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An Improved Splitting Function for Small-x Evolution

Matching together GLAP and BFKL

G. Altarelli CERN Based on G.A., R. Ball, S.Forte hep-ph/9911273 (NPB <u>575</u>,313) hep-ph/0001157 (lectures) hep-ph/0011270 (NPB <u>599</u>,383) hep-ph/0104246

More specifically on hep-ph/0109178 (NPB <u>621</u>,359) and on hep-ph/0306156 (NPB <u>674</u>,459), hep-ph/0310016

Related work (same physics, similar conclusion, different techniques): Ciafaloni, Colferai, Salam, Stasto [see also Thorne] Our goal is to construct a relatively simple, closed form, improved anomalous dimension $\gamma_l(\alpha,N)$ (or splitting function $P_l(\alpha,x)$)

 $P_{I}(\alpha, x)$ should

- reduce to perturbative results at large x
- contain BFKL corrections at small x
- include running coupling effects
- be sufficiently simple to be included in fitting codes

and of course

closely reflect the trend of the data



Recall: $\gamma(N) = \int_0^1 x^N P(x) dx$ $P(x) = \frac{1}{x} (\ln 1/x)^n \qquad \longrightarrow \qquad \gamma(N) = \frac{n!}{N^{n+1}}$

At 1-loop:

$$\alpha \cdot \gamma_{1l}(N) = \alpha \cdot \left[\frac{1}{N} - A(N)\right]$$

This corresponds to the "double scaling" behavior at small x:

$$G(\xi, t) \sim \exp\left[\sqrt{\frac{4n_C}{\pi\beta_0}} \cdot \xi \cdot \frac{\log Q^2 / \Lambda^2}{\log \mu^2 / \Lambda^2}\right] \qquad \beta(\alpha) = -\beta_0 \alpha^2 + \dots$$

A. De Rujula et al '74/Ball, Forte

Amazingly supported by the data

In principle the BFKL approach provides a tool to control $(\alpha/N)^n$ corrections to $\gamma(N, \alpha)$, that is $1/x(\alpha \log 1/x)^n$ to splitting functions. Define t- Mellin transf.:

$$G(\xi, M) = \int_{-\infty}^{+\infty} e^{-Mt} G(\xi, t) dt$$

with inverse:

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{Mt} G(\xi, M) \frac{dM}{2\pi i}$$

ξ-evolution eq.n (BFKL) [at fixed α]:

$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha)G(\xi, M)$$

with $\chi(M, \alpha) = \alpha \cdot \chi_0(M) + \alpha^2 \cdot \chi_1(M) + ...$
 \bigstar known

Bad behaviour, bad convergence

At 1-loop:

$$\psi(M) = \frac{\Gamma'(M)}{\Gamma(M)}$$

$$\alpha \chi_0(M) = \frac{\alpha n_C}{\pi} \int_0^1 [z^{M-1} + z^{-M} - 2] \frac{dz}{1-z} = \frac{\alpha n_C}{\pi} \cdot [2\psi(1) - \psi(M) - \psi(1-M)]$$
Near M=0:

$$\alpha \chi_0(M) \sim \frac{\alpha n_C}{\pi} [\frac{1}{M} + 2\zeta(3)M^2 + 2\zeta(5)M^4 +]$$

At M=1/2

$$\lambda_0 = \alpha \chi_0 \left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi} 4 \ln 2 = \alpha c_0 \sim 2.65 \alpha \sim 0.5$$

The minimum value of $\alpha \chi_0$ at M=1/2 is the Lipatov intercept:

$$\lambda_0 = \alpha \chi_0 \left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi} 4 \ln 2 = \alpha c_0 \sim 2.65 \alpha \sim 0.5$$

It corresponds to (for x->0):

 $xP(x) \sim x^{-\lambda 0}$

Too hard, not supported by data



In the region of t and x where both

$$\frac{d}{dt}G(N, t) = \gamma(N, \alpha)G(N, t)$$
$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha)G(\xi, M)$$

are approximately valid, the "duality" relation holds:

$$\chi(\gamma(\alpha, N), \alpha) = N$$

Proof:

Note: γ is leading twist while χ is all twist.

