DIFFERENTIAL EQUATIONS

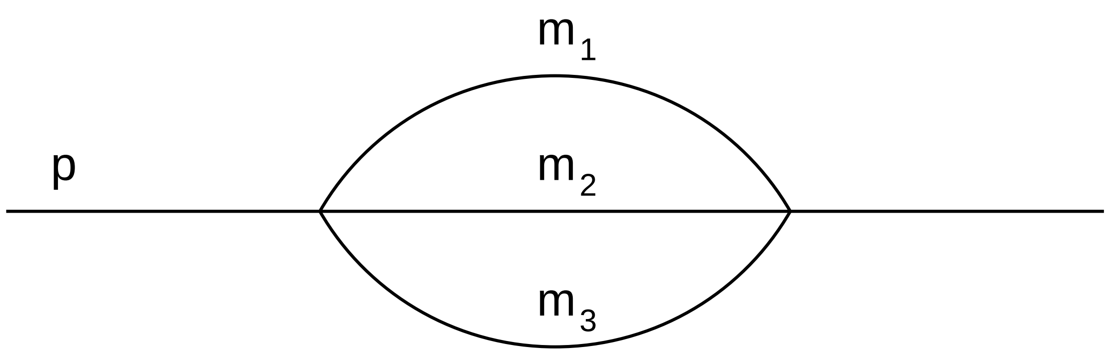
for the EQUAL MASS 2-LOOP SUNRISE

at ARBITRARY MOMENTUM TRANSFER

Presented by E. Remiddi
(from S. Laporta & E. Remiddi, in preparation)

Zinnowitz 27 April 2004

- the differential equations
- $d = 4 \quad \Longleftarrow \quad d = 2$
- expansion of the equations at $d = 2$; Euler's formal solution
- solving the homogeneous equation - the expansions
- solving the homogeneous equation - interpolation & matching
- integration constants & expansions at the sensible points
- Conclusion: the solution in closed analytic form.



The 2-loop sunrise graph and its 2 M.I.s at $m_i = 1$

$$S(d,p^2) = \frac{1}{\Gamma^2\left(3-\frac{d}{2}\right)} \int \frac{d^d k_1}{4\pi^{\frac{d}{2}}} \int \frac{d^d k_2}{4\pi^{\frac{d}{2}}} \frac{1}{(k_1^2+1)(k_2^2+1)[(p-k_1-k_2)^2+1]}$$

$$S_1(d,p^2) = \frac{1}{\Gamma^2\left(3-\frac{d}{2}\right)} \int \frac{d^d k_1}{4\pi^{\frac{d}{2}}} \int \frac{d^d k_2}{4\pi^{\frac{d}{2}}} \frac{1}{(k_1^2+1)^2(k_2^2+1)[(p-k_1-k_2)^2+1]}$$

the linear system of 1st order diff-eq.s for the 2 M.I.s, $p^2 = z = -u$,

$$\begin{aligned} z\frac{d}{dz}S(d,z) &= (d-3)S(d,z) + 3S_1(d,z) \ , \\ z(z+1)(z+9)\frac{d}{dz}S_1(d,z) &= \frac{1}{2}(d-3)(8-3d)(z+3)S(d,z) \\ &+ \frac{1}{2}\left[(d-4)z^2 + 10(2-d)z + 9(8-3d)\right]S_1(d,z) \\ &+ \frac{1}{2}\frac{z}{(d-4)^2} \end{aligned}$$

the equivalent 2nd order equation for $S(d,z)$

$$\begin{aligned} & z(z+1)(z+9) \quad \tfrac{d^2}{dz^2}S(d,z) \\ & +\frac{1}{2}\left[(12-3d)z^2 + 10(6-d)z + 9d\right] \quad \tfrac{d}{dz}S(d,z) \\ & +\frac{1}{2}(d-3)\left[(d-4)z - d - 4\right] \quad S(d,z) \quad = \frac{3}{2}\frac{1}{(d-4)^2} \end{aligned}$$

