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Loops&Legs'04

# Evaluating multiloop Feynman integrals by Mellin-Barnes representation

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- Mellin-Barnes representation as a tool to evaluate master integrals
- Examples: massless triple box and massive double boxes
- Perspectives and related mathematical problems

A given Feynman graph $\Gamma \rightarrow$ tensor reduction (one-loop graphs: G. Passarino and M. Veltman, Nucl. Phys. **B160** (1979) 151,..., two-loop two- and three-point graphs: S. Actis, A. Ferroglia, G. Passarino, M. Passera and S. Uccirati, hep-ph/0402132) $\rightarrow$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators.

$$F(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

$d = 4 - 2\epsilon$

Methods: analytical, numerical, semianalytical ...

A straightforward analytical strategy:
to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

An advanced strategy:
to derive, without calculation, and then apply integration by parts (IBP) and Lorentz-invariance (LI) identities between the given family of Feynman integrals as *recurrence relations*.

A general integral from the given family is expressed as a linear combination of some basic (*master*) integrals.

The whole problem of evaluation$\rightarrow$

- solution of the reduction procedure
- evaluation of the master Feynman integrals.

Recent attempts to make the reduction procedure systematic:

- Laporta's idea $\rightarrow$ [S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087; T. Gehrmann and E. Remiddi [Nucl. Phys.

**B601** (2001) 248, 287]: "When increasing the total dimension of the denominator and numerator in Feynman integrals from a given family, the total number of IBP and LI equations grows faster than the number of independent Feynman integrals (labelled by the powers of propagators and powers of independent scalar products in the numerators). Therefore this system of equations sooner or later becomes overconstrained, and one obtains the possibility to perform a reduction to master integrals"

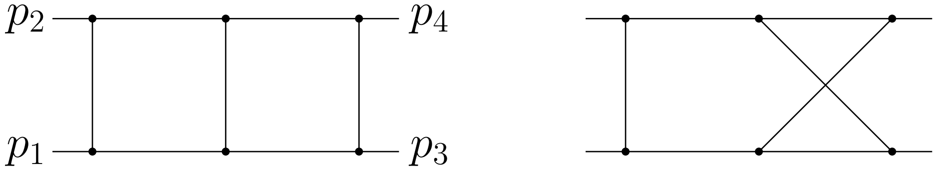
- shifting dimension [O.V. Tarasov, Nucl. Phys. **B 480** (1996) 397; Phys. Rev. D54 (1996) 6479]
- Baikov's method [P.A. Baikov, Phys. Lett. **B385** (1996) 404; Nucl. Instrum. Methods **A389** (1997) 347; V.A. Smirnov and M. Steinhauser, Nucl. Phys. **B 672** (2003) 199]

Evaluating master integrals:
Feynman/alpha parameters, Mellin–Barnes (MB) representation [V.A. Smirnov, J.B Tausk], DE (method of differential equations [A.V. Kotikov, Phys. Lett. **B254** (1991) 158; **B259** (1991) 314; **B267** (1991) 123; E. Remiddi, Nuovo Cim. **110A** (1997) 1435; T. Gehrmann and E. Remiddi, Nucl. Phys. **B580** (2000) 485])

MB representation:

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z) \; .$$

The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right.




Massless on-shell ($p_i^2 = 0,\ i = 1, 2, 3, 4$) double boxes: done in 1999-2000, with multiple subsequent applications.

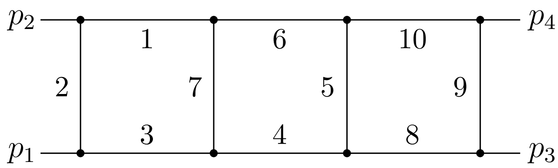
Reduction using shifting dimension

Master integrals: MB.

more loops, more legs, more parameters. . .

triple boxes

#loops + #legs $= 3 + 4 = 7 \gg 1$.




The general planar triple box Feynman integral

$$
\begin{aligned}
T(a_1, \ldots, a_{10}; s, t; \epsilon) \ &= \ \int\!\!\int\!\!\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l\, \mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}} \\
&\times \frac{1}{[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}} \\
&\times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}} \,,
\end{aligned}
$$

where $k^2 = k^2 + i0$, $s = s + i0$, etc., $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, and $k, l$ and $r$ are loop momenta.

Sevenfold MB representation of the general planar triple box

$$
\begin{aligned}
T(a_1,\ldots,a_8;s,t,m^2;\epsilon) \;&=\; \frac{\left(i\pi^{d/2}\right)^3(-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}} \\
&\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)} \\
&\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)} \\
&\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6) \\
&\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7) \\
&\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7) \\
&\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7) \\
&\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7) \\
&\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,
\end{aligned}
$$

where $a_{589(10)}=a_5+a_8+a_9+a_{10}, a_{13}=a_1+a_3,\ \ldots,$ and $a=a_{12\ldots(10)}$.