Still the two perturbative exp.ns are related and improve each other.

Non perturbative terms in χ correspond to power or exp. suppressed terms in γ .

Proof of duality

Take a second Mellin transform:

 $MG(N,M) = \gamma(N)G(N,M) - S(M)$ $MG(N,M) = \chi(M)G(N,M) - T(N)$ $MG(N,M) = \chi(M)G(N,M) - T(N)$ $MG(N,M) = T(N)/(\chi(M)-N)$ $MG(N,M) = T(N)/(\chi(M)-N)$

At fixed N the pole at $\chi(M_0(N)) = N$ fixes the large t behaviour of the inverse Mellin transform G(N,t):

 $G(N,t) \sim \exp[-M_0(N)t]$ or $M_0(N) = \gamma(N)$ $\begin{cases} & \swarrow \\ & \chi(\gamma(N)) = N \\ & \text{Similarly} \end{cases}$ the duality relations





Example: if
$$\chi(M, \alpha) = \alpha \left[\frac{1}{M} + \frac{1}{1-M}\right] \implies$$

 $\alpha \left[\frac{1}{\gamma} + \frac{1}{1-\gamma}\right] = N \implies \gamma = \frac{1}{2} \left[1 \pm \sqrt{1 - \frac{4\alpha}{N}}\right]$

For example at 1-loop: $\chi_0(\gamma_s(\alpha, N)) = N/\alpha$ $\chi_0 \text{ improves } \gamma \text{ by adding a series of terms in } (\alpha/N)^n:$ $\chi_0 \rightarrow \gamma_s \left(\frac{\alpha}{N}\right) \qquad \gamma_s \left(\frac{\alpha}{N}\right) = \sum_k c_k \left(\frac{\alpha}{N}\right)^k$

$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \dots - \text{double count.}$$

This is the naive result from GLAP+(LO)BFKL The data discard such a large raise at small x



Similarly it is very important to improve χ by using γ_{1l}

Near M=0,
$$\chi_0 \sim 1/M$$
, $\chi_1 \sim -1/M^2$

Duality + momentum cons. ($\gamma(\alpha, N=1)=0$)



$$\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \dots - \text{double count.}$$

Double Leading Expansion

The minimum value of $\alpha \chi_0$ at M=1/2 is the Lipatov intercept:

$$\lambda_0 = \alpha \chi_0 \left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi} 4 \ln 2 = \alpha c_0 \sim 2.65 \alpha \sim 0.5$$

It corresponds to (for x->0):

 $xP(x) \sim x^{-\lambda 0}$

Too hard, not supported by data



$$\gamma(N, \alpha) = \alpha \cdot \gamma_{1l}(N) + ... \sim \alpha \cdot \left[\frac{1}{N} - A(N)\right]$$

Momentum conservation: $\gamma(1, \alpha)=0$ A(1)=1Duality: $\gamma(\chi(M)) = M \longrightarrow \alpha \cdot \left[\frac{1}{\chi} - A(\chi)\right] = M \longrightarrow$ $\begin{array}{c} \longrightarrow \quad \chi = \frac{\alpha}{M + \alpha A(\chi)} \quad \longrightarrow \quad \chi(M \sim 0) \sim \frac{\alpha}{M + \alpha A(1)} \sim \frac{\alpha}{M + \alpha} \\ \chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \dots \text{ -double count.} \end{array}$ $\chi_0(M) = \alpha \cdot \left[\frac{1}{M} + 0(M^2) \right]$

$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s \left(\frac{\alpha}{N}\right) + \dots \text{-double count.}$$

 $\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s \left(\frac{\alpha}{M}\right) + \dots \text{-double count.}$