and $$S_1(d,z) = \frac{1}{3}\left[-(d-3) + z\frac{d}{dz}\right]S(d,z)$$

any integral in d-dimensions can be expressed as a combination of integrals in $d-2$ dimensions (O. Tarasov);
for the 2-loop sunrise

$$S(2+d,z) = \frac{1}{12(d-1)(3d-2)(3d-4)} \times \left\{ 2(d-4)^2(z+1)(z+9)\left[1+(z-3)\frac{d}{dz}\right] S(d,z) + (d-2)(d-4)^2(87+22z-z^2) S(d,z) - \frac{36}{(d-2)^2} + \frac{3z-63}{(d-2)} \right\}$$

if $d = 4 + \eta$, for $\eta \to 0$

$$
\begin{aligned}
S(2+\eta, z) &= S^{(0)}(2,z) + \eta\, S^{(1)}(2,z) + ... \\
S(4+\eta, z) &= \frac{1}{\eta^2} S^{(-2)}(4,z) + \frac{1}{\eta} S^{(-1)}(4,z) + S^{(0)}(4,z) + \eta S^{(1)}(4,z) + ...
\end{aligned}
$$

the $S^{(k)}(4,z)$ can be *algebraically* obtained through the $2 + d \to d$ identity from the $S^{(k)}(2,z)$

$$
\begin{aligned}
S^{(-2)}(4,z) &= -\frac{3}{8} && \textit{for free,}\quad S^{(-2)}(2,z) = 0 \\
S^{(-1)}(4,z) &= \frac{9}{16} + \frac{z}{32} && \textit{for free,}\quad S^{(-1)}(2,z) = 0 \\
S^{(0)}(4,z) &= \frac{1}{12}(z+1)(z+9)\left(1 + (z-3)\frac{d}{dz}\right) S^{(0)}(2,z) \\
&\quad - \frac{1}{128}(72 + 13z)\ . \\
S^{(1)}(4,z) &= ......
\end{aligned}
$$

the $S^{(k)}(2,z)$, $k=0,1,..$ satisfy a system of chained diff-eq.s, obtained by the expansion in $(d-2)$ of the diff-eq. for $S(d,z)$:

$$\Big\{ \frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{d}{dz} + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right] \Big\} S^{(k)}(2,z) = N^{(k)}(2,z)$$

$$\begin{aligned}
N^{(0)}(2,z) &= \frac{1}{24z} - \frac{3}{64(z+1)} + \frac{1}{192(z+9)} = \frac{3}{8z(z+1)(z+9)} \\
N^{(1)}(2,z) &= \left(-\frac{1}{2z} + \frac{1}{z+1} + \frac{1}{z+9}\right)\frac{dS^{(0)}(2,z)}{dz} \\
&+ \left(\frac{5}{18z} - \frac{1}{8(z+1)} - \frac{11}{72(z+9)}\right) S^{(0)}(2,z) \\
&+ \frac{1}{24z} - \frac{3}{64(z+1)} + \frac{1}{192(z+9)} \\
N^{(2)}(2,z) &= .....
\end{aligned}$$

the associated homogeneous equation is the same for any k

$$\left\{ \frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{d}{dz} + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right] \right\}\Psi(z) = 0$$

If $\Psi_1(z), \Psi_2(z)$ are two linearly independent solutions of the homogeneous equation, the solutions of the inhomogeneous equations are given by the variation-of-constants method of Euler

$$\begin{aligned} S^{(k)}(2,z) = & \quad \Psi_1(z)\left(\Psi_1^{(k)} - \int_0^z \frac{dw}{W(w)}\Psi_2(w)N^{(k)}(2,w)\right) \\ & + \quad \Psi_2(z)\left(\Psi_2^{(k)} + \int_0^z \frac{dw}{W(w)}\Psi_1(w)N^{(k)}(2,w)\right) \end{aligned}$$

where $W(z)$ is the Wronskian,
$\Psi_1^{(k)}, \Psi_2^{(k)}$ two integration constants

The formal solution becomes a substancial explicit analytic formula when:

- the homogeneous solutions $\Psi_1(z), \Psi_2(z)$ are found;
- the 2 constants $\Psi_1^{(k)}, \Psi_2^{(k)}$ are fixed (for each k);
- the Wronskian $W(z)$ is evaluated.

