ϵ expansion in the continuum

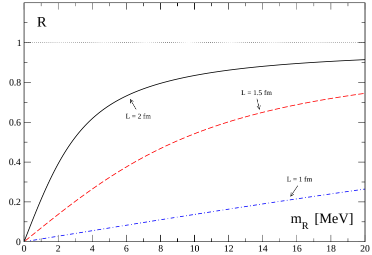
$$G(x) = \frac{1}{V}\sum_p \frac{e^{ipx}}{p^2 + M_\pi^2} = \frac{1}{VM_\pi^2} + \frac{1}{V}\sum_{p\neq 0}\frac{e^{ipx}}{p^2+M_\pi^2},$$

- Infrared divergences $\Rightarrow$ resum to all orders zero-modes
- Modify the power counting of the p regime

$$\frac{1}{T} = O(\epsilon), \quad \frac{1}{L} = O(\epsilon), \quad M_\pi = O(\epsilon^2).$$

The order parameter, vanishes in the chiral limit at fixed finite volume (Gasser,Leutwyler:1987)

$$R = \frac{\langle \bar{q}q\rangle}{\langle \bar{q}q\rangle_\infty}$$



Power counting

$$\frac{1}{T} = \mathrm{O}(\epsilon) \quad \frac{1}{L} = \mathrm{O}(\epsilon) \quad m = \mathrm{O}(\epsilon^4) \quad \Sigma = (250\mathrm{MeV})^3 \quad m\Sigma V \lesssim 1$$

$$m \simeq 15\mathrm{MeV} \qquad L = 1.5\mathrm{fm}$$

$$m \simeq 6\mathrm{MeV} \qquad L = 2\mathrm{fm}$$

$$m \simeq 2\mathrm{MeV} \qquad L = 3\mathrm{fm}$$

What about Wilson fermions?

GSM: $m \sim a\Lambda^2$ Aoki (LCE): $m \sim a^2\Lambda^3$

$0.08\mathrm{fm} \lesssim a \lesssim 0.04\mathrm{fm}$ $\Lambda \simeq 250\mathrm{Mev}$

$a \simeq 0.08\mathrm{fm}:$ $a\Lambda^2 \simeq 25\mathrm{MeV}$ $a^2\Lambda^3 \simeq 3\mathrm{MeV}$

$a \simeq 0.04\mathrm{fm}:$ $a\Lambda^2 \simeq 12\mathrm{MeV}$ $a^2\Lambda^3 \simeq 1\mathrm{MeV}$

$$a = 0.04\text{fm} \quad \frac{L}{a} = 48 \quad m = 6MeV \quad \textit{GSM IDEAL (?)}$$

$$a = 0.08\text{fm} \quad \frac{L}{a} = 24 \quad m = 6MeV \quad \textit{Between GSM and LCE (?)}$$

There is a transition region

We need to understand better

We need to check with numerical simulations

Generic small masses $m \sim a \sim \epsilon^4$

$$\mathcal{L}^{(2)}_{W\chi} = \frac{F^2}{4}\left\{\mathrm{Tr}\left[\partial_\mu U(x)^\dagger \partial_\mu U(x)\right] - 2B_0 \mathrm{Tr}\left[\mathcal{M}^\dagger U(x) + \mathcal{M} U(x)^\dagger\right] - 2aW_0 \mathrm{Tr}\left[U(x) + U(x)^\dagger\right]\right\},$$

$$\mathcal{M} \to \mathcal{M}' = \mathcal{M} + \frac{W_0}{B_0} a.$$

NO cutoff effects at LO (reabsorbed in definition of the mass)

$$\begin{aligned}\mathcal{L}^{(4)}_{W\chi} &= \mathcal{L}^{(4)}_{\chi} + a\widetilde{W}\mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U)\mathrm{Tr}(U + U^\dagger) - 2aB_0 W \mathrm{Tr}(\mathcal{M}'^\dagger U + U^\dagger \mathcal{M}')\mathrm{Tr}(U + U^\dagger) + \\ &- a^2 W' \left[\mathrm{Tr}(U + U^\dagger)\right]^2 - 2aB_0 H' \mathrm{Tr}(\mathcal{M}' + \mathcal{M}'^\dagger).\end{aligned}$$

NO cutoff effects at NLO

Wilson fermions are "effectively" automatic O(a) improved

(up to NNLO corrections)

Hic Sunt Leones $m \sim a^2 \sim \epsilon^4$

Aoki region

$$\mathcal{L}^{(2)}_{W\chi} = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U(x)^\dagger \partial_\mu U(x)\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}'^\dagger U(x) + \mathcal{M}' U(x)^\dagger\right] - a^2 W' \left[\mathrm{Tr}\left(U(x) + U(x)^\dagger\right)\right]^2 .$$

- Cutoff effects already at leading order

Aoki scenario: different pattern of SSB not

$SU(2) \times SU(2) \rightarrow SU(2)$

but $SU(2) \rightarrow U(1)$

Sh.-Sin. : No zero modes $\Rightarrow$ no need for resummation!!

