac{-2y(t)}{1-y^2(t)}\ln y(t) \qquad (13)
\end{aligned}
$$

Close path upwards to the left, so the infinite series of residua of

$$\Gamma[1+r]$$

at $r=-n, n=1,2,\cdots$ arises with weight function

$$G(r) = (-t)^{-1-r}\ \frac{\Gamma^3[-r]}{\Gamma[-2r]}$$

and the sum may be done with Mathematica, see p.33.

$$
\begin{aligned}
\mathtt{V312m}[\mathtt{0},\mathtt{y}] &= \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]} \\
&\quad \frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1+r]\right] \qquad (14) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{t^n}{\binom{2n}{n}(2n+1)} S_1(n),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathtt{V3l2m}[1,\mathtt{y}] &= \frac{1/4}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\;\frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&\Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\
&+\gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\
&-4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\
&+4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\
&-\Psi[-z]\Psi[1+z]) + \Psi[1+z]^2\Big] \\
&= const\frac{1}{4}\sum_{n=0}^{\infty}\frac{(t)^n}{\binom{2n}{n}(2n+1)}\left[S_1(n)^2 + \zeta_2 - S_2(n)\right]. \qquad (15)
\end{aligned}
$$

Here, $\Psi[r] =$ ... and $\Psi[1,r] =$ ..., and the harmonic numbers $S_k(n)$ are

$$
S_k(n) = \sum_{i=1}^{n}\frac{1}{i^k},
$$

see e.g. talk by S.Moch.

Experimentally,

$$\texttt{V3l2m}[2,\texttt{y}] \quad = \quad \frac{1/12}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-t)^{-1-z}\ \frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z)]}$$
$$\Big[a(z)+c_1(z)\Psi(0,1+z)+\Psi(2,1+z)+2\Psi(0,1+z)^2+\Psi(0,1+z)^3$$
$$+\,3\Psi(0,1+z)\Psi(1,1+z)+d_1(z)[\Psi(1,1+z)+2\Psi(0,1+z)^2]\Big] \qquad (16)$$

with some longer coefficients $cc1,\ d_1(z),\ aa$:

```
cc1 =
3*EulerGamma^2 - 6*EulerGamma*Log[-s] + 3*Log[-s]^2
 + 12*PolyGamma[0, -2*z]^2
+  6*PolyGamma[0, -2*z]*(2*(EulerGamma - Log[-s]) - 4*PolyGamma[0, -z])
 - 12*(EulerGamma - Log[-s])*PolyGamma[0, -z]
 +  12*PolyGamma[0, -z]^2 - 12*PolyGamma[1, -2*z] + 6*PolyGamma[1, -z]
```

and

$$d_1(z) \quad = \quad 3*EulerGamma-3*Log[-s]+6*PolyGamma[0,-2*z]-6*PolyGamma[0,-z] \qquad (17)$$

Finally,

```
aa =
EulerGamma^3 - 3*EulerGamma^2*Log[-s] + 3*EulerGamma*Log[-s]^2
- Log[-s]^3 + 8*PolyGamma[0, -2*z]^3
+  12*PolyGamma[0, -2*z]^2*(EulerGamma - Log[-s] - 2*PolyGamma[0, -z])
 + 12*(EulerGamma - Log[-s])*PolyGamma[0, -z]^2
 - 8*PolyGamma[0, -z]^3 - 12*EulerGamma*PolyGamma[1, -2*z]
 + 12*Log[-s]*PolyGamma[1, -2*z]
+  6*EulerGamma*PolyGamma[1, -z] - 6*Log[-s]*PolyGamma[1, -z]
 -  6*PolyGamma[0, -z]*(EulerGamma^2 - 2*EulerGamma*Log[-s] + Log[-s]^2
- 4*PolyGamma[1, -2*z] + 2*PolyGamma[1, -z])
+  6*PolyGamma[0, -2*z]*(EulerGamma^2 - 2*EulerGamma*Log[-s] + Log[-s]^2
 - 4*(EulerGamma - Log[-s])*PolyGamma[0, -z]
+    4*PolyGamma[0, -z]^2 - 4*PolyGamma[1, -2*z] + 2*PolyGamma[1, -z])
 + 8*PolyGamma[2, -2*z] - 2*PolyGamma[2, -z]
```

$$\mathtt{V3l2m}[2,\mathtt{y}] = \frac{1}{1}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)} \Big[\frac{1}{12}S_1[n]^3 - \frac{1}{4}S_1[n]S_2[n] + \frac{1}{4}\zeta_2 S_1[n] + \frac{1}{6}S_3[n] - \frac{1}{6}\zeta_3\Big]. \tag{18}$$

Sum this up!!

