Heavy Flavour Production in DIS
Two–Loop Massive Operator Matrix Elements and Beyond

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1. Introduction
2. The Method
3. The Calculation
4. Results
5. Comparison to Former Results
6. Conclusion

based on:

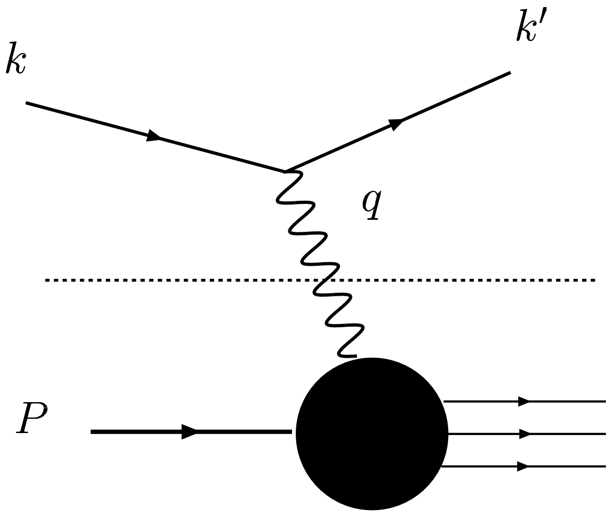
J.B., A. De Freitas, W.L. van Neerven, and S. Klein, Nucl. Phys. **B755** (2006) 272.

I. Bierenbaum, J.B., and S. Klein, Phys. Lett. **B648** (2007) 195; Nucl. Phys. **B780** (2007) 40

and in preparation.;

I. Bierenbaum, J.B., S. Klein, and C. Schneider, `arXiv:0707.4759 [math-ph]`.

Deep–Inelastic Scattering (DIS):



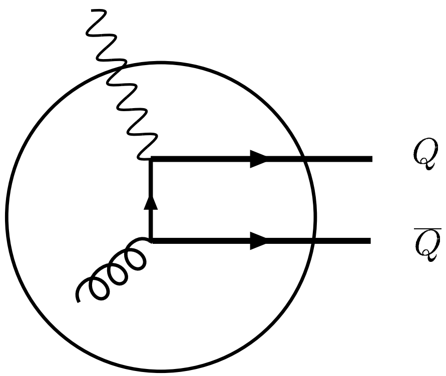
$\rightarrow L^{\mu\nu}$

$\rightarrow W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjørken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

Heavy–flavor production: LO–process: photon–gluon fusion



Hadronic Tensor for heavy quark production via single photon exchange:

$$W_{\mu\nu}^{Q\bar{Q}}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle_{Q\bar{Q}}$$

