Perturbative and Non-Perturbative Renormalization of GPDs

Martha Constantinou

Physics Department, University of Cyprus

In collaboration with:

ETMC members,

H. Panagopoulos, A. Skouroupathis, F. Stylianou

March 19, 2009

OUTLINE

A Non-perturbative renormalization

- 3-pt Green's functions
- 2-pt Green's functions
- Renormalization conditions

B Perturbative renormalization

- $\mathcal{O}(a^2)$ corrections to the fermion propagator - Z_q
- Numerical results for Z_q
- $\mathcal{O}(a^2)$ corrections to extended fermion bilinears - Z_Γ

3-pt Functions

Physical basis:

$$G_{\alpha\delta}(p) = \frac{a^{12}}{V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle u_\alpha(x) \mathcal{O}^{\Gamma}(z) \bar{d}_\delta(y) \rangle$$

$$\boxed{\begin{aligned} \mathcal{O}^{\mathrm{V}}(z) &= \bar{u}(z)\, \gamma^{\{\mu} \overleftrightarrow{D}^{\nu\}} d(z) \\ \mathcal{O}^{\mathrm{A}}(z) &= \bar{u}(z)\, \gamma^5\, \gamma^{\{\mu} \overleftrightarrow{D}^{\nu\}} d(z) \\ \mathcal{O}^{\mathrm{T}}(z) &= \bar{u}(z)\, \gamma^5\, \sigma^{\mu\,\{0} \overleftrightarrow{D}^{\nu\}} d(z) \end{aligned}}$$

- No mixing with lower dimension operators
- No disconnected diagrams

$$\mathcal{O}^{\Gamma}(z) \equiv \bar{u}(z)\, J^{\Gamma}(z, z') \qquad z, z' : \text{ nearest neighbors}$$

$$\mathcal{O}^{\{\mu\nu\}} = \frac{1}{2}\left(\mathcal{O}^{\mu\nu} + \mathcal{O}^{\nu\mu}\right) - \frac{1}{4}\delta_{\mu\nu} \sum_{\rho} \mathcal{O}^{\rho\rho}$$

Wick contraction

$$G_{\alpha\delta}(p) = \frac{a^{12}}{V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle \mathcal{U}_{\alpha\beta}(x,z)\, J^{\Gamma}_{\beta\gamma}(z,z')\, \mathcal{D}_{\gamma\delta}(z',y)\rangle^{G}$$

$\mathcal{U}(x,z)$, $\mathcal{D}(z',y)$: up, down propagators

- Method A ($Z_{\mathcal{O}^{00}}$ at $q^2 = 0$, Zhaofeng)

★ Fixed value for z ($z = 0$)

$$G_{\alpha\delta}(p) = a^{8} \sum_{x,y} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle \mathcal{U}_{\alpha\beta}(x,0)\, J^{\Gamma}_{\beta\gamma}(0,z')\, \mathcal{D}_{\gamma\delta}(z',y)\rangle^{G}$$

z' neighbors of $z = 0$

★ Point to all propagators: $\mathcal{S}(z,x)$, $\mathcal{S}(z',x)$

★ Dirac equation solvable with point source at the fixed value of z

★ Translation invariance $\leftrightarrow$ Average over gauge field configurations

★ Less inversions, but larger statistical errors

- Method B

Twisted basis:

$$G^{ad}_{\alpha\delta}(p) = \frac{a^{12}}{4\,V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \; \langle \left[(\hat{1} + i\,\gamma^5)\mathcal{U}(x,z)(\hat{1} + i\,\gamma^5) \right]^{ab}_{\alpha\beta} \; \tilde{J}^{bc}_{\beta\gamma}(z,z') \left[(\hat{1} - i\,\gamma^5)\mathcal{D}(z',y)(\hat{1} - i\,\gamma^5) \right]^{cd}_{\gamma\delta} \; \rangle^G$$

exact relation:

