$\pi^+\pi^+$ scattering length from twisted lattice QCD

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Scattering Lengths

- a plane wave of momentum $\vec{k}$ scatters into a spherical wave

$$\psi(\vec{x}) \to e^{i\vec{k}\cdot\vec{x}} + \frac{1}{r} e^{ikr} f(k,\theta)$$

- each angular momentum contributes

$$\psi(\vec{x}) \to \sum_l (2l+1) \frac{P_l(\cos\theta)}{2ikr} \left[(1 + 2ikf_l(k)) e^{ikr} - e^{il\pi} e^{-ikr} \right]$$

- unitarity for each angular momentum constrains the amplitude

$$S_l(k) = 1 + 2ikf_l(k) \qquad |S_l(k)| = 1 \qquad S_l(k) = e^{2i\delta_l(k)}$$

- scattering length $\delta_{l=0}(k) = -ak$ for small k (in general $\delta_l(k) \propto k^{2l+1}$)

$$\sigma_{\text{tot}}(k \to 0) = \lim_{k\to 0} \frac{4\pi}{k^2} \sum_l \sin^2 \delta_l(k) = 4\pi a^2$$

Current Values

- LO chiral perturbation theory prediction

$$m_\pi a_s = -\frac{m_\pi^2}{8\pi f_\pi^2} \approx -0.04483$$

- experimental measurement of $K^+ \to \pi^+ \pi^- e^+ \nu_e$ from E865 at BNL

$$m_\pi a_s = -0.0454(31)(10)(8)$$

- phenomenological analysis by CGL

$$m_\pi a_s = -0.0444(10)$$

- indirect lattice calculation by MILC

$$m_\pi a_s = -0.0432(6)$$

- only one previous full QCD, light quark lattice calculation by NPLQCD

$$m_\pi a_s = -0.04330(42)$$

- current ETMC result

$$m_\pi a_s = -0.04384(36)$$

Lüscher's Finite Size Method

- Lüscher relates finite size effect in π–π channel to scattering length

$$E_{\pi\pi} - 2m_\pi = -\frac{4\pi a_s}{m_\pi L^3}\left[1 + c_1\frac{a_s}{L} + c_2\left(\frac{a_s}{L}\right)^2\right] + O(L^{-6})$$

- method requires m_π, which we get from a two-point

$$C_\pi(t) = \langle(\pi^+)^\dagger(t)\pi^+(0)\rangle \longrightarrow A_\pi e^{-m_\pi t}$$

- also requires lowest E in π–π $(I = 2)$ channel, which comes from

$$C_{\pi\pi}(t) = \langle(\pi^+\pi^+)^\dagger(t)(\pi^+\pi^+)(0)\rangle \longrightarrow A_{\pi\pi} e^{-Et}$$

where $(\pi^+\pi^+)(t) = \pi^+(t+1)\pi^+(t)$

- a ratio gives $\delta E = E_{\pi\pi} - 2m_\pi$ directly

$$\frac{C_{\pi\pi}(t)}{C_\pi(t)^2} \to \frac{A_{\pi\pi}e^{-Et}}{(A_\pi e^{-m_\pi t})^2} = \frac{A_{\pi\pi}}{A_\pi A_\pi}e^{-(E-2m_\pi)t} = Ae^{-\delta Et}$$

Constant Contributions to $C_{\pi\pi}$

- the transfer matrix formalism applied to a two-point gives

$$C_{\mathcal{O}}(t) = \langle \mathcal{O}^\dagger(t)\mathcal{O}(0)\rangle = \mathrm{Tr}(e^{-\hat{H}(T-t)}\hat{O}^\dagger e^{-\hat{H}t}\hat{O})/\mathrm{Tr}(e^{-\hat{H}T})$$

- the spectral resolution of a two-point function is

$$C_{\mathcal{O}}(t) = \sum_{n,m} e^{-E_n T} e^{-(E_m - E_n)t} \langle n|\hat{O}^\dagger|m\rangle\langle m|\hat{O}|n\rangle \,/\, \mathcal{Z}(T)$$