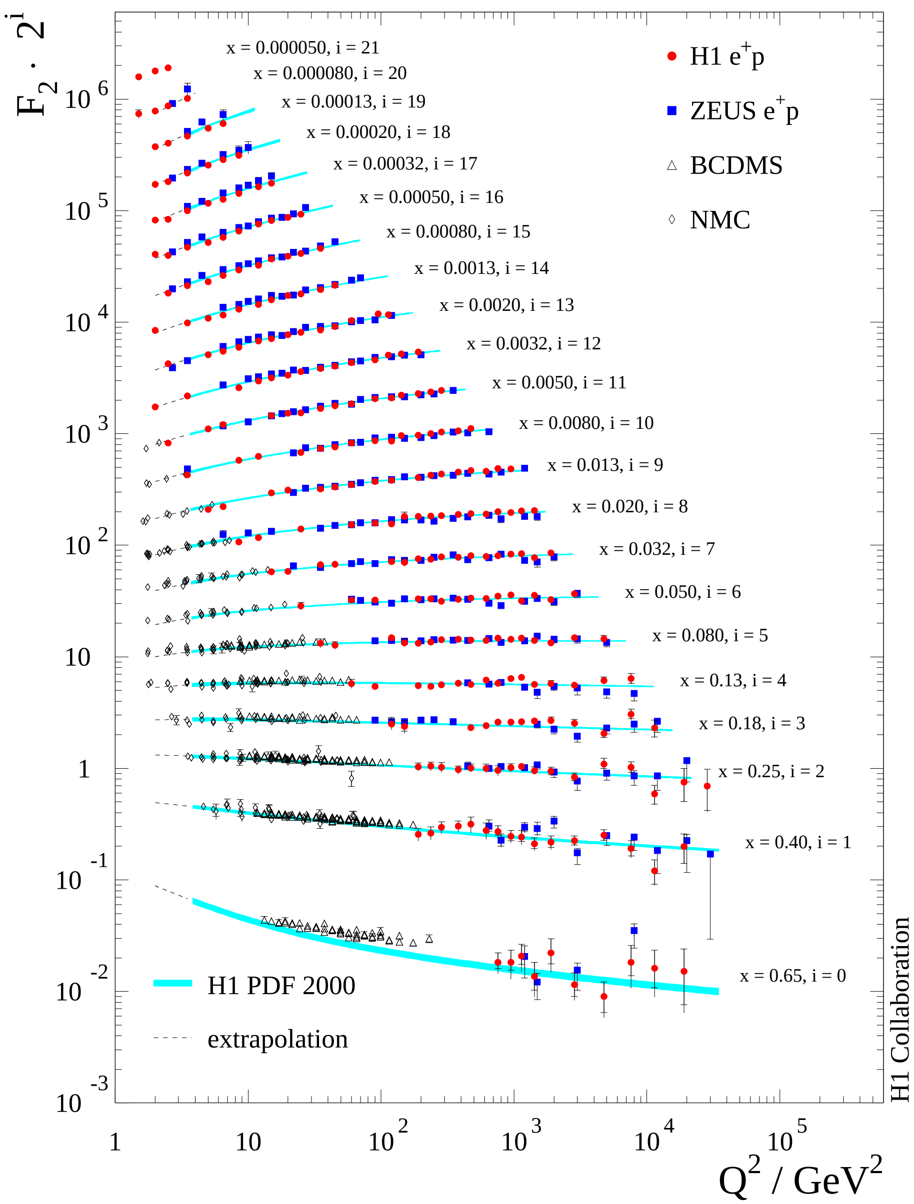
$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L^{Q\bar{Q}}(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2^{Q\bar{Q}}(x,Q^2)$$