The master triple box:

$$
\begin{aligned}
&T(1,1,\ldots,1;s,t;\epsilon) \\
&\quad=\frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}}\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
&\quad\times\frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
&\quad\times\frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
&\quad\times\Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
&\quad\times\Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
&\quad\times\Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)\,.
\end{aligned}
$$

Radcor& LL'02:

the Regge asymptotics (in the limit $t/s \to 0$) of the master planar triple box [V.A. Smirnov, Phys.Lett. **B547** (2002) 239]:

$$
\begin{aligned}
&T(1,1,\ldots,1;s,t;\epsilon) \\
&= -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}}\left\{\frac{16}{9\epsilon^6} - \frac{5L}{3\epsilon^5} - \frac{3\pi^2}{2\epsilon^4} - \left[\frac{11\pi^2}{12}L + \frac{131\zeta(3)}{9}\right]\frac{1}{\epsilon^3}\right. \\
&+ \left[\frac{49\zeta(3)}{3}L - \frac{1411\pi^4}{1080}\right]\frac{1}{\epsilon^2} - \left[\frac{503\pi^4}{1440}L - \frac{73\pi^2\zeta(3)}{4} + \frac{301\zeta(5)}{15}\right]\frac{1}{\epsilon} \\
&+ \left.\left[\frac{223\pi^2\zeta(3)}{12} + 149\zeta(5)\right]L - \frac{624607\pi^6}{544320} + \frac{167\zeta(3)^2}{9} + O(\epsilon)\right\},
\end{aligned}
$$

where $L = \ln s/t$

Analytical evaluation [V.A. Smirnov, Phys. Lett. **B567** (2003)]:

- The standard procedure of taking residues and shifting contours, with the goal to obtain a sum of integrals where one may expand integrands in Laurent series in $\epsilon$.
- The first and the second Barnes lemmas $\to$ no more than twofold MB integrals of gamma functions and their derivatives.
- Closing contour in the complex plane and summation in terms of HPL.

Result:

$$T(1,1,\ldots,1;s,t;\epsilon) = -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} \frac{c_j(x,L)}{\epsilon^j} \,,$$

where $x=-t/s$, $L=\ln(s/t)$, and

$$c_6 = \frac{16}{9}\,, \quad c_5 = -\frac{5}{3}L\,, \quad c_4 = -\frac{3}{2}\pi^2\,,$$

$$c_3 = 3(H_{0,0,1}(x) + LH_{0,1}(x)) + \frac{3}{2}(L^2+\pi^2)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3\,,$$

$$\begin{aligned}
c_2 &= -3\left(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)\right) \\
&\quad -L\left(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)\right) - \frac{3}{2}(L^2+\pi^2)H_{1,1}(x) \\
&\quad -\left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4\,,
\end{aligned}$$

$$\begin{aligned}
c_1 &= 3\left(81H_{0,0,0,0,1}(x) + 41H_{0,0,0,1,1}(x) + 37H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x)\right. \\
&\quad +33H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29H_{1,0,0,0,1}(x) \\
&\quad \left.+H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\right) + L\left(177H_{0,0,0,1}(x) + 85H_{0,0,1,1}(x)\right. \\
&\quad \left.+73H_{0,1,0,1}(x) + 3H_{0,1,1,1}(x) + 61H_{1,0,0,1}(x) + 3H_{1,0,1,1}(x) + 3H_{1,1,0,1}(x)\right) \\
&\quad +\left(\frac{119}{2}L^2 + \frac{139}{12}\pi^2\right)H_{0,0,1}(x) + \left(\frac{47}{2}L^2 + 20\pi^2\right)H_{0,1,1}(x) \\
&\quad +\left(\frac{35}{2}L^2 + 14\pi^2\right)H_{1,0,1}(x) + \frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1}(x) \\
&\quad +\left(\frac{23}{2}L^3 + \frac{83}{12}\pi^2 L - 96\zeta_3\right)H_{0,1}(x) + \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_{1,1}(x) \\
&\quad +\left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right)H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5\,,
\end{aligned}$$