- for $\mathcal{O} = \pi^+\pi^+$, the standard contribution is from $n = \Omega$ and $m = \pi^+\pi^+$

$$C_{\mathcal{O}}(t) \to e^{-E_{\pi\pi}t}\langle\Omega|(\hat{\pi}^+\hat{\pi}^+)^\dagger|\pi^+\pi^+\rangle\langle\pi^+\pi^+|\hat{\pi}^+\hat{\pi}^+|\Omega\rangle$$

- but there is also $n = \pi^-$ and $m = \pi^+$

$$C_{\mathcal{O}}(t) \to e^{-E_\pi T}\langle\pi^-|(\hat{\pi}^+\hat{\pi}^+)^\dagger|\pi^+\rangle\langle\pi^+|\hat{\pi}^+\hat{\pi}^+|\pi^-\rangle$$

- this constant contribution to $C_{\pi\pi}$ is only suppressed by $e^{-m_\pi T}$ but you must compare it to $e^{-E_{\pi\pi}t} \approx e^{-2m_\pi t}$

Constant Contributions to C_π^2

- the ground state contribution to C_π is

$$C_\pi(t) \to A \cosh(m(t - T/2))$$

- the ratio requires $C_\pi^2(t)$

$$C_\pi^2(t) \to A^2 \cosh^2(m(t - T/2))$$

- but even this term has a constant contribution

$$C_\pi^2(t) \to \frac{1}{2} A^2 \left(\cosh(2m(t - T/2)) + 1\right)$$

Xu's Ratio

- to eliminate the constant in $C_{\pi\pi}(t)$ and $C_\pi^2(t)$, Xu defines

$$R(t+1/2) = \frac{C_{\pi\pi}(t) - C_{\pi\pi}(t+1)}{C_\pi(t)^2 - C_\pi(t+1)^2}$$

- this ratio is dominated by the correct π or π–π states asymptotically

$$R(t+1/2) \to A_R \left\{ \cosh(\delta E t') + \sinh(\delta E t') \cdot \tanh(m_\pi t') \right\}$$

where $t' = t + 1/2 - T$

- so $R(t+1/2)$ is fit to determine δE directly (and A_R)

Lattice Calculation

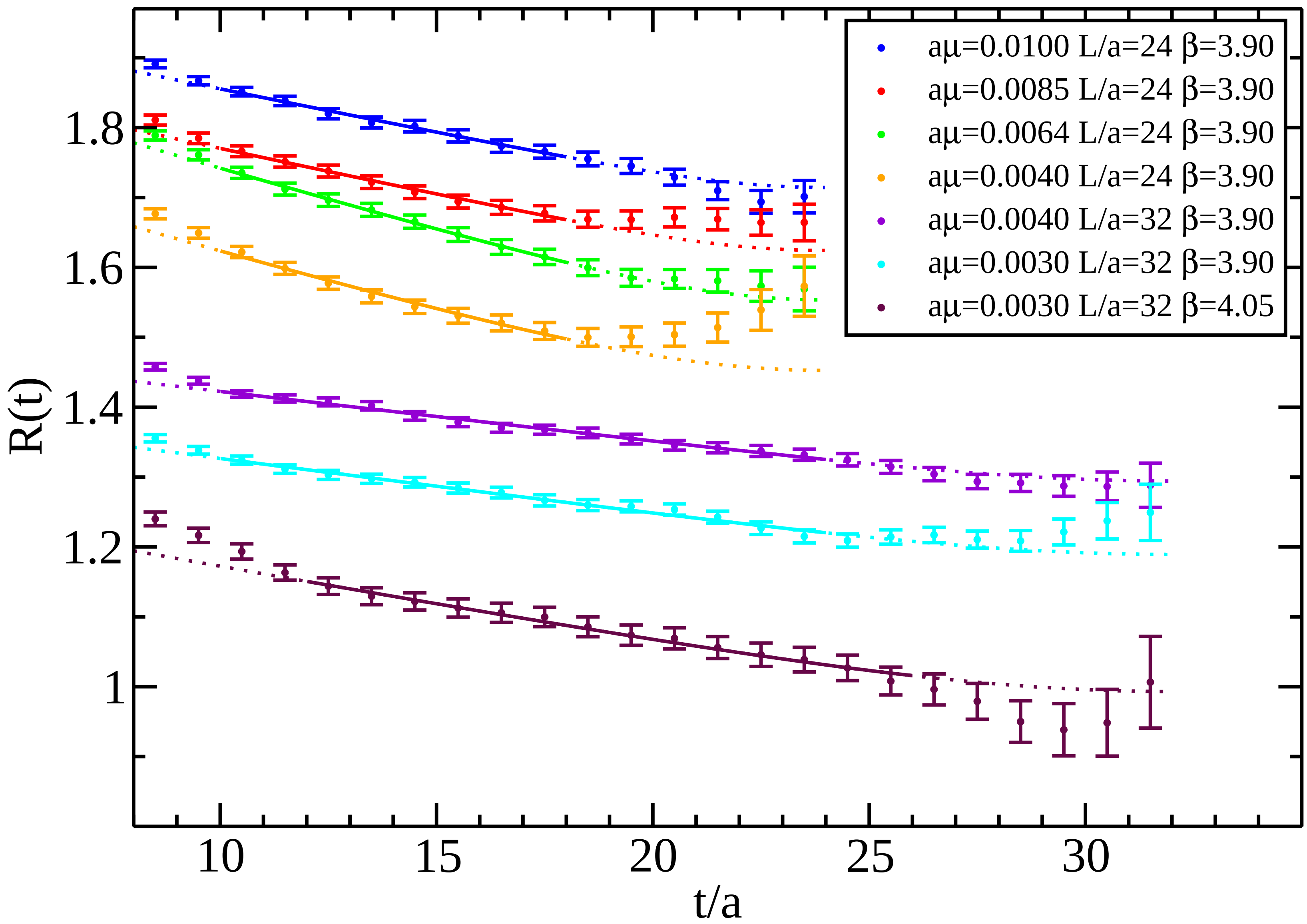
- $N_F = 2$ single spacing study, with tests for a and V dependence