The Wronskian is not a problem (Liouville's formula);

$$W(z) = \Psi_1(z)\frac{d\Psi_2(z)}{dz} - \Psi_2(z)\frac{d\Psi_1(z)}{dz}$$

satisfies the equation

$$\frac{d}{dz}W(z) = -\left(\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right)W(z)$$

which gives $$W(z) = \frac{C}{z(z+1)(z+9)}$$

$W(z)$ is known (up to the multiplicative but irrelevant constant C)

Main problem: finding the two solutions

$$\Psi_1(z), \Psi_2(z)$$

of the homogeneous equation.

Method:

- find the expansions at the singular points of the homogeneous equation;
- find "interpolating solutions" for joining smoothly the expansions

by inspection, the singular points of the homogeneous equation are

$$z = 0, -1, -9, \infty$$

at each point, there is a regular solution and another solution with a logarithmic singularity at that point

- around $z = 0$

there are 2 independent solutions, which can be written as

$$\begin{aligned}\Psi_1^{(0)}(z) &= \psi_1^{(0)}(z) \\ \Psi_2^{(0)}(z) &= \ln z\, \psi_1^{(0)}(z) + \psi_2^{(0)}(z) \ ;\end{aligned}$$

with the initial conditions $\quad \psi_1^{(0)}(0) = 1 \ , \quad \psi_2^{(0)}(0) = 0 \ ,$

the hde (homogeneous differential equation) gives (recursively) the expansion

$$\begin{aligned}\psi_1^{(0)}(z) &= 1 - \frac{1}{3}z + \frac{5}{27}z^2 + ... \\ \psi_2^{(0)}(z) &= -\frac{4}{9}z + \frac{26}{81}z^2 + ...\end{aligned}$$

the expansions converge for $|z| \leq 1$ (next singularity at $z = -1$);
for $1 > z > 0$ the $\Psi_i^{(0)}(z)$ are real;
for $0 > z > -1$, if $z = -(u + i\epsilon)$, $\ln z = \ln u - i\pi$,
so that $\Psi_2^{(0)}(z)$ develops an imaginary part.

- around $z = -1$

the 2 independent solutions can be written as

$$\begin{aligned} \Psi_1^{(1)}(z) &= \psi_1^{(1)}(z) \\ \Psi_2^{(1)}(z) &= \ln(z+1)\ \psi_1^{(1)}(z) + \psi_2^{(1)}(z)\ ; \end{aligned}$$

with suitable initial conditions the hde gives

$$\begin{aligned} \psi_1^{(1)}(z) &= 1 + \frac{1}{4}(z+1) + \frac{5}{32}(z+1)^2 + ... \\ \psi_2^{(1)}(z) &= +\frac{3}{8}(z+1) + \frac{33}{128}(z+1)^2 + ... \end{aligned}$$

the expansions converge for $|z+1| \leq 1$ (next singularity at $z = 0$);
for $0 > z > -1$ the $\Psi_i^{(1)}(z)$ are real ;
for $-1 > z > -2$ and $z = -(u + i\epsilon)$, $\ln(z+1) = \ln(u-1) - i\pi$,
so that $\Psi_2^{(1)}(z)$ develops an imaginary part.

- around $z = -9$

the 2 independent solutions are

$$\begin{aligned} \Psi_1^{(9)}(z) &= \psi_1^{(9)}(z) \\ \Psi_2^{(9)}(z) &= \ln(z+9)\ \psi_1^{(9)}(z) + \psi_2^{(9)}(z)\ ; \end{aligned}$$

with

$$\begin{aligned} \psi_1^{(9)}(z) &= 1 + \frac{1}{12}(z+9) + \frac{7}{864}(z+9)^2 + ... \\ \psi_2^{(9)}(z) &= +\frac{5}{72}(z+9) + \frac{97}{10368}(z+9)^2 + ... \end{aligned}$$

the expansions converge for $|z+9| \leq 8$ (next singularity at $z = -1$);
for $-1 > z > -9$ the $\Psi_i^{(9)}(z)$ are real ;
for $-9 > z > -17$ and $z = -(u + i\epsilon)$, $\ln(z+9) = \ln(u-9) - i\pi$,
so that $\Psi_2^{(9)}(z)$ develops an imaginary part.