$$
\begin{aligned}
c_0 = &-\left(951H_{0,0,0,0,0,1}(x) + 819H_{0,0,0,0,1,1}(x) + 699H_{0,0,0,1,0,1}(x) + 195H_{0,0,0,1,1,1}(x)\right.\\
&+547H_{0,0,1,0,0,1}(x) + 231H_{0,0,1,0,1,1}(x) + 159H_{0,0,1,1,0,1}(x) + 3H_{0,0,1,1,1,1}(x)\\
&+363H_{0,1,0,0,0,1}(x) + 267H_{0,1,0,0,1,1}(x) + 195H_{0,1,0,1,0,1}(x) + 3H_{0,1,0,1,1,1}(x)\\
&+123H_{0,1,1,0,0,1}(x) + 3H_{0,1,1,0,1,1}(x) + 3H_{0,1,1,1,0,1}(x) + 147H_{1,0,0,0,0,1}(x)\\
&+303H_{1,0,0,0,1,1}(x) + 231H_{1,0,0,1,0,1}(x) + 3H_{1,0,0,1,1,1}(x) + 159H_{1,0,1,0,0,1}(x)\\
&+3H_{1,0,1,0,1,1}(x) + 3H_{1,0,1,1,0,1}(x) + 87H_{1,1,0,0,0,1}(x) + 3H_{1,1,0,0,1,1}(x)\\
&\left.+3H_{1,1,0,1,0,1}(x) + 3H_{1,1,1,0,0,1}(x)\right)\\
&-L\left(729H_{0,0,0,0,1}(x) + 537H_{0,0,0,1,1}(x) + 445H_{0,0,1,0,1}(x) + 133H_{0,0,1,1,1}(x)\right.\\
&+321H_{0,1,0,0,1}(x) + 169H_{0,1,0,1,1}(x) + 97H_{0,1,1,0,1}(x) + 3H_{0,1,1,1,1}(x)\\
&+165H_{1,0,0,0,1}(x) + 205H_{1,0,0,1,1}(x) + 133H_{1,0,1,0,1}(x) + 3H_{1,0,1,1,1}(x)\\
&\left.+61H_{1,1,0,0,1}(x) + 3H_{1,1,0,1,1}(x) + 3H_{1,1,1,0,1}(x)\right)\\
&-\left(\frac{531}{2}L^2 + \frac{89}{4}\pi^2\right)H_{0,0,0,1}(x) - \left(\frac{311}{2}L^2 + \frac{619}{12}\pi^2\right)H_{0,0,1,1}(x)\\
&-\left(\frac{247}{2}L^2 + \frac{307}{12}\pi^2\right)H_{0,1,0,1}(x) - \left(\frac{71}{2}L^2 + 32\pi^2\right)H_{0,1,1,1}(x)\\
&-\left(\frac{151}{2}L^2 - \frac{197}{12}\pi^2\right)H_{1,0,0,1}(x) - \left(\frac{107}{2}L^2 + 50\pi^2\right)H_{1,0,1,1}(x)\\
&-\left(\frac{35}{2}L^2 + 14\pi^2\right)H_{1,1,0,1}(x) - \frac{3}{2}\left(L^2 + \pi^2\right)H_{1,1,1,1}(x)\\
&-\left(\frac{119}{2}L^3 + \frac{317}{12}\pi^2 L - 455\zeta_3\right)H_{0,0,1}(x) - \left(\frac{47}{2}L^3 + \frac{179}{12}\pi^2 L\right.\\
&\left.-120\zeta_3\right)H_{0,1,1}(x) - \left(\frac{35}{2}L^3 + \frac{35}{12}\pi^2 L - 156\zeta_3\right)H_{1,0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L\right.\\
&\left.-3\zeta_3\right)H_{1,1,1}(x) - \left(\frac{69}{8}L^4 + \frac{101}{8}\pi^2 L^2 - 291\zeta_3 L + \frac{559}{90}\pi^4\right)H_{0,1}(x)\\
&-\left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right)H_{1,1}(x)\\
&-\left(\frac{27}{40}L^5 + \frac{25}{8}\pi^2 L^3 - \frac{183}{2}\zeta_3 L^2 + \frac{131}{60}\pi^4 L - \frac{37}{12}\pi^2\zeta_3 + 57\zeta_5\right)H_1(x)\\
&+\left(\frac{223}{12}\pi^2\zeta_3 + 149\zeta_5\right)L + \frac{167}{9}\zeta_3^2 - \frac{624607}{544320}\pi^6\,.
\end{aligned}
$$

$\zeta_3 = \zeta(3), \zeta_5 = \zeta(5)$ and $\zeta(z)$ is the Riemann zeta function.

The functions $H_{a_1,a_2,\ldots,a_n}(x) \equiv H(a_1, a_2, \ldots, a_n; x)$, with $a_i = 1, 0, -1$, are HPL [E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) 725] which are recursively defined by

$$H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \ldots, a_n; t)\,,$$

where

$$f(\pm 1; t) = 1/(1 \mp t), \quad f(0; t) = 1/t,$$
$$H(\pm 1; x) = \mp \ln(1 \mp x), \quad H(0; x) = \ln x\,,$$

with $a_i = 1, 0, -1$.