β	$a\mu$	L/a	N_{traj}	$a\delta E(10^{-3})$	$m_\pi a_s$	Δ_{FV}
m_π dependence						
3.9	0.0100	24	479	7.25(57)(44)	-0.298(20)(16)	0.0005
3.9	0.0085	24	487	7.66(69)(19)	-0.267(21)(06)	0.0006
3.9	0.0064	24	553	9.67(91)(05)	-0.254(21)(02)	0.0008
3.9	0.0040	24	955	9.94(98)(64)	-0.175(15)(10)	0.0011
3.9	0.0030	32	562	4.31(40)(02)	-0.137(11)(01)	0.0004
V dependence						
3.9	0.0040	32	490	3.92(30)(08)	-0.165(11)(03)	0.0003
a dependence						
4.05	0.0030	32	375	6.59(90)(53)	-0.161(19)(12)	0.0010

- the above are Xu's numbers

- my results for the same plateaus agree but with slightly larger errors

Ratio Fits


aμ=0.0100 L/a=24 β=3.90
aμ=0.0085 L/a=24 β=3.90
aμ=0.0064 L/a=24 β=3.90
aμ=0.0040 L/a=24 β=3.90
aμ=0.0040 L/a=32 β=3.90
aμ=0.0030 L/a=32 β=3.90
aμ=0.0030 L/a=32 β=4.05
R(t)
1.8
1.6
1.4
1.2
1
10
15
20
25
30
t/a

Chiral Perturbation Theory

- NLO chiral perturbation theory gives

$$m_\pi a_s = -\frac{m_\pi^2}{8\pi f_\pi^2}\left\{1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\left(\frac{m_\pi^2}{\mu^2}\right) - 1 - l_{\pi\pi}(\mu)\right]\right\}$$

- where the counter term is related to conventional LECs by

$$l_{\pi\pi}(\mu) = 64\pi^2(4l_1^r(\mu) + 4l_2^r(\mu) + l_3^r(\mu) - l_4^4(\mu))$$