G. Altarelli



A considerable improvement is obtained by including running coupling effects

Recall that the x-evolution equation was at fixed $\boldsymbol{\alpha}$

$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha)G(\xi, M)$$

In the following:

- Summary of general results
- Airy approximation
- Application to our problem

The implementation of running coupling in BFKL is not simple. In M-space α becomes an operator

$$\alpha(t) = \frac{\alpha}{1 + \beta_0 \alpha t} \Rightarrow \frac{\alpha}{1 - \beta_0 \alpha \frac{d}{dM}}$$

In leading approximation:

$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha)G(\xi, M)$$

$$\int \frac{d}{d\xi}G(\xi, M) = \frac{\alpha}{1 - \beta_0 \alpha} \chi_0(M)G(\xi, M)$$

A perturbative expansion in β_0 leads to validity of duality with modified χ and γ :

$$\Delta \chi_{1}(M) = \beta_{0} \frac{\chi_{0}''(M)\chi_{0}(M)}{2\chi_{0}'(M)} \qquad \Delta \gamma_{ss}(N) = -\beta_{0} \frac{\chi_{0}''(\gamma_{s})\chi_{0}(\gamma_{s})}{2\chi_{0'}^{2}(\gamma_{s})}$$

But this expansion fails near M=1/2: $\chi_0'(1/2)=0$

By taking a second MT the equation can be written as [F(M) is a boundary condition]

$$\left(1-\beta_0 \alpha \frac{d}{dM}\right) NG(N,M) + F(M) = \alpha \chi_0(M)G(N,M)$$

It can be solved iteratively

$$G(N, M) = \frac{F(M)}{N - \alpha \chi_0(M)} + \frac{\alpha \beta_0}{N - \alpha \chi_0(M)} \frac{d}{dMN} \frac{F(M)}{N - \alpha \chi_0(M)} + \dots$$

or in closed form:

$$G(N, M) = H(N, M) +$$

+ $\int_{M_0}^{M} dM \exp\left[\frac{M-M}{\beta_0 \alpha} - \frac{1}{\beta_0 N} \int_{M}^{M} \chi_0(M') dM''\right] \frac{F(M)}{\beta_0 \alpha N}$

H(*N*,*M*) is a homogeneous eq. sol. that vanishes faster than all pert. terms and can be dropped.

The following properties can be proven:

• From G(N,M) we can obtain G(N,t) and evaluate it by saddle point expansion. The perturbative G(N,t) is reproduced and satisfies duality (in terms of modified χ and γ according to the perturbative results singular at $\chi'(1/2)=0$) and factorisation (no t-dep. from the boundary condition).

From G(N,M) we can get G(ξ,M). This presents unphysical oscillations when χ>0 for all M.

These problems can be studied by using the Airy expansion: The asymptotics is fixed by the behaviour of χ near the minimum, where a quadratic form is taken:

Lipatov; Collins,Kwiecinski; Thorne; Ciafaloni, Taiuti,Mueller

$$\chi_{eff}(M) = c + \frac{1}{2}k\left(M - \frac{1}{2}\right)^2$$

G.A., R. Ball, S.Forte, hep-ph/0109178 (NPB 621,359)

For a quadratic kernel the explicit solution is $G(N, t) = K(N) \exp \frac{1}{2\beta_0 \alpha(t)} \cdot Ai[z(\alpha(t), N)]$ where $z(\alpha(t), N) = \left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}} \cdot \frac{1}{\beta_0} \cdot \left[\frac{1}{\alpha(t)} - \frac{c}{N}\right]$ $K(N) = \exp \frac{-1}{2\beta_0 \alpha} \cdot \left(\frac{2\beta_0 N}{k}\right)^{\frac{1}{3}} \cdot \frac{1}{\pi N}$