T. Binoth and G. Heinrich, Nucl. Phys. **B680** (2004) 375: confirmation with the help of numerical integration based on a sector decomposition in the space of alpha parameters [T. Binoth and G. Heinrich, Nucl. Phys. **B585** (2000) 741]

Any massless planar on-shell triple box can be evaluated by this procedure.

The possibility to evaluate three-loop virtual corrections to various scattering processes?

Studying cross order relations in $N = 4$ supersymmetric gauge theories [C. Anastasiou, L.J. Dixon, Z. Bern and D.A. Kosower, Phys. Rev. Lett. **91** (2003) 251602; hep-th/0402053]

("one more triple box is needed")

One leg off-shell, $p_1^2 = q^2 \neq 0$, $p_i^2 = 0,\ i = 2,3,4$:
done in 2000-2002

Reduction à la Laporta, Gehrmann & Remiddi [T. Gehrmann and E. Remiddi, Nucl. Phys. **B601**]

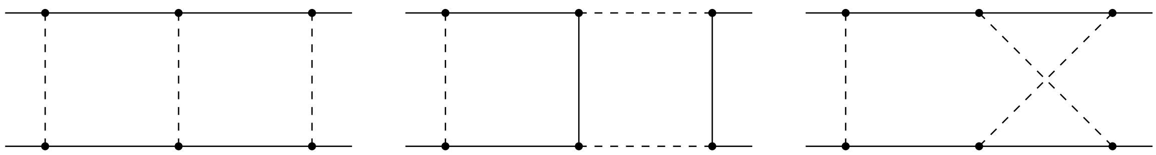
Master integrals: MB (first results [V.A. Smirnov, Phys. Lett. **B491** (2000) 130; **B500** (2001) 330.]) and DE (systematic evaluation [T. Gehrmann and E. Remiddi, Nucl. Phys. **B601** (2001) 248; **B601** (2001) 287]).

All results are expressed in terms of two-dimensional harmonic polylogarithms which generalize harmonic polylogarithms [E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) 72].

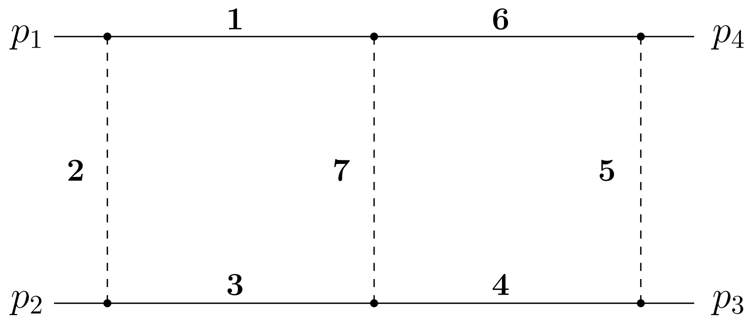
Applications to the process $e^+e^- \to$ 3jets: LL'04, talk by N. Glover

This combination of reduction à la Laporta, Gehrmann & Remiddi and DE was successfully applied in numerous calculations, e.g., various classes of vertex diagrams [P. Mastrolia and E. Remiddi, Nucl. Phys. **B664** (2003) 341; U. Aglietti and R. Bonciani, Nucl. Phys. **B668** (2003) 3; R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. **B661** (2003) 289; **B676** (2004) 399; hep-ph/0311145; U. Aglietti and R. Bonciani, hep-ph/0401193; U. Aglietti, R. Bonciani, G. Degrassi and A. Vicini, hep-ph/0404071]

Massive on-shell 2-boxes, $p_i^2 = m^2,\ i = 1,2,3,4$



Planar diagram of the first type




$$
\begin{aligned}
&B_{PL,1}(a_1,\ldots,a_8;s,t,m^2;\epsilon) \\
&\quad = \int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(k^2-m^2)^{a_1}[(k+p_1)^2)]^{a_2}[(k+p_1+p_2)^2-m^2]^{a_3}} \\
&\quad \times \frac{[(k+p_1+p_2+p_3)^2]^{-a_8}}{[(l+p_1+p_2)^2-m^2)]^{a_4}[(l+p_1+p_2+p_3)^2]^{a_5}(l^2-m^2)^{a_6}[(k-l)^2]^{a_7}}
\end{aligned}
$$