Better to stay away from this area

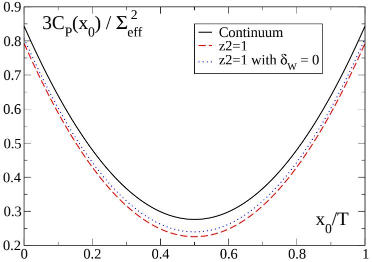
Transition region

$$S = S_2^{(0)} + S_2^{(2)} + \delta S_4$$

$$\mathcal{Z} = \int \mathcal{D}[U]\, \mathrm{e}^{-S_2^{(0)}} \left\{1 + S_2^{(2)} + \delta S_4\right\}$$

Computed

$$\langle S^0 \rangle, \qquad \frac{1}{L^3}\int d^3x \langle S^0(x) S^0(0)\rangle = C_{\mathrm{S}}(x_0), \qquad \frac{1}{L^3}\int d^3x \langle P^a(x) P^b(0)\rangle = \delta^{ab} C_{\mathrm{P}}(x_0),$$



Summary

- In the Aoki region there could be LCE
- In the GSM region Wilson fermions are effectively automatic O(a) improved up to NNLO
- In the transition region tool to analyze cutoff effects
- Computed correlators including O(a^2) and O(am)
  - Proper power counting identified a =O(ϵ^3) (Bär,Necco,Schaefer:2008)
  - Correlator computed up to relative O(ϵ^3) corrections
- Certain linear combinations are free from O(a^2) errors

Let us do it for twisted mass

Wilson twisted mass

$$\mathcal{L} = \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu \widetilde{U}(x)\partial_\mu \widetilde{U}^\dagger(x)\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[\widetilde{\mathcal{M}}^\dagger \widetilde{U}(x) + \widetilde{U}^\dagger(x)\widetilde{\mathcal{M}}\right]$$

$$\widetilde{\mathcal{M}} = m\mathbb{1} + i\mu_{\mathrm{q}}\tau^3.$$

Equivalence of tmQCD and QCD

$$\widetilde{U} = U_{\mathrm{V}} U, \qquad U_{\mathrm{V}} = \mathrm{e}^{i\tau^3\omega_0}.$$

Haar measure is invariant

$$\mathcal{Z} = \int \mathcal{D}[\widetilde{U}] \mathrm{e}^{-S[\widetilde{U}]} = \int \mathcal{D}\,[U]\,\mathrm{e}^{-S[U_{\mathrm{V}} U]}$$

QCD partition function $\Rightarrow$

$$\widetilde{\mathcal{M}}^{\dagger} U_{\mathrm{V}} = \mathcal{M} = M \mathbb{1}.$$

Given our parametrizations this happens if

$$s_0 \equiv \sin \omega_0 = \frac{\mu_{\mathrm{q}}}{M}, \qquad c_0 \equiv \cos \omega_0 = \frac{m}{M}$$

Same partition function with $M = \sqrt{m^2 + \mu_{\mathrm{q}}^2}$

Transition region

- δS_4 not invariant
- Different group integrals $\rightsquigarrow$ careful with flavour structure
- Computed at NLO for Wtm

$$\mathcal{Z}, C_{\mathrm{PP}}^{1,2}, C_{\mathrm{PP}}^{3}, C_{\mathrm{AP}}^{1,2}, C_{\mathrm{VP}}^{1,2},$$

- Computed the shift in the twist angle induced by NLO
- Missing

$$C_{\mathrm{AA}}, C_{\mathrm{VV}}, \ldots$$

Reweighting

Exact reweighting to better sample configuration space (long history)