- for large z, i.e. $|z| > 9$, use $y = 1/z$;

the solutions then are

$$\begin{aligned} \Psi_1^{(\infty)}(z) &= y\psi_1^{(\infty)}(y) \\ \Psi_2^{(\infty)}(z) &= y\left(\ln y\ \psi_1^{(\infty)}(y) + \psi_2^{(\infty)}(y)\right) \ ; \end{aligned}$$

with

$$\begin{aligned} \psi_1^{(\infty)}(y) &= 1 - 3y + 15y^2 + ... \\ \psi_2^{(\infty)}(y) &= -4y + 26y^2 + ... \end{aligned}$$

the expansions converge for $|y| < 1/9,\ |z| > 9$;
for $z > 0$ the $\Psi_i^{(\infty)}(z)$ are real ;
for $-9 > z > -\infty$, and $z = -(u + i\epsilon)$, $\ln y = -\ln u + i\pi$
so that $\Psi_2^{(\infty)}(z)$ develops an imaginary part.

- Next problem: build 2 solutions in the whole $-\infty < z < \infty$ range by joining smoothly the previous local solutions.

- Hint: at $d = 2$, for $-z = u \geq 9$ the imaginary part of $S(d, z)$ (up to a multiplicative constant is (Cutkosky-Veltman rule)

$$J(u) = \int_4^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{R_4(u,b)}}$$

where $$R_4(u,b) = b(b-4)(b-(\sqrt{u}-1)^2)(b-(\sqrt{u}+1)^2) \; ;$$

$J(u)$ is a solution of the homogenous equation
(as the inhomogeneous part has no imaginary part).

- The explicit verification leads to a family of similar solutions for the various ranges of u; they can all be expressed as `complete elliptic integral of the first kind` of suitable (non trivial) argument
`[ S. Groote, A.A. Pivovarov hep-ph/0003115v2 ]`

- `if` $\quad 0 < u < 1,$ $\quad$ `so that` $\quad 0 < (1-\sqrt{u})^2 < (1+\sqrt{u})^2 < 4$

the two interpolating solutions can be choosen as

$$J_1^{(0,1)}(u) = \int_0^{(1-\sqrt{u})^2} \frac{db}{\sqrt{-R_4(u,b)}} = \int_{(\sqrt{u}+1)^2}^{4} \frac{db}{\sqrt{-R_4(u,b)}}$$

$$= \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K\left(\frac{(1-\sqrt{u})^3(3+\sqrt{u})}{(1+\sqrt{u})^3(3-\sqrt{u})}\right)$$

$$J_2^{(0,1)}(u) = \int_{(\sqrt{u}-1)^2}^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{R_4(u,b)}}$$

$$= \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K\left(\frac{16\sqrt{u}}{(1+\sqrt{u})^3(3-\sqrt{u})}\right)$$

• `if` $1 < u < 9,$ `so that` $0 < (\sqrt{u}-1)^2 < 4 < (\sqrt{u}+1)^2$ `choose`

$$J_1^{(1,9)}(u) = \int_0^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{-R_4(u,b)}} = \int_4^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{-R_4(u,b)}}$$

$$= \frac{1}{\sqrt{16\sqrt{u}}} K\left(\frac{(\sqrt{u}-1)^3(3+\sqrt{u})}{16\sqrt{u}}\right)$$

$$J_2^{(1,9)}(u) = \int_{(\sqrt{u}-1)^2}^{4} \frac{db}{\sqrt{R_4(u,b)}}$$

$$= \frac{1}{\sqrt{16\sqrt{u}}} K\left(\frac{(\sqrt{u}+1)^3(3-\sqrt{u})}{16\sqrt{u}}\right)$$