$$
\begin{aligned}
B_{PL,1}(a_1,\ldots,a_8;s,t,m^2;\epsilon) \;&=\; \frac{\left(i\pi^{d/2}\right)^2 (-1)^a}{\prod_{j=2,4,5,6,7}\Gamma(a_j)\Gamma(4-a_{4567}-2\epsilon)(-s)^{a-4+2\epsilon}} \\
\times \frac{1}{(2\pi i)^6}\int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=1}^{5} \mathrm{d}z_j &\left(\frac{m^2}{-s}\right)^{z_1+z_5} \left(\frac{t}{s}\right)^{w} \frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}
\end{aligned}
$$

$$
\times \frac{\Gamma(4 - a_{13} - 2a_{28} - 2\epsilon + z_2 + z_3)\Gamma(a_{1238} - 2 + \epsilon + z_4 + z_5)\Gamma(a_7 + w - z_4)}{\Gamma(4 - a_{46} - 2a_{57} - 2\epsilon - 2w - 2z_1 - z_2 - z_3)}
$$
$$
\times \frac{\Gamma(a_{4567} - 2 + \epsilon + w + z_1 - z_4)\Gamma(a_8 - z_2 - z_3 - z_4)\Gamma(-w - z_2 - z_3 - z_4)}{\Gamma(4 - a_{1238} - 2\epsilon + w - z_4)\Gamma(a_8 - w - z_2 - z_3 - z_4)}
$$
$$
\times \frac{\Gamma(a_5 + w + z_2 + z_3 + z_4)\Gamma(2 - a_{567} - \epsilon - w - z_1 - z_2)}{\Gamma(4 - a_{13} - 2a_{28} - 2\epsilon + z_2 + z_3 - 2z_5)}
$$
$$
\times \Gamma(2 - a_{457} - \epsilon - w - z_1 - z_3)\Gamma(2 - a_{128} - \epsilon + z_2 - z_5)\Gamma(-z_1)\Gamma(-z_5)
$$
$$
\times \Gamma(2 - a_{238} - \epsilon + z_3 - z_5)\Gamma(4 - a_{46} - 2a_{57} - 2\epsilon - 2w - z_2 - z_3)\,.
$$

Analytical evaluation of a master double box [V.A. Smirnov, Phys. Lett. **B524** (2002) 129]:

$$
B_{PL,1}(1, \ldots, 1, 0; s, t, m^2; \epsilon)
= -\frac{\left(i\pi^{d/2} \mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^2 x^2}{s^2(-t)^{1+2\epsilon}} \left[ \frac{b_2(x)}{\epsilon^2} + \frac{b_1(x)}{\epsilon} + b_{01}(x) + b_{02}(x, y) + O(\epsilon) \right] ,
$$

where $x = 1/\sqrt{1 - 4m^2/s}$, $y = 1/\sqrt{1 - 4m^2/t}$, and

$$
\begin{aligned}
b_2(x) &= 2 \ln^2 \frac{1 - x}{1 + x} , \\
b_1(x) &= -8 \left[ \mathrm{Li}_3\left(\frac{1 - x}{2}\right) + \mathrm{Li}_3\left(\frac{1 + x}{2}\right) + \mathrm{Li}_3\left(\frac{-2x}{1 - x}\right) + \mathrm{Li}_3\left(\frac{2x}{1 + x}\right) \right] \\
&\quad + 4 \ln \frac{1 - x}{1 + x} \left[ \mathrm{Li}_2\left(\frac{1 - x}{2}\right) - \mathrm{Li}_2\left(\frac{-2x}{1 - x}\right) \right] \\
&\quad - (4/3) \ln^3(1 - x) + 4 \ln^2(1 - x) \ln(1 + x) \\
&\quad - 6 \ln(1 - x) \ln^2(1 + x) + (2/3) \ln^3(1 + x) \\
&\quad + 4 \ln 2 \left( \ln(1 - x) \ln(1 + x) + \ln^2(1 + x) \right) \\
&\quad - 2 \ln^2 2 \left( \ln(1 - x) + 3 \ln(1 + x) \right) \\
&\quad - (\pi^2/3)(4 \ln 2 - \ln(1 - x) - 3 \ln(1 + x)) + (8/3) \ln^3 2 + 14\zeta(3) ,
\end{aligned}
$$

The finite part, $b_0(x, y)$, includes polylogarithms, e.g.,

$$
\mathrm{Li}_3\left(\frac{-(1 - x)y}{1 + xy}\right) .
$$

Planar 2-loop box diagrams with one-loop insertion: R. Bonciani, A. Ferroglia, P. Mastrolia and E. Remiddi, Nucl. Phys. **B681** (2004) 261.

Double boxes with two reduced lines: J. Gluza, talk at LL'04.