From
$$G(N, t) = K(N) \exp \frac{1}{2\beta_0 \alpha(t)} \cdot Ai[z(\alpha(t), N)]$$

one obtains G(x,t) by inv. MT

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{N\xi} G(N, t) \frac{dN}{2\pi i}$$

The asymptotics is dominated by the saddle condition:

$$\xi = -\frac{1}{Ai[z(\alpha(t), N)]} \cdot \frac{d}{dN} Ai[z(\alpha(t), N)]$$

For c>0 at not too large ξ this is satisfied at large N. When ξ increases N gets smaller. Then oscillations start, d/dN changes sign and the real saddle is lost. G(ξ ,t) starts oscillating, in agreement with the general analysis.

Ai[z(N)]



The dual anom. dim. γ_A is given by

$$\gamma_{A}(\alpha(t), N) = \frac{d}{dt}\log G(N, t) = \frac{1}{2} + \left(\frac{2\beta_{0}N}{k}\right)^{\frac{1}{3}}\frac{Ai'(z)}{Ai(z)}$$

$$\xrightarrow{z \text{ large}} \frac{1}{2} - \sqrt{\frac{2}{k}\left(\frac{N}{\alpha(t)} - c\right)} - \frac{1}{4} \cdot \frac{\beta_{0}\alpha}{1 - \frac{\alpha}{N}c} + \dots$$



The splitting fnct is completely free of oscillations at all x!!

The oscillations get factorised into the initial condition



The effect of running on χ is a softer small-x behaviour



As an effect of running, the small-x asymptotics is much softened:





The Airy result is free of the perturbative β_0 singularities.

At NLL order we can add the full γ_A and subtract its large N limit:

$$\chi_{0} \rightarrow \gamma_{s} \qquad \chi_{1} \rightarrow \gamma_{ss}$$

$$\gamma(\alpha, N) \approx \gamma_{s} \left(\frac{\alpha}{N}\right) + \alpha \gamma_{ss} \left(\frac{\alpha}{N}\right) + \alpha \Delta \gamma_{ss} \left(\frac{\alpha}{N}\right) + \gamma_{A}(\alpha, N) - \frac{1}{2} + \sqrt{\frac{2}{k} \left(\frac{N}{\alpha} - c\right)} + \frac{1}{4} \cdot \frac{\beta_{0} \alpha}{1 - \frac{\alpha}{N} c}$$

The last term cancels the sing. of $\alpha \Delta \gamma_{ss}$ (N= αc corresponds to M=1/2)

The goal of our recent work is to use these results to construct a relatively simple, closed form, improved anom. dim. $\gamma_I(\alpha,N)$ or splitting funct.n $P_I(\alpha,x)$

G.A., R. Ball, S.Forte, hep-ph/ 0306156 (NPB <u>674</u>,459), 0310016

 $P_{I}(\alpha, x)$ should

- reduce to pert. result at large x
- contain BFKL corr's at small x
- include running coupling effects (Airy)
- be sufficiently simple to be included in fitting codes and of course
- closely follow the trend of the data G. Altarelli

Improved anomalous dimension

1st iteration: optimal use of $\gamma_{11}(N)$ and $\chi_0(M)$

$$\gamma_{I}(\alpha, N) = \alpha \gamma_{1I}(N) + \gamma_{s}\left(\frac{\alpha}{N}\right) - \frac{\alpha n_{c}}{\pi N} + \gamma_{A}(\alpha, N) - \frac{1}{2} + \sqrt{\frac{2}{k_{0}}\left(\frac{N}{\alpha} - c_{0}\right)} + \frac{1}{4}\beta_{0}\alpha - \text{mom sub}$$

Properties:

Pert. Limit
$$\alpha \rightarrow 0$$
, N fixed
 $\gamma_{I}(\alpha, N) \longrightarrow \alpha \gamma_{1I}(N) + 0(\alpha^{2})$

• Limit α ->o , α /N fixed

$$\gamma_{I}(\alpha, N) \longrightarrow \alpha \gamma_{1l}(N) + \gamma_{s}\left(\frac{\alpha}{N}\right) - \frac{\alpha n_{c}}{\pi N} + O(\alpha \alpha/N)$$

$$\begin{array}{ccc} & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ &$$



Here is the same plot for the corresponding splitting fncts.