- `if` $9 < u < \infty,$ `so that` $0 < 4 < (\sqrt{u}-1)^2 < (\sqrt{u}+1)^2$ `choose`

$$J_1^{(9,\infty)}(u) = \int_0^4 \frac{db}{\sqrt{-R_4(u,b)}} = \int_{(\sqrt{u}-1)^2}^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{-R_4(u,b)}}$$

$$= \frac{1}{\sqrt{(\sqrt{u}-1)^3(\sqrt{u}+3)}} K\left(\frac{16\sqrt{u}}{(\sqrt{u}-1)^3(\sqrt{u}+3)}\right)$$

$$J_2^{(9,\infty)}(u) = \int_4^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{R_4(u,b)}}$$

$$= \frac{1}{\sqrt{(\sqrt{u}-1)^3(\sqrt{u}+3)}} K\left(\frac{(\sqrt{u}+1)^3(\sqrt{u}-3)}{(\sqrt{u}-1)^3(\sqrt{u}+3)}\right)$$

transformations of the argument

the singular points of the homogeneous equation

$$\begin{aligned} z &= 0, -1, -9, \infty \\ u &= 0, +1, +9, \infty \end{aligned}$$

are mapped into themselves by the conformal transformations

$$u \to \frac{9}{u}, \quad u \to \frac{9-u}{1-u}, \quad u \to 9\frac{1-u}{9-u}$$

correspondingly, if $J(u)$ is a solution, one finds that

$$\frac{1}{u} J\left(\frac{9}{u}\right), \quad \frac{1}{1-u} J\left(\frac{9-u}{1-u}\right), \quad \frac{1}{9-u} J\left(9\frac{1-u}{9-u}\right)$$

are also solutions – hence linear combinations of the "standard" $J_i(u)$.

$$\text{if} \quad 0 < u < 1 \qquad J_1^{(9,\infty)}\left(\frac{9}{u}\right) = u\frac{\sqrt{3}}{9}J_2^{(0,1)}(u)$$

$$J_2^{(9,\infty)}\left(\frac{9}{u}\right) = u\frac{\sqrt{3}}{3}J_1^{(0,1)}(u)$$

$$\text{from which} \quad \frac{J_2^{(9,\infty)}\left(\frac{9}{u}\right)}{J_1^{(9,\infty)}\left(\frac{9}{u}\right)} = 3\frac{J_1^{(0,1)}(u)}{J_2^{(0,1)}(u)}$$

$$\text{or, in terms of K's} \quad \frac{K(1-a)}{K(a)} = 3\frac{K(1-b)}{K(b)}$$

the relation between the arguments of the $K's$

$$a = a\left(v = \frac{9}{u}\right) = \frac{16\sqrt{v}}{(\sqrt{v}-1)^3(\sqrt{v}+3)}$$

$$b = b(u) = \frac{16\sqrt{u}}{(1+\sqrt{u})^3(3-\sqrt{u})}$$

is the parametric form of a *modular equation* of degree 3

$$\text{if} \quad 9 < u < \infty \qquad J_1^{(9,\infty)}\left(9\frac{u-1}{u-9}\right) = (u-9)\frac{\sqrt{3}}{18}J_2^{(9,\infty)}(u)$$

$$J_2^{(9,\infty)}\left(9\frac{u-1}{u-9}\right) = (u-9)\frac{\sqrt{3}}{12}J_1^{(9,\infty)}(u)$$

$$\text{from which} \quad \frac{J_2^{(9,\infty)}\left(9\frac{u-1}{u-9}\right)}{J_1^{(9,\infty)}\left(9\frac{u-1}{u-9}\right)} = \frac{3}{2}\frac{J_1^{(9,\infty)}(u)}{J_2^{(9,\infty)}(u)}$$

$$\text{or, in terms of K's} \quad \frac{K(1-a)}{K(a)} = \frac{3}{2}\frac{K(1-b)}{K(b)}$$

the relation between the arguments of the $K's$

$$a = a\left(v = 9\frac{u-1}{u-9}\right) = \frac{16\sqrt{v}}{(\sqrt{v}-1)^3(\sqrt{v}+3)}$$

$$b = b(u) = \frac{(\sqrt{u}+1)^3(\sqrt{u}-3)}{(1+\sqrt{u})^3(3-\sqrt{u})}$$

is the parametric form of a *modular equation* of degree 3/2 [?]