A master double box with a numerator [G. Heinrich and V.A. Smirnov, to be published]:

$$B_{PL,1}(1,\ldots,1,-1;s,t,m^2;\epsilon) = -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^2 x^2}{s^2(m^2)^{2\epsilon}}\left[\frac{b_2(x)}{\epsilon^2}+\frac{b_1(x)}{\epsilon}+b_0(x,y)+O(\epsilon)\right] ,$$

where

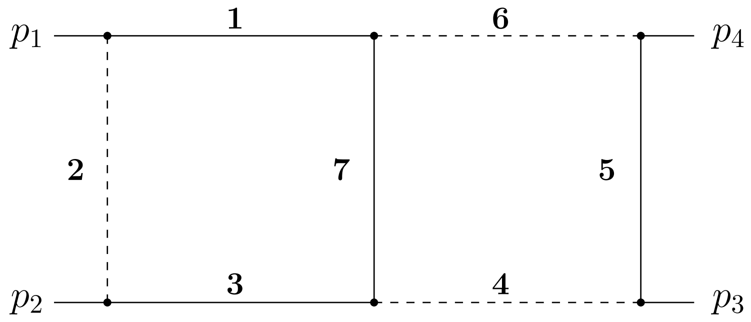
$$\begin{aligned}
b_2(x) &= \frac{3}{2}\ln^2\frac{1-x}{1+x} ,\\
b_1(x) &= \mathrm{Li}_3\left(\frac{(1-x)^2}{(1+x)^2}\right)-\left(2\ln\frac{x}{1+x}+2\ln 2-\ln\frac{-s}{m^2}\right)\mathrm{Li}_2\left(\frac{(1-x)^2}{(1+x)^2}\right)\\
&+2\ln\frac{1-x}{1+x}\mathrm{Li}_2\left(\frac{1-x}{2}\right)+\frac{1}{6}\ln^3(1-x)+\frac{3}{2}\ln^2(1-x)\ln(1+x)\\
&-\frac{5}{2}\ln(1-x)\ln^2(1+x)+\frac{5}{6}\ln^3(1+x)-2\ln 2\,\ln(1-x)\ln\frac{1-x}{1+x}\\
&-\frac{\pi^2}{6}\ln\frac{-s}{m^2}-2\ln^2\frac{1-x}{1+x}\,\ln\frac{-t}{m^2}-\frac{\pi^2}{6}\left(3\ln(1-x)-2\ln x-\ln(1+x)\right)\\
&+\ln^2 2\,\ln\frac{1-x}{1+x}+\frac{\pi^2}{3}\ln 2-\zeta(3) .
\end{aligned}$$

The finite part, $b_0(x,y)$ includes polylogarithms and HPL

$$\frac{x^2}{s^2}H\left(-1,0,0,1;\frac{1-x}{1+x}\right)$$

Numerical control of finite MB and numerical confirmation by the method of [T. Binoth and G. Heinrich, Nucl. Phys. **B585** (2000) 741]

Planar diagram of the second type




$$
\begin{aligned}
&B_{PL,2}(a_1,\ldots,a_8;s,t,m^2;\epsilon) \\
&\times = \frac{\left(i\pi^{d/2}\right)^2(-1)^a}{\prod_{j=2,4,5,6,7}\Gamma(a_j)\Gamma(4-a_{4567}-2\epsilon)(-s)^{a-4+2\epsilon}} \\
&\times \frac{1}{(2\pi i)^6}\int_{-i\infty}^{+i\infty}\prod_{j=1}^{6}\mathrm{d}z_j\left(\frac{m^2}{-s}\right)^{z_5+z_6}\left(\frac{t}{s}\right)^{z_1}\prod_{j=1}^{6}\Gamma(-z_j)\frac{\Gamma(a_2+z_1)\Gamma(a_4+z_2+z_4)}{\Gamma(a_3-z_2)\Gamma(a_1-z_3)} \\
&\times \frac{\Gamma(4-a_{445667}-2\epsilon-z_2-z_3-2z_4)\Gamma(a_6+z_3+z_4)}{\Gamma(4-a_{445667}-2\epsilon-z_2-z_3-2z_4-2z_5)\Gamma(6-a-3\epsilon-z_4-z_5)} \\
&\times \frac{\Gamma(8-a_{13}-2a_{245678}-4\epsilon-2z_1-z_2-z_3-2z_4-2z_5)}{\Gamma(8-a_{13}-2a_{245678}-4\epsilon-2z_1-z_2-z_3-2z_4-2z_5-2z_6)} \\
&\times \frac{\Gamma(2-a_{456}-\epsilon-z_4-z_5)\Gamma(2-a_{467}-\epsilon-z_2-z_3-z_4-z_5)}{\Gamma(a_{45678}-2+\epsilon+z_2+z_3+z_4+z_5)} \\
&\times \Gamma(a_{4567}+\epsilon-2+z_2+z_3+z_4+z_5)\Gamma(a_{45678}-2+\epsilon+z_1+z_2+z_3+z_4+z_5) \\
&\times \Gamma(4-a_{1245678}-2\epsilon-z_1-z_2-z_4-z_5-z_6) \\
&\times \Gamma(4-a_{2345678}-2\epsilon-z_1-z_3-z_4-z_5-z_6)\Gamma(a-4+2\epsilon+z_1+z_4+z_5+z_6)\,.
\end{aligned}
$$