Answer is known to us from another technique: differential equations;
see our Bhabha webpage, file

master.m

On-shell example: B4l2m, the 1-loop on-shell box

```
den = (x4 d4 + x5 d5 + x6 d6 + x7 d7 // Expand) /. kinBha /. m^2 -> 1 //  Expand

Q = -Coefficient[den, k]/2 // Simplify
    =  p3 x4 + p2 x5 - p1 (x4 + x6)

M = Coefficient[den, k^2] // Simplify
    = x4 + x5 + x6 + x7 -> 1

J = den /. k -> 0 // Simplify
    = t x4

F[x] = (Q^2 - J  M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4 // Expand
    = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7

B4l2ma = mb[(x5+x6)^2, -tx7x4 - sx5x6, nu, ga]

B4l2mb = B4l2ma /. (-sx5x6 - tx4x7)^(-ga - nu) ->
            mb[(-s)x5x6, (-t)x7x4, nu+ga, de]
            /.((-s)x5x6)^de_ -> (-s)^de x5^de x6^de
            /.((x56^2)^ga    -> (x5 + x6)^(2ga)
```

```
=
   (inv2piI^2(-s)^de x5^de x6^de ((x5 + x6)^(2ga)((-t)x4x7)^(-de-ga-nu)
        Gamma[-de] Gamma[-ga] Gamma[de + ga + nu] /Gamma[nu]

B4l2mc = B4l2mb /. (x5 + x6)^(2ga) ->
            mb[x5, x6, -2ga, si]
            /. ((-t)x4x7)^si_ -> (-t)^si x4^si x7^si // ExpandAll
=
   1/(Gamma[-2ga] Gamma[nu])
inv2piI^3  (-s)^de (-t)^(-de - ga - nu)
            x4^(-de - ga - nu) x5^(de + si) x6^(de + 2 ga - si) x7^(-de - ga - nu)
            Gamma[-de] Gamma[-ga] Gamma[ de + ga + nu] Gamma[-si] Gamma[-2ga + si]

B4l2md =  xfactor4[a4, x4, a5, x5, a6, x6, a7, x7] B4l2mc
=
.... (-s)^de (-t)^(-de-ga-nu)
x4^(-1+a4-de-ga-nu) x5^(-1+a5+de+si) x6^(-1+a6+de+2ga-si) x7^(-1+a7- de-ga-nu)

B4l2me =
  B4l2md /.
      x4^B4_  x5^B5_  x6^B6_  x7^B7_ -> xint4[x4^B4 x5^B5 x6^B6 x7^B7]
= ...
```

```
B4l2mf =  B4l2me /.
      Gamma[a6 + de + 2 ga - si]Gamma[-si]Gamma[ a5 + de + si] Gamma[-2 ga + si]
    ->  barne1[si, a5 + de, -2 ga, a6 + de + 2 ga, 0]
```

This finishes the evaluation of the MB-representation for `B4l2m`.

Package: AMBRE.m (K. Kajda, with support by J. Gluza and TR)

B4l2m

```
F[x] = (Q^2 - J  M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4 // Expand
     = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7
```

`B4l2m`, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite ϵ:

$$\begin{aligned} \texttt{B4l2m} = \texttt{Box}(t,s) &= \frac{e^{\epsilon\gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \\ & \frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2+\epsilon+z_1+z_2]\Gamma^2[1+z_1]\Gamma[-z_1]\Gamma[-z_2] \\ & \Gamma^2[-1-\epsilon-z_1-z_2] \frac{\Gamma[-2-2\epsilon-2z_1]}{\Gamma[-2-2\epsilon-2z_1-2z_2]} \end{aligned} \tag{19}$$

`Mathematica` package `MB` used for analytical expansion $\epsilon \to 0$:

[Czakon:2005rk]

$$\texttt{B4l2m} = -\frac{1}{\epsilon}\texttt{I1} + \ln(-s)\texttt{I1} + \epsilon\left(\frac{1}{2}\left[\zeta(2) - \ln^2(-s)\right]\texttt{I1} - 2\texttt{I2}\right). \tag{20}$$

with I1 being also the divergent part of the vertex function $C_0(t;m,0,m)/s = \texttt{V3l2m}/s$ (as is well-known):

$$\texttt{I1} = \frac{e^{\epsilon\gamma_E}}{st}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_1 \left(\frac{m^2}{-t}\right)^{z_1} \frac{\Gamma^3[-z_1]\Gamma[1+z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s}\frac{2y}{1-y^2}\ln(y) \tag{21}$$

with $y = (\sqrt{1-4m^2/t}-1)/(\sqrt{1-4m^2/t}+1)$: close contour to left, take residua at $(1+z_1) = -n$, sum up with Mathematica:

```
Residue[F[x]Gamma[1 + x], {x, -n}] // InputForm  =   -(-1)^n F[-n]/n!

Sum[s^(n) Gamma[n + 1]^3/(n!Gamma[2 + 2n]), {n, 0, Infinity}] // InputForm
=  (4*ArcSin[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])
```

The I2 is more complicated:

$$\texttt{I2} = \frac{e^{\epsilon\gamma_E}}{t^2}\frac{1}{(2\pi i)^2} \int_{-\frac{3}{4}-i\infty}^{-\frac{3}{4}+i\infty} dz_1 \left(\frac{-s}{-t}\right)^{z_1} \Gamma[-z_1]\Gamma[-2(1+z_1)]\Gamma^2[1+z_1]$$
$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz_2 \left(\frac{m^2}{-t}\right)^{z_2} \Gamma[-z_2]\frac{\Gamma^2[-1-z_1-z_2]}{\Gamma[-2(1+z_1+z_2)]}\Gamma[2+z_1+z_2]. \tag{22}$$

The expansion of `B412m` at small m^2 and fixed value of t

With

$$m_t = \frac{-m^2}{t}, \tag{23}$$

$$r = \frac{s}{t}, \tag{24}$$

Look, under the integral, at $(-m^2/t)^{z_2}$,

and close the path to the right.

Seek the residua from the poles of Γ-functions with the smallest powers in m^2 and try to sum the resulting series.

Automatize this, it is not too easy.

we have obtained a compact answer for `I2` with the additional aid of `XSUMMER`

[Moch:2005uc]

. The box contribution of order ϵ in this limit becomes:

$$\begin{aligned} \mathtt{B412m}[t, s, m^2; +1] = \frac{1}{st}\Big\{ & 4\zeta_3 - 9\zeta_2 \ln(m_t) + \frac{2}{3}\ln^3(m_t) + 6\zeta_2\ln(r) - \ln^2(m_t)\ln(r) \\ & + \frac{1}{3}\ln^3(r) - 6\zeta_2\ln(1+r) + 2\ln(-r)\ln(r)\ln(1+r) - \ln^2(r)\ln(1+r) \\ & + 2\ln(r)\mathrm{Li}_2(1+r) + 2\mathrm{Li}_3(-r)\Big\} + \mathcal{O}(m_t). \end{aligned} \tag{25}$$

Some routines in mathematica which were used:

```
(* Barnes' first lemma: \int d(si) Gamma(si1p+si)Gamma(si2p+si)Gamma(si1m-si)Gamma(si2m-si)
                        with 1/inv2piI = 2 Pi I *)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
  1/inv2piI  Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
      si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m]

 (* Mellin-Barnes integral: (A+B)^(-nu) = 1/(2 Pi I) \int d(si) a^si b^(-nu - si)
                            Gamma[-si]Gamma[nu+si]/Gamma[nu]  *)

mb[a_, b_, nu_, si_]:=inv2piI a^si b^(-nu-si)Gamma[-si]Gamma[nu+si]/Gamma[nu]

 (* After the k-integration, the integrand for \int\prod(dxi xi^(ai -  1))\delta(1-\sum xi)
              will be (L=1 loop) : xfactorn F^(-nu) Q(xi).pe with  nu = a1 + .. + an - d/2 *)

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
  I Pi^(d/2) (-1)^(a1 + a2 + a3) x1^(a1 - 1) x2^(a2 - 1)x3^(a3 - 1)Gamma[
        a1 + a2 + a3 - d/2]/(Gamma[a1]Gamma[a2]Gamma[a3])

 (* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 *)

xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_)  ] :=
  Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1]  / Gamma[a1 + a2 + a3 + 3]
```

.