The evaluation of the $\Psi_i^{(u_j)}(z)$ and the $J_i^{(u_j,u_k)}(u)$ at the singular points $u_j = (0, 1, 9, \infty)$ is almost elementary; that is the key for constructing the solutions of the homogeneous equation.

- for $\infty > z > 0$ define

$$\begin{aligned}\Psi_1(z) &= \Psi_1^{(0)}(z)\\ \Psi_2(z) &= \Psi_2^{(0)}(z)\end{aligned}$$

The continuation to the timelike region $-z = u > 0$ is performed with the usual $u + i\epsilon$ prescription; for $z \to 0^-$ (or $u \to 0^+$)

$$\begin{aligned}\lim_{z\to 0^-} \Psi_1(z - i\epsilon) &= 1\\ \lim_{z\to 0^-} \Psi_2(z - i\epsilon) &= \ln u - i\pi\end{aligned}$$

For the $J_i^{(0,1)}(u)$ one finds in the same point

$$\begin{aligned}\lim_{u\to 0^+} J_1^{(0,1)}(u) &= \frac{1}{\sqrt{3}}\left(-\frac{1}{2}\ln u + \ln 3\right)\\ \lim_{u\to 0^+} J_2^{(0,1)}(u) &= \frac{\pi}{\sqrt{3}}\end{aligned}$$

- the match at $u = 0^+$ gives for the whole range $0 > z > -1$, i.e. $0 < u < 1$

$$\begin{aligned} \Psi_1(z - i\epsilon) &= \frac{\sqrt{3}}{\pi} J_2^{(0,1)}(u) \\ \Psi_2(z - i\epsilon) &= -2\sqrt{3} J_1^{(0,1)}(u) + \frac{\sqrt{3}}{\pi}(2\ln 3 - i\pi) J_2^{(0,1)}(u) \end{aligned}$$

- one can now move to $u = 1^-$; one finds

$$\begin{aligned} \lim_{u \to 1^-} J_1^{(0,1)}(u) &= \frac{\pi}{4} \\ \lim_{u \to 1^-} J_2^{(0,1)}(u) &= -\frac{3}{4}\ln(1-u) + \frac{9}{4}\ln 2 \end{aligned}$$

- by comparison with the values at $z = -1$ of the $\Psi_i^{(1)}(z)$ one obtains

$$\begin{aligned} J_1^{(0,1)}(u) &= \frac{\pi}{4}\Psi_1^{(1)}(z) \\ J_2^{(0,1)}(u) &= \frac{9}{4}\ln 2\ \Psi_1^{(1)}(z) - \frac{3}{4}\Psi_2^{(1)}(z) \end{aligned}$$

- by replacing the $J_i^{(0,1)}(u)$ with the $\Psi_i^{(1)}(z)$ around $z=-1$ the $\Psi_i(z)$ become

$$\begin{aligned}
\Psi_1(z-i\epsilon) &= \frac{9\sqrt{3}}{4\pi}\ln 2\ \Psi_1^{(1)}(z-i\epsilon) - \frac{3\sqrt{3}}{4\pi}\Psi_2^{(1)}(z-i\epsilon) \\
\Psi_2(z-i\epsilon) &= \frac{\sqrt{3}}{4}\left(\frac{18}{\pi}\ln 2\ln 3 - 2\pi - i9\ln 2\right)\Psi_1^{(1)}(z-i\epsilon) \\
&+ \frac{3\sqrt{3}}{4\pi}(-2\ln 3 + i\pi)\Psi_2^{(1)}(z-i\epsilon)
\end{aligned}$$

- move across $z=-u=-1$ from $z=-1^+, u=1^-$ to $z=-1^-, u=1^+$;

$$\lim_{z\to -1}\Psi_1^{(1)}(z-i\epsilon) = 1\ , \qquad \lim_{z\to -1}\Psi_2^{(1)}(z-i\epsilon) = \ln(u-1) - i\pi$$