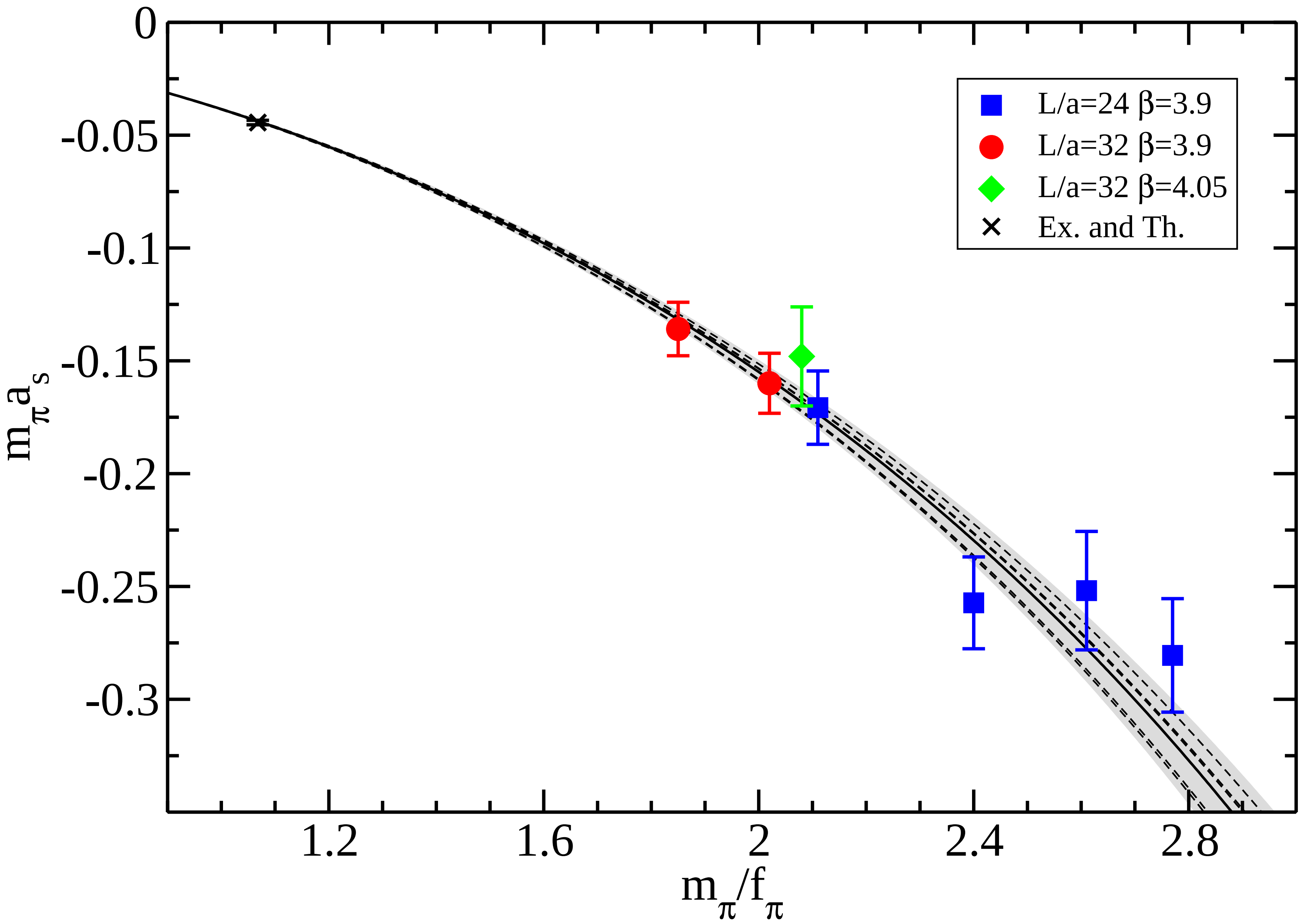
- we can eliminate the scale dependence by choosing $\mu = f_{\pi,phys}$ and extrapolating in $x = m_\pi/f_\pi$

$$m_\pi a_s = -\frac{x^2}{8\pi}\left\{1 + \frac{x^2}{16\pi^2}\left[3\ln\left(x^2\right) - 1 - l_{\pi\pi}\right]\right\}$$