A master planar double box of the second type [G. Heinrich and V.A. Smirnov, to be published]:

$$B_{PL,2}(1,\ldots,1,0;s,t,m^2;\epsilon)$$
$$= -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^2 x^2y^2}{s^2(-t)^{1+2\epsilon}}\left[\frac{b_2(x,y)}{\epsilon^2}+\frac{b_1(x,y)}{\epsilon}+b_0(s,t,m^2)+O(\epsilon)\right],$$

where

$$b_2(x) = \ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y},$$

$$b_1(x) = -2\ln\frac{1-y}{1+y}\left[\mathrm{Li}_2\left(\frac{1-x}{2}\right)-\mathrm{Li}_2\left(\frac{1+x}{2}\right)+\mathrm{Li}_2(x)-\mathrm{Li}_2(-x)\right]$$
$$+2\ln\frac{1-x}{1+x}\left[\mathrm{Li}_2\left(\frac{1-y}{2}\right)-\mathrm{Li}_2\left(\frac{1+y}{2}\right)+\mathrm{Li}_2(y)-\mathrm{Li}_2(-y)\right]$$
$$+\ln\frac{1-x}{1+x}\ln\frac{1-y}{1+y}\left[4\ln\frac{y}{x}+\ln\frac{1-x}{1-y}+\ln\frac{1+x}{1+y}\right].$$

$b_0(x)$ includes, e.g., $\mathrm{Li}_3\left(\frac{-(1+x)y}{1-xy}\right)$ and other polylogarithms as well as the following three two-parametric integrals of elementary functions:

$$\int_0^1\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\frac{\sqrt{x_1}\mathrm{Arsh}\left[(\sqrt{-t}\sqrt{x_1}\sqrt{1-x_2})/(2\sqrt{x_1+x_2-x_1x_2})\right]}{\sqrt{1-x_1}(4-sx_1)x_2\sqrt{(4-t)x_1(1-x_2)+4x_2}}$$
$$\times(\ln(-s)+2\ln x_1-\ln\left(-sx_1^2+(1-x_1)(4-sx_1)x_2\right),$$

$$\int_0^1\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\frac{\ln^2(x_1+x_2-x_1x_2)}{\sqrt{1-x_1}(4-sx_1)\sqrt{1-x_2}(4-tx_2)},$$

$$\int_0^1\int_0^1 \mathrm{d}x_1\mathrm{d}x_2\frac{\ln(x_1+x_2-x_1x_2)}{\sqrt{1-x_1}(4-sx_1)\sqrt{1-x_2}(4-tx_2)}$$
$$\times(2\ln 2-\ln(-s)+\ln(1-x_1)-\ln x_1+\ln(4-sx_1)+\ln(1-x_2))$$

Numerical control of MB integrals and numerical confirmation by the method of Binoth and Heinrich.

Functions in results:

HPL or 2dHPL depending on special combinations of $s, t$ and $m^2$, or a new class of functions?

U. Aglietti and R. Bonciani, hep-ph/0401193; U. Aglietti, R. Bonciani, G. Degrassi and A. Vicini, hep-ph/0404071: generalized HPL which are defined similarly to HPL, with other basic functions, in particular,

$$1/\sqrt{t(t+4)}$$

However, for example,

$$H(-r,-1;x) = \int_0^x \frac{\mathrm{d}t}{\sqrt{t(t+4)}}$$

equals

$$2\mathrm{Li}_2\left(z, \frac{\pi}{3}\right) + \frac{1}{2}\ln^2 z - \frac{\pi^2}{18},$$

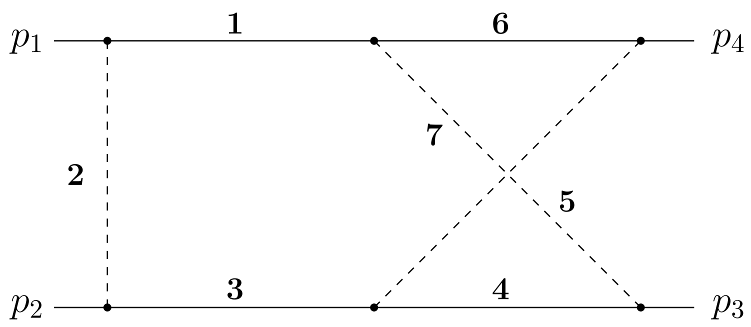
where

$$z = \frac{\sqrt{4+x} - \sqrt{x}}{\sqrt{4+x} - \sqrt{x}}$$

and $\mathrm{Li}_2(r,\theta) = \mathrm{Re}\left[\mathrm{Li}_2\left(re^{i\theta}\right)\right]$ is the dilogarithm of complex argument.