$$\boxed{\mathcal{U}(x,z) = \gamma^5\,\mathcal{D}^\dagger(z,x)\,\gamma^5}$$

$$G^{ad}_{\alpha\delta}(p) = -\frac{a^{12}}{4\,V} \sum_{z} \langle \left[(\hat{1} - i\,\gamma^5) \sum_{x} \mathcal{D}^\dagger(z,x)\mathrm{e}^{-\mathrm{i}p\cdot x}(\hat{1} - i\,\gamma^5) \right]^{ab}_{\alpha\beta} \; \tilde{J}^{bc}_{\beta\gamma}(z,z') \left[(\hat{1} - i\,\gamma^5) \sum_{y} \mathcal{D}(z',y)\mathrm{e}^{\mathrm{i}p\cdot y}(\hat{1} - i\,\gamma^5) \right]^{cd}_{\gamma\delta} \; \rangle^G$$

- Method B

Twisted basis:

$$G^{ad}_{\alpha\delta}(p) = \frac{a^{12}}{4\,V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \, \langle \left[(\hat{1} + i\,\gamma^5) \boxed{\mathcal{U}(x,z)} (\hat{1} + i\,\gamma^5) \right]^{ab}_{\alpha\beta} \tilde{J}^{bc}_{\beta\gamma}(z,z') \left[(\hat{1} - i\,\gamma^5) \mathcal{D}(z',y) (\hat{1} - i\,\gamma^5) \right]^{cd}_{\gamma\delta} \rangle^G$$

exact relation:

$$\boxed{\mathcal{U}(x,z) = \gamma^5\, \mathcal{D}^\dagger(z,x)\, \gamma^5}$$

$$G^{ad}_{\alpha\delta}(p) = -\frac{a^{12}}{4\,V} \sum_{z} \langle \left[(\hat{1} - i\,\gamma^5) \sum_{x} \mathcal{D}^\dagger(z,x) \mathrm{e}^{-\mathrm{i}p\cdot x} (\hat{1} - i\,\gamma^5) \right]^{ab}_{\alpha\beta} \tilde{J}^{bc}_{\beta\gamma}(z,z') \left[(\hat{1} - i\,\gamma^5) \sum_{y} \mathcal{D}(z',y) \mathrm{e}^{\mathrm{i}p\cdot y} (\hat{1} - i\,\gamma^5) \right]^{cd}_{\gamma\delta} \rangle^G$$

- Method B

Twisted basis:

$$G^{ad}_{\alpha\delta}(p) = \frac{a^{12}}{4\,V}\sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}\ \langle \left[(\hat{1}+i\,\gamma^5)\boxed{\mathcal{U}(x,z)}(\hat{1}+i\,\gamma^5)\right]^{ab}_{\alpha\beta} \tilde{J}^{bc}_{\beta\gamma}(z,z') \left[(\hat{1}-i\,\gamma^5)\mathcal{D}(z',y)(\hat{1}-i\,\gamma^5)\right]^{cd}_{\gamma\delta} \rangle^G$$

exact relation:

$$\boxed{\mathcal{U}(x,z) = \gamma^5\,\mathcal{D}^\dagger(z,x)\,\gamma^5}$$

$$G^{ad}_{\alpha\delta}(p) = -\frac{a^{12}}{4\,V}\sum_{z} \langle \left[(\hat{1}-i\,\gamma^5)\sum_x \mathcal{D}^\dagger(z,x)\mathrm{e}^{-\mathrm{i}p\cdot x}(\hat{1}-i\,\gamma^5)\right]^{ab}_{\alpha\beta} \tilde{J}^{bc}_{\beta\gamma}(z,z') \left[(\hat{1}-i\,\gamma^5)\sum_y \mathcal{D}(z',y)\mathrm{e}^{\mathrm{i}p\cdot y}(\hat{1}-i\,\gamma^5)\right]^{cd}_{\gamma\delta} \rangle^G$$

Evaluation of $\breve{\mathcal{D}}$:

$$\sum_z \mathcal{K}^{ac}_{\alpha\gamma}(x,z) \underbrace{\sum_y \mathrm{e}^{ip\cdot y}\mathcal{D}^{cb}_{\beta\gamma}(z,y)}_{\breve{\mathcal{D}}^{cb}_{\beta\gamma}(z,p)} = \mathrm{e}^{ip\cdot x}\,\delta_{\alpha\beta}\,\delta_{ab}$$