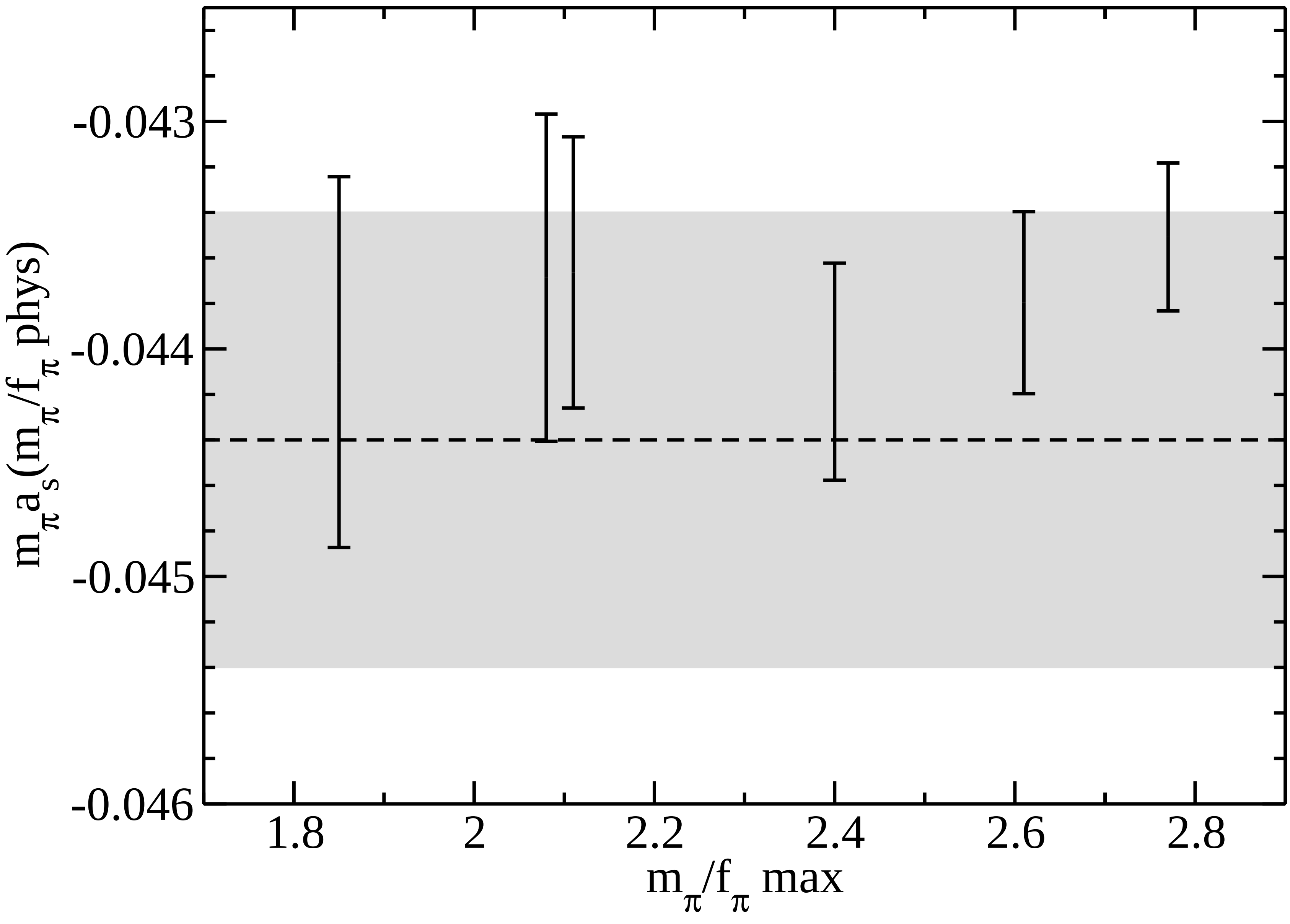
where $l_{\pi\pi} = l_{\pi\pi}(\mu = f_{\pi,phys})$