- $u=1^+$ is in the range $1<u<9$ of the $J_i^{(1,9)}(u)$; at $u=1^+$

$$\lim_{u\to 1^+} J_1^{(1,9)}(u) = \frac{\pi}{4}\ , \qquad \lim_{u\to 1^+} J_2^{(1,9)}(u) = -\frac{3}{4}\ln(u-1) - \frac{9}{4}\ln 2$$

- in the range $-1 > z > -9$:

express the $\Psi_i^{(1)}(z - i\epsilon)$, and therefore also the $\Psi_i(z - i\epsilon)$ in terms of the $J_i^{(1,9)}(u)$,
evaluate the $J_i^{(1,9)}(u)$ at $u = 9^-$,
express the $J_i^{(1,9)}(u)$ in terms of the $\Psi_i^{(9)}(z)$
replace the $J_i^{(1,9)}(u)$ in the $\Psi_i(z - i\epsilon)$ with the $\Psi_i^{(9)}(z)$
.... and so on till $z = -\infty$, $u = \infty$

- the continuation of the two $\Psi_i(z)$ to the whole $z = -u$ range is so obtained:

– the constant in the Wronskian is fixed as $C = 9$;
– the knowledge of behaviours & expansions at the singular points allows the construction of a (fast & precise) numerical routine.

• now fixing the integration constants;
back to the (sofar formal) solution, zeroth order in $(d-2)$ (for simplicity); $N^{(0)}(2,w)/W(w) = 1/24$ and

$$\begin{aligned} S^{(0)}(2,z) &= \Psi_1(z)\left(\Psi_1^{(0)} - \frac{1}{24}\int_0^z dw\Psi_2(w)\right) \\ &+ \Psi_2(z)\left(\Psi_2^{(0)} + \frac{1}{24}\int_0^z dw\Psi_1(w)\right) \end{aligned}$$

in the interval $0 > z > -9$ the solution $S^{(0)}(2,z)$ is real, while $\Psi_i(z), \Psi_j(w)$ in Euler's formula can develop an imaginary part;
$\Longrightarrow$ those (qualitative) constraints fix (quantitatively) the integration constants $\Psi_i^{(0)}$

- `if` $0 > z > -1$ or $0 < u < 1$, `then`

$$\mathrm{Im} S^{(0)}(2,z) = -\sqrt{3}\Psi_2^{(0)} J_2^{(0,1)}(u) = 0$$

$\Longrightarrow$ $\Psi_2^{(0)} = 0$ `in fact` $\Psi_2^{(k)} = 0$ `for any` k

- `if` $-1 > z > -9$ or $1 < u < 9$, `then`

$$\mathrm{Im} S^{(0)}(2,z) = \frac{3}{\pi} \left(\sqrt{3}\Psi_1^{(0)} - \frac{1}{4}\int_0^1 dv\ J_1^{(0,1)}(v) \right) J_1^{(1,9)}(u) = 0$$

$$\Longrightarrow \quad \Psi_1^{(0)} = \frac{\sqrt{3}}{12}\int_0^1 dv\ J_1^{(0,1)}(v) = \frac{\sqrt{3}}{12}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) .$$

$S^{(0)}(2,z)$ is now analytically known.

- at $z = 0$ value & expansions

$$
\begin{aligned}
S^{(0)}(2,0) &= \Psi_1^{(0)} = \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
S^{(0)}(2,z) &\simeq \Psi_1^{(0)}\ \psi_1^{(0)}(z) \\
&\quad + \frac{1}{24} z - \frac{23}{864} z^2 + \ldots.\ , \\
S^{(0)}(4,z) &\simeq \Psi_1^{(0)} \left(\frac{3}{2} + \frac{1}{3} z - \frac{1}{27} z^2 + \ldots. \right) \\
&\quad - \frac{21}{32} - \frac{3}{128} z + \frac{11}{1728} z^2 + \ldots.
\end{aligned}
$$