$\mathcal{K}$: Dirac operator for down quark

- ★ Perform the summation over z
- ★ Dirac equation solved with momentum source
- ★ $\sharp$ of inversion depends on the $\sharp$ of momenta considered
- ★ Application of any operator
- ★ High statistical accuracy is expected

Choice for momentum within the plateau for $Z_{\mathcal{O}^{00}}$: $p_\rho = \frac{2\pi}{L_\rho}(4,2,2,2)$

2-pt Functions

Twisted basis:

up quark

$$\mathcal{S}^{ab}_{\alpha\beta}(p) = -\frac{a^8}{4\,V}\sum_z \langle\, \mathrm{e}^{\mathrm{i}p\cdot z}\left[(\hat{1} - i\,\gamma^5)\breve{\mathcal{D}}^\dagger(z,p)(\hat{1} - i\,\gamma^5)\right]^{ab}_{\alpha\beta}\rangle^G$$

down quark

$$\mathcal{S}^{ab}_{\alpha\beta}(p) = \frac{a^8}{4\,V}\sum_z \langle\, \mathrm{e}^{-\mathrm{i}p\cdot z}\left[(\hat{1} - i\,\gamma^5)\breve{\mathcal{D}}(z,p)(\hat{1} - i\,\gamma^5)\right]^{ab}_{\alpha\beta}\rangle^G$$

Amputated Green's Function

Bare $\quad \Gamma(p) = \mathcal{S}^{-1}(p) G(p) \mathcal{S}^{-1}(p)$

Renormalized $\quad \Gamma_R(p) = Z_q^{-1}\, Z_{\mathcal{O}}\, \Gamma(p)$

Renormalization Conditions:

RI-MOM scheme:

$$Z_q = \frac{i}{12} \frac{\mathrm{Tr}\left[\frac{1}{a}\sin(a\,p_\rho)\,\gamma_\rho \cdot \mathcal{S}^{-1}(p)\right]}{\frac{1}{a^2}\sum_\rho \sin^2(a\,p_\rho)}\Bigg|_{p^2=\bar{\mu}^2}$$

Trace over color and spin indices

$$Z_q^{-1}\, Z_{\mathcal{O}}\, \frac{1}{12}\mathrm{Tr}\left[\Gamma(p)\,\Gamma_{\mathrm{Born}}^{-1}(p)\right]\Big|_{p^2=\bar{\mu}^2} = 1$$

Γ_{Born}: tree-level value of $\Gamma(p)$

$$\frac{1}{L^2} \ll \Lambda_{\mathrm{QCD}}^2 \ll \bar{\mu}^2 \ll \frac{1}{a^2}$$

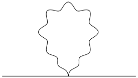
- Reliable perturbation theory (Wilson coefficients)
- Small $\mathcal{O}(a)$ lattice effects

So far...

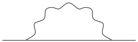
- We generated the momentum sources
- We are working on the code for the gauge fixing
- We are ready to do the inversions
- We know how to calculate the 3-pt functions
- Starting point: 50 configurations and momentum $p_\rho = \frac{2\pi}{L_\rho}(4, 2, 2, 2)$

Correction to the fermion propagator - Z_q

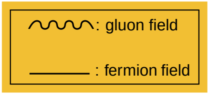
- Twisted mass fermions and Symanzik improved gluons
- Up to 2^{nd} order in the lattice spacing a
- General covariant gauge λ
- General values for momentum p and lattice spacing a
- General values for the masses m and μ



1



2



Technical Procedure

- Wick contraction of appropriate vertices
- Simplification of color dependence, Dirac matrices and tensors
- Exploitation of symmetries of the theory and of the diagrams
- Isolation of the logarithmic and non-Lorentz invariant terms:
  - 19 *new primitive divergent integrals, e.g.*

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(\hat{k}^2\right)\left(\widehat{k + a\,p}^{\,2} + a^2(m^2+\mu^2)\right)}, \qquad \hat{q}^2 = 4\sum_\rho \sin^2\!\left(\frac{q_\rho}{2}\right)$$