Another nice box with numerator, B5l3m($p_e.k_1$)

We used it for the determination if the small mass expansion.

$$
\begin{aligned}
&\texttt{B5l3m}(\mathrm{p_e}\cdot\mathrm{k_1}) \;=\; \frac{m^{4\epsilon}(-1)^{a_{12345}}\; e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta\\
&(-s)^{(4-2\,\epsilon)-a_{12345}-\alpha-\beta-\delta}\;(-t)^{\delta}\\
&\frac{\Gamma[-4+2\,\epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\,\epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\,\Gamma[-\beta]}{\Gamma[7-3\,\epsilon-a_{12345}-\alpha]\,\Gamma[5-2\,\epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\,\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]}\\
&\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\frac{\Gamma[4-2\,\epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]}\Big\{(p_e\cdot p_3)\;\Gamma[1+a_4+\delta]\;\Gamma[6-3\,\epsilon-a_{12345}-\alpha]\\
&\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\;\Gamma[3-\epsilon-a_{12}-\alpha]\;\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\\
&\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\;\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[a_1+\gamma]\;\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\;\Gamma[7-3\,\epsilon-a_{12345}-\alpha]\\
&\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\\
&\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\;\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\;\Gamma[4-2\,\epsilon-a_{1123}-\gamma]\\
&\Gamma[5-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\;\Gamma[1+a_1+\gamma]\Big]\;\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\,\epsilon-a_{12345}-\alpha]\;\Gamma[3-\epsilon-a_{12}-\alpha]\\
&\Gamma[5-2\,\epsilon-a_{1123}-\gamma]\;\Gamma[4-2\,\epsilon-a_{1123}-2\,\alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\;\Gamma[5-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\\
&\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\;\Gamma[4-2\,\epsilon-a_{1234}-\alpha-\beta-\delta]\\
&\Gamma[8-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\delta-\gamma]\Gamma[9-4\,\epsilon-a_{112233445}-2\,\alpha-2\,\beta-2\,\delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}
\end{aligned}
$$

.

B5l2m2

$$\texttt{B5l2m2} \;=\; \frac{m^{4\epsilon}(-1)^{a_{12345}}e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[4-2\epsilon-a_{13}](2\pi i)^3}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma(-s)^{2-\epsilon-a_{245}-\gamma-\alpha+\beta}(-t)^{\alpha}$$
$$\Gamma[-2+\epsilon+a_{13}+\beta]\Gamma[-\gamma]\Gamma[2-\epsilon-a_{245}-\gamma-\alpha]\Gamma[-\alpha]$$
$$\Gamma[a_2+\alpha]\Gamma[a_4+\alpha]\Gamma[4-2\epsilon-a_{113}-\beta]\Gamma[-2+\epsilon+a_{245}+\gamma+\alpha-\beta]\Gamma[a_1+\beta]$$
$$\frac{\Gamma[4-2\epsilon-a_{2245}-2\alpha+\beta]\Gamma[2-\epsilon-a_{24}-\gamma-\alpha+\beta]}{\Gamma[4-2\epsilon-a_{245}+\beta]\Gamma[4-2\epsilon-a_{22445}-2\gamma-2\alpha+\beta]}$$

This kind of expression now has to be evaluated:

- Check special cases of indices, set lines to 1 (by setting $a_i \to 0$ if possible)
- Extract the ϵ-dependence related to UV and IR singularities (see next pages)
- After that: may set $s<0,\ t<0$ and evaluate numerically Euclidean case
- Use sector decomposition for a numerical comparison - if you have a program for that
- Try to go Minkowskian in a numerical way (if you like this)
- Go on analytically, e.g. by taking residua $\to$ get nested infinite sums from the residua
- Try to sum them up