Note: for α_s =0.2 the pole in GLAP is ~0.191/N while the pole in γ_1 is ~0.014/(N-N₀) (only visible at very small x)



We can add the 2-loop perturbative result γ_{2l} :



Preview of our next paper (in advanced preparation)

BFKL kernel symmetric in k_1^2 , k_2^2 (gluon virtualities)

Mellin variable

$$\left(\frac{s}{k_1k_2}\right)$$
 in DIS

 $S Q^2 >> k^2 \qquad \left(\frac{3}{Q^2}\right)$

The symm. and the DIS kernels are related by:

$$\chi = \chi_{DIS} \qquad \chi_{DIS} \left(M + \frac{N}{2} \right) = \chi_{SYMM}(M)$$

Fadin, Lipatov

We use the underlying symmetry to improve the DL expansion (both 1/M and 1/(1-M) fixed) Ciafaloni, Salam





Naive symmetrization:

 $\chi_0(M) = [2\psi(1) - \psi(M) - \psi(1 - M)]$

$$\chi_s\left(\frac{\alpha}{M}\right) + \chi_s\left(\frac{\alpha}{1-M}\right) + \alpha\left[\chi_0(M) - \frac{1}{M} - \frac{1}{1-M}\right]$$
 Not relevant:
 χ_{DIS} not symm.!

In "symmetric" variables:

$$\chi_{s}\left(\frac{\alpha}{M+\frac{N}{2}}\right) + \chi_{s}\left(\frac{\alpha}{1-M+\frac{N}{2}}\right) + \alpha[\psi(1) + \psi(1+N) - \psi\left(M+\frac{N}{2}\right) - \psi\left(1-M+\frac{N}{2}\right) - \frac{1}{M+\frac{N}{2}} - \frac{1}{1-M+\frac{N}{2}}]$$

Note: N determined self-consistently

The actual function for DIS:

 $\chi_s\left(\frac{\alpha}{M}\right) + \chi_s\left(\frac{\alpha}{1-M+N}\right) + \alpha[\psi(1) + \psi(1+N) - \psi(M) - \psi(1-M+N) - \frac{1}{M} - \frac{1}{1-M+N}]$

to be called χ_{DLSymm,LO}(after mom. cons. subtraction)G. AltarelliSimilarly for NLO



We use $\chi_{\text{DLSymm,LO}}$ for implementing the Airy procedure:

Theorem: c, k are the same as for "symmetric" variables

$$\chi_{eff}(M) = c + \frac{1}{2}k\left(M - \frac{1}{2}\right)^2$$



G. Altarelli

Our most important competitors: Ciafaloni, Colferai, Salam, Stasto hep-ph/0307188. Also Thorne

Same physics: regularisation of M=0 pole in χ (and of M=1 pole using symmetrisation) and running coupling effects

Different resummation technique, no Airy expansion (num. sol of evol eqn.), and they include χ_1 but not γ_{2l}



In '01 we introduced N₀ as a parameter and fitted it

Here we compare LO with LO Well consistent!



The χ_1 leftover terms have ambiguities as large as the terms themselves The deviations for $1/x \sim 10^{-2}$ - 10^{-3} could be exp. visible (D. Haidt)

Summary and Conclusion

• We have constructed an improved an. dim. that reduces to the pert. result at large x and incorporates BFKL with running coupling effects at small x.

• We think we now know how to get the best use of the joint info from γ and χ .

• Properly introducing running coupling effects in the LO softens the asympt. small-x behaviour as shown by the data.

A clearer picture of the matching of GLAP and BFKL is finally emerging