- Convergent terms: Taylor expansion in p and a up to $\mathcal{O}(a^3p^3)$
- Numerical integration over the internal momentum k
- Extrapolation of results to $L \to \infty$
  - *only source of systematic errors*

Renormalization Condition:

$$Z_q = \frac{i}{4} \frac{\mathrm{Tr}\left[\frac{1}{a}\sin(a\,p_\rho)\,\gamma_\rho \cdot \mathcal{S}^{-1}(p)\right]}{\frac{1}{a^2}\sum_\rho \sin^2(a\,p_\rho)}\Bigg|_{p^2=\bar{\mu}^2}$$

Trace over spin indices

$\not{p}^3 \equiv \sum_\rho \gamma_\rho\, p_\rho^3$

$$Z_q = \frac{i}{4}\mathrm{Tr}\left[\left(\not{p} - \frac{a^2}{6}\not{p}^3\right) \cdot \mathcal{S}^{-1}(p)\right]\frac{1}{p^2}\left(1 + \frac{a^2}{3p^2}\sum_\rho p_\rho^4\right)\Bigg|_{p^2=\bar{\mu}^2}$$

General expression of $S^{(-1)}(p)$

Numerical Results:

- Tree-level Symanzik gluons
- Landau gauge
- $m = 0$
- $\mu = 0$

$$Z_q = 1 + \frac{C_F\, g^2}{\pi^2} \Bigg[- 0.813954545(4) \\ + a^2\, p^2 \left(0.071697632(3) - 0.012673612187\, \mathrm{Log}[a^2\, p^2]\right) \\ + a^2 \frac{\sum_\rho p_\rho^4}{p^2} \left(0.13165622(1) - 0.054513888125\, \mathrm{Log}[a^2\, p^2]\right) \Bigg]_{p^2 = \bar{\mu}^2}$$

What are the choices for the direction of the momentum?

$Z_q(\bar\mu = \frac{1}{a})$: Perturbative and non-perturbative comparison

β	Non-perturbative[1]	Perturbative[2]	Pert. average [3]
3.8	0.755(4)	0.771056601(4)	0.749939687(2)
3.9	0.757(3)	0.782134880(4)	0.762039787(2)
4.05	0.777(5)	0.796797308(3)	0.778054624(2)

[1] Non-perturbative with subtraction of $\mathcal{O}(a^2)$ contributions (by Petros)

[2] $a\,p^\rho = (1,0,0,0)$

[3] average over 15 momenta leading to $(a\,p)^2 = 1$:

$$4 \times (1,0,0,0)\ ,6 \times \frac{1}{\sqrt{2}}(1,1,0,0)\ ,4 \times \frac{1}{\sqrt{3}}(1,1,1,0)\ ,1 \times \frac{1}{2}(1,1,1,1)$$

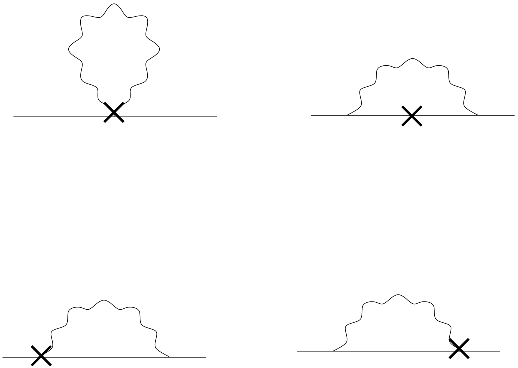
$\star$ Bare coupling for perturbative results

- Ambiguity on the choice of the momentum direction
- The same behavior occurs in other renormalization prescriptions

$$Z_q = \frac{i}{4}\mathrm{Tr}\left[\gamma^\rho \partial_\rho \left(\mathcal{S}^{-1}(p)\right)\right]\Big|_{p^2=\bar\mu^2}$$

Current Work

Correction to fermion extended operators (GPDs) - $Z_{\mathcal{O}}$



THANK YOU