Shrinking of lines; seek the ϵ-expansion

Go on with some study of the 2nd planar 2-box, B7l4m2 (see also Smirnov book 4.73):

$$B_{\mathrm{pl},2} = \frac{\mathrm{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{-t}{-s}\right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\}}{\prod_{l=19}^{24} \Gamma_l(\{z_i\}}$$

with $a = a_1 + ... + a_7$ and

$$\begin{aligned} z_i &= \mathrm{const} + i\, \Im m(z_i) \\ d &= 4 - 2\epsilon \\ \mathrm{const} &= \frac{(i\pi^{d/2})^2(-1)^a(-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d-a_{4567})} \end{aligned}$$

The integrand includes e.g.:

$$\begin{aligned} \Gamma_2 &= \Gamma(-z_2) \\ \Gamma_4 &= \Gamma(-z_4) \\ \Gamma_7 &= \Gamma(a_4 + z_2 + z_4) \\ \Gamma_8 &= \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \\ &\ldots \end{aligned}$$

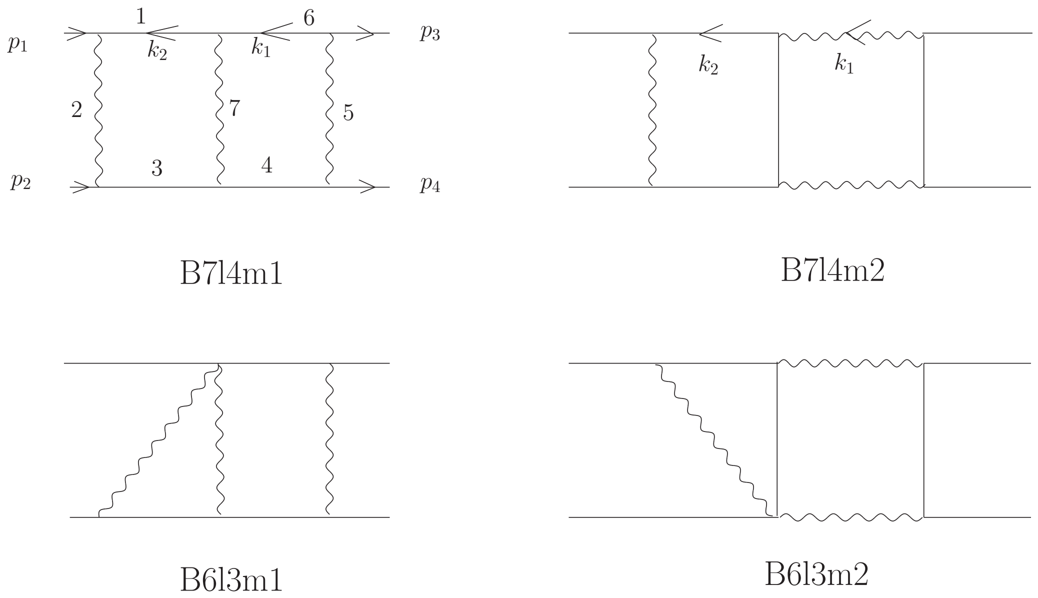


Figure 7: The planar 6- and 7-line topologies.

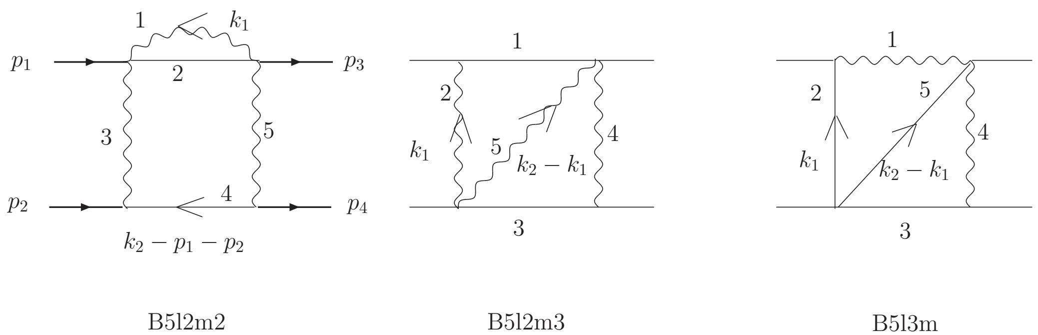


Figure 8: The 5-line topologies. B7l4m2: shrink line 1 get B6l3m2, then line 4 get B5l3m