- at $z=-1,\ u=1$ (mass shell–pseudothreshold) value & expansions

$$
\begin{aligned}
S^{(0)}(2,-1) &= \frac{1}{16}\int_0^1 dv\ J_2^{(0,1)}(v) = \frac{1}{64}\pi^2 \\
S^{(0)}(2,z) &\simeq \frac{\pi^2}{64}\psi_1^{(1)}(z) \\
&\quad -\frac{3}{64}(z+1) - \frac{3}{128}(z+1)^2 + ... \\
S^{(0)}(4,z) &\simeq \frac{\pi^2}{64}\left(-\frac{1}{2}(z+1)^2 - \frac{5}{8}(z+1)^3 + ...\right) \\
&\quad -\frac{59}{128} + \frac{3}{128}(z+1) + \frac{5}{64}(z+1)^2 + \frac{37}{384}(z+1)^3 + ...
\end{aligned}
$$

- at $z=-9,\ u=9$ (pseudothreshold) value & expansions

$$
\begin{aligned}
S^{(0)}(2,z) &\simeq s_0^{(9)}\left[\ln(z+9)\ \psi_1^{(9)}(z)+\psi_2^{(9)}(z)\right]+t_0^{(9)}\ \psi_1^{(9)}(z) \\
&\quad +\frac{1}{192}(z+9)+\frac{5}{6912}(z+9)^2+....
\end{aligned}
$$

$$
\begin{aligned}
S^{(0)}(4,z) &\simeq \left(s_0^{(9)}\ln(z+9)+t_0^{(9)}\right)\left(\frac{1}{54}(z+9)^2+\frac{1}{648}(z+9)^3+...\right) \\
&\quad +s_0^{(9)}\left(8-\frac{4}{9}(z+9)-\frac{4}{9}(z+9)^2+\frac{7}{5832}(z+9)^3+...\right) \\
&\quad +\frac{45}{128}-\frac{23}{384}(z+9)-\frac{1}{1728}(z+9)^2+\frac{1}{10368}(z+9)^3+...
\end{aligned}
$$

$$
\begin{aligned}
s_0^{(9)} &= -\frac{\sqrt{3}}{48}\pi \\
t_0^{(9)} &= -\frac{\sqrt{3}}{48}\left[-\pi\ln(72)+5\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]
\end{aligned}
$$

- when $u > 9$ (above threshold)

$$\mathrm{Im} S^{(0)}(2,-u) = \frac{1}{4}\pi J_2^{(9,\infty)}(u)$$

by recalling the id.'s for transformation of the arguments

$$\begin{aligned} S^{(0)}(2,0) &= \frac{\sqrt{3}}{12}\int_0^1 dv J_1^{(0,1)}(v) = \frac{1}{4}\int_9^\infty \frac{dv}{v} J_2^{(9,\infty)}(v) \\ S^{(0)}(2,-1) &= \frac{1}{16}\int_0^1 dv J_2^{(0,1)}(v) = \frac{1}{4}\int_9^\infty \frac{dv}{v-1} J_2^{(9,\infty)}(v) \end{aligned}$$

according to the dispersion relation for $S^{(0)}(2,z)$

- for large positive z (u spacelike)

$$
\begin{aligned}
S^{(0)}(2,z) &= \frac{3}{16z}\left[\ln^2 z\ \psi_1^{(\infty)}\left(\frac{1}{z}\right) - 2\ln z\ \psi_2^{(\infty)}\left(\frac{1}{z}\right)\right. \\
&\quad \left. + \frac{2}{z} - \frac{1}{2z^2} + ...\right]
\end{aligned}
$$

$$
\begin{aligned}
S^{(0)}(4,z) &= \ln^2 z\left(\frac{3}{32} + \frac{3}{16z} - \frac{3}{16z^2} + ...\right) \\
&\quad + \ln z\left(\frac{1}{32}z + \frac{9}{32z} + \frac{3}{8z^2} + ...\right) \\
&\quad - \frac{13}{128}z - \frac{15}{32} - \frac{3}{64z} + \frac{29}{64z^2} + ...
\end{aligned}
$$

Conclusion

$S^{(0)}(2, z)$ **is known in closed analytic form**

- the behaviours at all the sensible points $u = (0, 1, 9, \infty)$ are known;
- the expansions needed for the fast & precise numerical evaluation can be immediately obtained from the differential equation up to any required order