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[U] \mathrm{e}^{-S_G[U]} \det\left(Q_+ Q_-\right) \mathcal{O} \quad Q_\pm = \gamma_5 D_\mathrm{W} \pm i\mu_\mathrm{q},$$

$$\det\left(Q_+ Q_-\right) = \frac{\det\left[Q_+ Q_- P_{n,\tilde{\epsilon}}\left(Q_+ Q_-\right)\right]}{\det\left[P_{n,\tilde{\epsilon}}\left(Q_+ Q_-\right)\right]}, \quad P_{n,\tilde{\epsilon}}\left(Q_+ Q_-\right) \simeq \left[Q_+ Q_-\right]^{-1},$$

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} W \rangle_P}{\langle W \rangle_P}, \quad W = \det\left[Q_+ Q_- P_{n,\tilde{\epsilon}}\left(Q_+ Q_-\right)\right] \simeq \prod_{\lambda_i < \tilde{\epsilon}} \left[\lambda_i P_{n,\tilde{\epsilon}}\left(\lambda_i\right)\right],$$

Stochastic reweighting (long history)

- Make more use of existing gauges
- Avoid instabilities with algorithms
- . . .

Stochastic reweighting

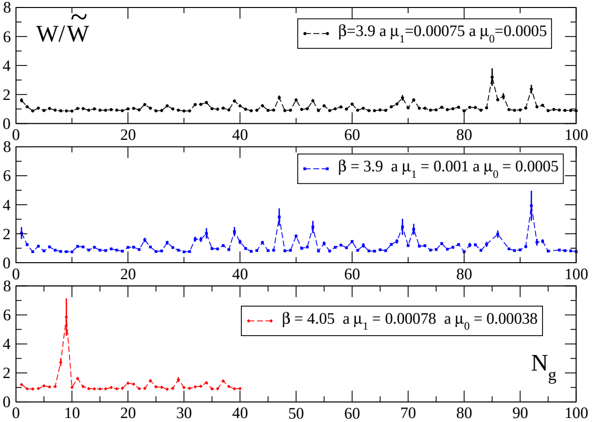
$$\det A_1 = \det \frac{A_1}{A_0} \det A_0, \qquad A_k = Q_{\mathrm{W}}^2 + \mu_k^2$$

$$W = \det \frac{A_1}{A_0} = \frac{1}{Z_\eta} \int d\eta d\eta^\dagger \mathrm{e}^{-\eta^\dagger \frac{A_0}{A_1} \eta}$$

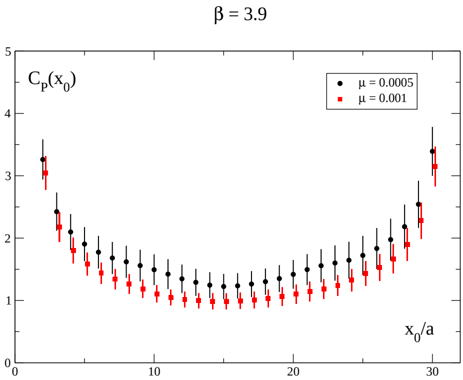
$$W = \frac{1}{Z_\eta} \int d\eta d\eta^\dagger \mathrm{e}^{-\eta^\dagger \eta} \mathrm{e}^{\eta^\dagger \left[1 - \frac{A_0}{A_1}\right] \eta}$$

We need to invert the “new” operator

$$[W]_\eta = \frac{1}{N_\eta} \sum_{i=1}^{N_\eta} \mathrm{e}^{\eta_i^\dagger \left[1 - \frac{A_0}{A_1}\right] \eta_i}$$


W/W̃
β=3.9 a μ1=0.00075 a μ0=0.0005
β = 3.9 a μ1 = 0.001 a μ0 = 0.0005
β = 4.05 a μ1 = 0.00078 a μ0 = 0.00038
Ng

Does it work?



Can be improved

- Use a single stochastic estimator on a smaller mass difference

$$W = \det \frac{A_n}{A_0} = \det \frac{A_n}{A_{n-1}} \det \frac{A_{n-1}}{A_{n-2}} \cdots \det \frac{A_1}{A_0}$$

- ...
- Attenzione: data are correlated

To do list

- Might be useful in the p regime
- Further checks and add data points in the ϵ regime
- Reweight in κ. Important to retune κ_{c}
- Stochastic low mode averaging
- New gauge configurations. Thermalizing $\beta = 4.05$ $L = 24$.
- Use new WχPT to fit data
- Precise determination of Σ, F and W'
- With bigger volumes B_K, g_{27}, ...
- Need to discuss computer and man power available in the future

A provocation

- If we want to discuss what to do next:
- I add to the list $N_f = 2 + 1$
- with Wtm and *overlap* fermions

- Thanks to:
- Jaume, Marianne and all the Grenoble team!!