- note, this is a resummation but only differs at NNLO

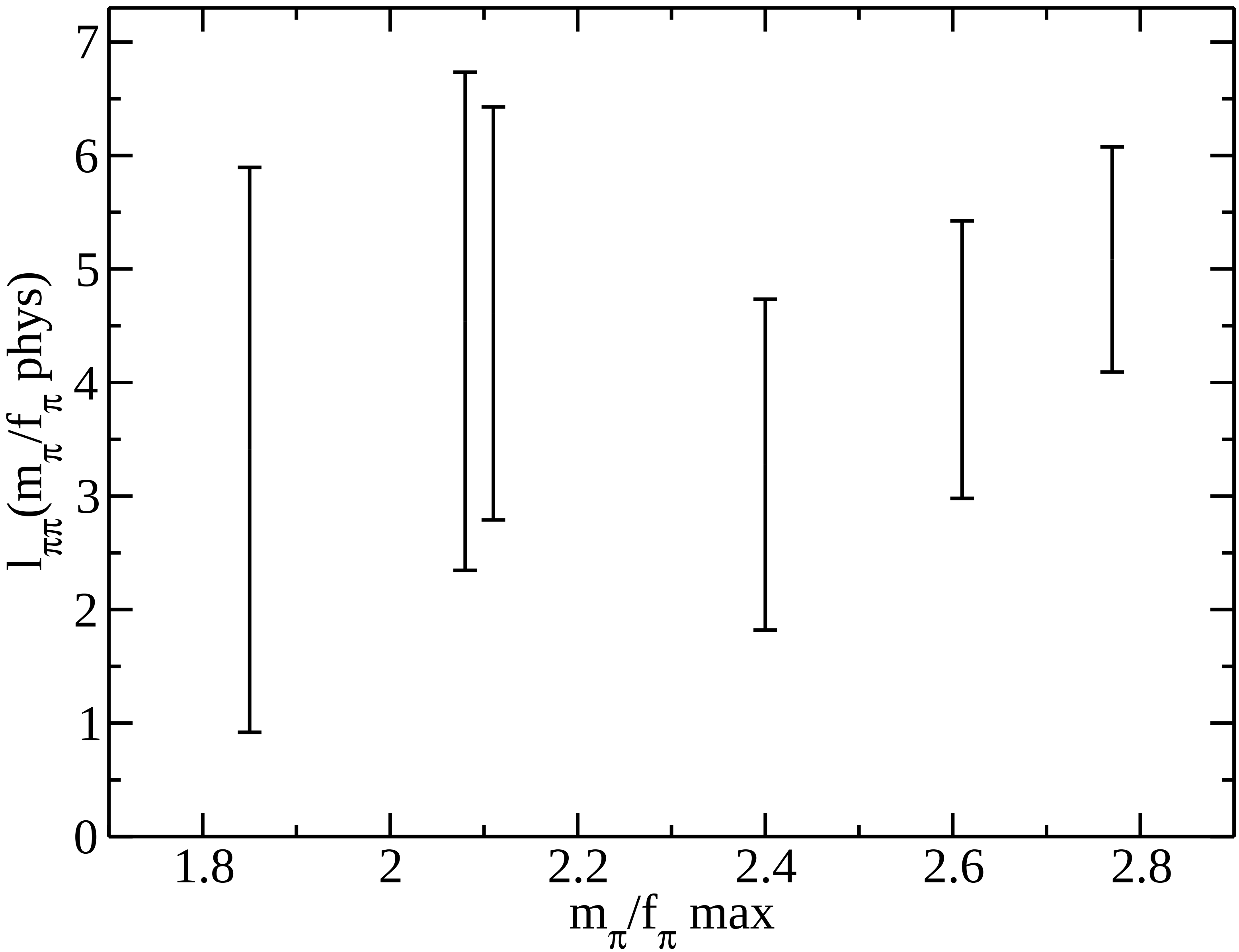
Chiral Perturbation Theory Fits



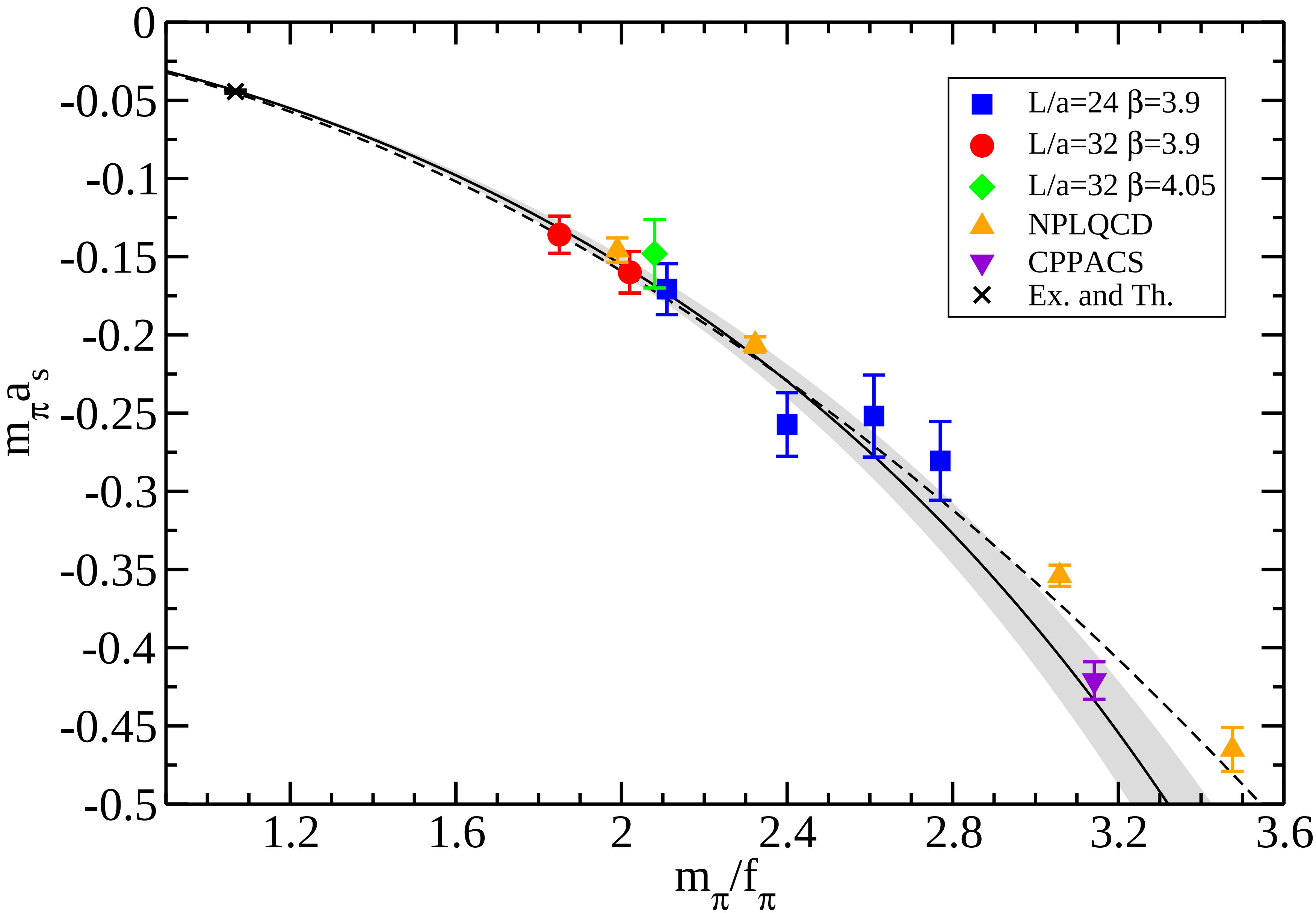
$m_\pi a_s$



$l_{\pi\pi}$



World's Full QCD $m_\pi a_s$ Calculations



Summary

- CGL result

$$m_\pi a_s = -0.0444(10)$$

- NPLQCD result

$$m_\pi a_s = -0.04330(42) \qquad l(\mu = f_{phy}) = 6.2(1.2)$$

- Xu's numbers

$$m_\pi a_s = -0.04384(28)(23) \qquad l(\mu = f_{phy}) = 4.14(84)(74)$$

- me

$$m_\pi a_s = -0.04380(40) \qquad l(\mu = f_{phy}) = 4.20(1.22) \qquad m_\pi / f_\pi < 2.61$$

$$m_\pi a_s = -0.04351(33) \qquad l(\mu = f_{phy}) = 5.08(99) \qquad m_\pi / f_\pi < 2.77$$

- a couple questions: ratios at large t, and the errors