Example:
derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and then setting $a_4 = 0$.
The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

$\to$ Start with the z_2, z_4 integrations by
taking the residues for closing the integration contours to the right:

$$\begin{aligned}
I_{2,4} &= \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4+z_2+z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \\
&= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,..} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4+z_2+n)}{\Gamma(a_4)} R(z_i) \\
&= \sum_{n,m=0,1,..} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4+n+m)}{\Gamma(a_4)} R(z_i) \to_{a_4=0} 1 \times R(z_i)
\end{aligned}$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$
we got a 4-fold Mellin-Barnes integral for topology B5l3m (by "shrinking of lines")
with $24 - 3 = 21$ z_i-dependent Γ-functions which may yield residua within four-fold sums.

The MB-representation has to be calculated explicitely at fixed indices, e.g.

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0$$

General Tasks, first two steps automated by `MB.m`:

- Find a region of definiteness of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

An important tool is the command FindInstance of Mathematica 5:
It allows to solve a system of inequalities.
Here an example for B7l4m3, the non-planar massive double box:

```
sol = FindInstance[
    Cases[B7l4m3 ... Gamma[x_] -> x > 0 /. ep -> -1/10, {z1, z2, z3, z4, z5, z6, z7, z8}]
```

The result is:

```
 {z1 ->  -1/20, z2 ->  -1/40,  z3 ->   -1/20,  z4 -> -29/32,
  z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 ->  -5/64}
```

Really, all arguments are positive:

```
G1[11/160] G10[1/320] G11[3/40] G12[3/40] G13[41/80] G14[37/40] G15[1/20] G16[1/40]
G17[1/20] G18[29/32] G19[67/80] G2[7/160] G20[83/160] G21[273/320] G3[7/80] G4[139/160]
G5[143/160] G6[1/320] G7[41/80] G8[1/80] G9[43/80]
```

Now set $\epsilon = 0$:

```
      G1[11/160] G1
 {z1 ->  -1/20, z2 ->  -1/40,  z3 ->   -1/20,  z4 -> -29/32,
  z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 ->  -5/64,   ep -> 0}
```

Determine again the arguments of the Gamma-functions; observe:
2 arguments are negative now: those for G3 and G8

```
G1[11/160] G10[33/320] G11[3/40] G12[7/40] G13[57/80] G14[37/40] G15[1/20] G16[1/40]
G17[1/20] G18[29/32] G19[67/80] G2[7/160] G20[83/160] G21[273/320] G4[123/160]
G5[127/160] G6[1/320] G7[5/16] G9[27/80] G3[-9/80] G8[-31/80]
```

Perform the corresponding shifts of integration curve, add the residua and again perform the test for the arguments of the new, lower-dimensional MB-integrals.

We derived an algorithmic solution for isolating the singularities in $1/\epsilon$
The automatization of that: MB.m (M. Czakon)

$$
\begin{aligned}
B5l3md2 \quad &\rightarrow \quad MB(\text{4-dim,fin}) + MB_3(\text{3-dim,fin}) \\
&+ \quad MB_{36}(\text{2-dim}, \epsilon^{-1}, \text{fin}) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1}, \text{fin}) \\
&+ \quad MB_5(\text{3-dim,fin})
\end{aligned}
$$

After these preparations e.g.:

$$
\begin{aligned}
MB_{365}(\text{1-dim}, \epsilon^{-2}) \quad &\sim \quad \frac{1}{\epsilon^2}\frac{1}{2\pi i}\int dz_6 \frac{(-s)^{(-z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} \\
&= \quad \frac{1}{\epsilon^2}\sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} \\
&= \quad -\frac{1}{\epsilon^2}\frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \\
&= \quad \frac{1}{\epsilon^2}\frac{-x}{4(1-x^2)}H[0,x]
\end{aligned}
$$

Here residua were taken at $z_6 = -n-1, n = 0, 1, .., $ and $H[0,x] = \ln(x)$ and $x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$.

Summary

- We have introduced to the representation of L-loop N-point Feynman integrals of general type
- The determination of the ϵ-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, ... so you have something to do if you like to ...

Problem: Determine the small mass limit of B5l2m2 or of any other of the 2-loop boxes for Bhabha scattering.
Prof. Gluza may check your solution.
He leaves soon.