Shall we need 2dGHPL?

Non-planar diagram




Eightfold MB representation of the general non-planar double box

$$B_{NP}(a_1,\ldots,a_8;s,t,u,m^2;\epsilon) \;=\; \frac{\left(i\pi^{d/2}\right)^2(-1)^a}{\prod_{j=2,4,5,6,7}\Gamma(a_j)\Gamma(4-a_{4567}-2\epsilon)(-s)^{a-4+2\epsilon}}$$
$$\times\frac{1}{(2\pi i)^8}\int_{-i\infty}^{+i\infty}\prod_{j=1}^{8}\mathrm{d}z_j\left(\frac{m^2}{-s}\right)^{z_5+z_6}\left(\frac{t}{s}\right)^{z_7}\left(\frac{u}{s}\right)^{z_8}\prod_{j=1}^{7}\Gamma(-z_j)$$
$$\frac{\Gamma(a_5+z_2+z_4)\Gamma(a_7+z_3+z_4)\Gamma(4-a_{455677}-2\epsilon-z_2-z_3-2z_4)}{\Gamma(a_1-z_2)\Gamma(a_3-z_3)\Gamma(a_8-z_4)}$$
$$\times\frac{\Gamma(2-a_{567}-\epsilon-z_2-z_4-z_5)\Gamma(2-a_{457}-\epsilon-z_3-z_4-z_5)}{\Gamma(4-a_{455677}-2\epsilon-z_2-z_3-2z_4-2z_5)}$$
$$\times\frac{\Gamma(a_8+z_1-z_4+z_7)\Gamma(8-a_{13}-2a_{245678}-4\epsilon-z_2-z_3-2z_5-2z_7-2z_8)}{\Gamma(6-a-3\epsilon-z_5)}$$
$$\times\frac{\Gamma(-a_8-z_1+z_4-z_7-z_8)\Gamma(4-a_{2345678}-2\epsilon-z_2-z_5-z_6-z_7-z_8)}{\Gamma(8-a_{13}-2a_{245678}-4\epsilon-z_2-z_3-2z_5-2z_6-2z_7-2z_8)}$$
$$\times\frac{\Gamma(4-a_{1245678}-2\epsilon-z_3-z_5-z_6-z_7-z_8)\Gamma(a_{28}+z_1-z_4+z_7+z_8)}{\Gamma(a_{245678}-2+\epsilon+z_1+z_2+z_3+z_5+z_7+2z_8)}$$
$$\times\Gamma(a_{4567}+\epsilon-2+z_2+z_3+z_4+z_5+z_8)\Gamma(a-4+2\epsilon+z_5+z_6+z_7+z_8)$$
$$\times\Gamma(a_{245678}-2+\epsilon+z_1+z_2+z_3+z_5+2z_7+2z_8)\,.$$

It is natural to treat the Mandelstam variables not restricted by the physical condition $s+t+u=4m^2$.

A non-planar master planar double box: [G. Heinrich and V.A. Smirnov, to be completed and published]:

$$B_{NP}(1,\ldots,1,0;s,t,m^2;\epsilon)$$
$$= \frac{\left(i\pi^{d/2}\right)^2 xz}{stu} \ln\frac{1-x}{1+x} \ln\frac{1-z}{1+z} \frac{1}{\epsilon^2} + O\left(\frac{1}{\epsilon}\right) \; ,$$

where $z = 1/\sqrt{1-4m^2/u}$.

Two-parametric integrals already in the $1/\epsilon$ part.

Numerical control of MB integrals.

Hopefully, the problem of evaluation of massive on-shell double boxes will be analytically solved. At least one can use MB representation for the evaluation of master integrals in this and various other problems.

Advantages of the method based on MB representation:

- The MB representation can be derived for a general Feynman integral corresponding to a given graph (i.e. with general integer power of the propagators).
- Resolution of singularities in $\epsilon$ is much simpler than in alpha and Feynman parametric integrals.
- After the resolution of singularities in $\epsilon$ one can always switch to numerical evaluation, at least in order to check analytical results. The convergence along imaginary axis is always perfect (it is sufficient to take integration between $-10i$ and $+10i$ instead of $(-i\infty, +i\infty)$ to have accuracy of 10 digits.
- Automation of calculations based on MB representation looks promising.