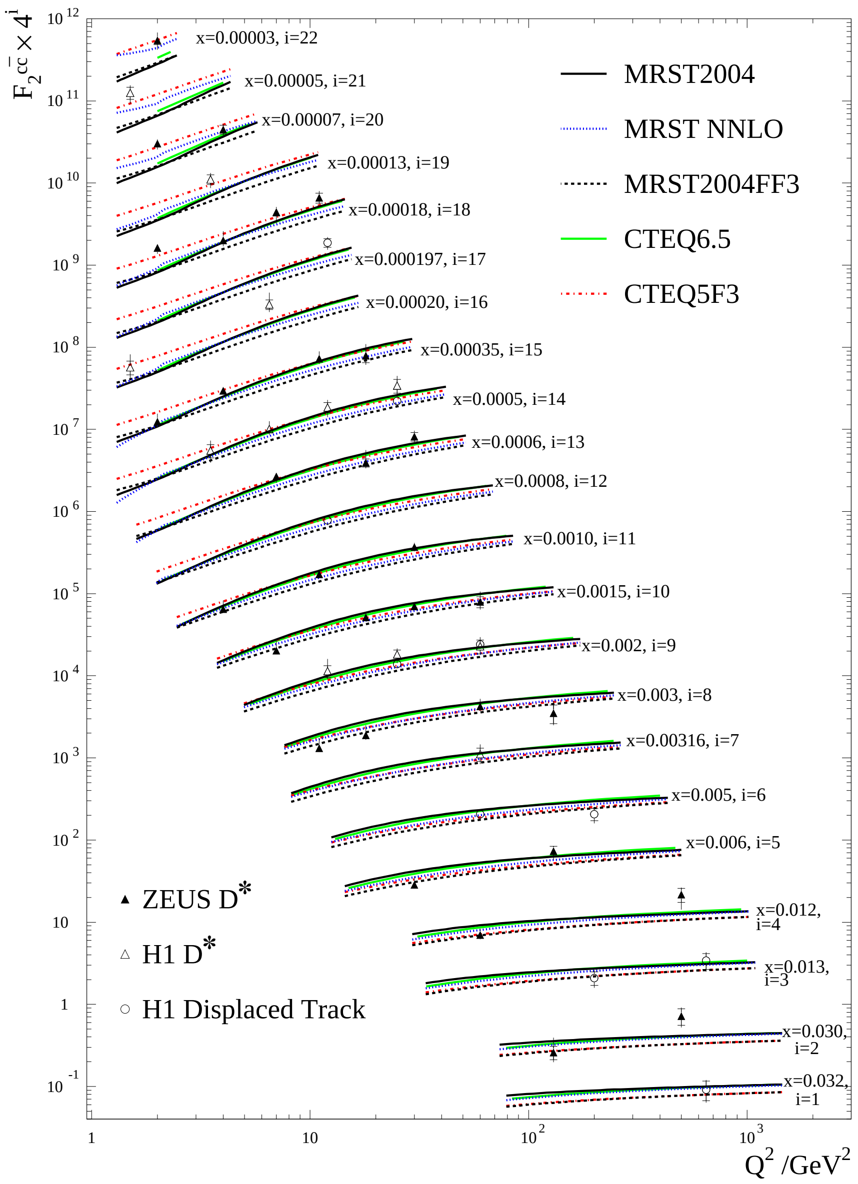
$$- \frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1^{Q\bar{Q}}(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2^{Q\bar{Q}}(x,Q^2)\right] .$$

Bjørken scaling, F_i depends only on x, Q^2–independent
scaling violation, F_i becomes Q^2–dependent

Goal of heavy flavour improved calculation:

- Increase accuracy of perturbative description of structure functions
- More precise definition of the Gluon and Sea Quark Distributions
- QCD analysis and determination of Λ_{QCD} from DIS data





[Thompson, 2007]

Unpolarized DIS :

- LO : [Witten, 1976; Babcock & Sivers, 1978; Shifman, Vainshtein, Zakharov 1978; Leveille & Weiler, 1979]
- NLO : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]
  asymptotic : [Buza, Matiounine, Smith, Migneron, van Neerven, 1996]

Polarized DIS :

- LO : [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]
- NLO : asymptotic: [Buza, Matiounine, Smith, van Neerven, 1997]

Mellin–Space Expressions:
[Alekhin, Blümlein, 2003].

massless RGE and Light–Cone Expansion in Bjørken–Limit $\{Q^2,\ \nu\} \rightarrow \infty$, x fixed:

$$\lim_{\xi^2 \to 0} \left[J(\xi), J(0)\right] \propto \sum_{i,N,\tau} c_{i,\tau}^N(\xi^2, \mu^2) \xi_{\mu_1} \ldots \xi_{\mu_N} O_{i,\tau}^{\mu_1 \ldots \mu_m}(0, \mu^2)$$

Operators: Flavour non-singlet, singlet and pure singlet; consider leading twist-2 operators

mass factorization between Wilson coefficients and parton densities;

$$F_i(x, Q^2) = \sum_j \underbrace{C_i^j\left(x, \frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{non-perturbative}}$$

with $\quad [f \otimes g](z) = \int_0^1 dz_1 \int_0^1 dz_2 \ \ \delta(z - z_1 z_2) \ f(z_1) g(z_2)$.

(massless) RGE: Altarelli–Parisi evolution equations for pdfs ($\mu^2 = Q^2$):

$$\frac{d}{d \ \ln \ Q^2} f_g(x, Q^2) = \sum_{l=0}^{\infty} a_s^{(l+1)}(Q^2) \int_x^1 \frac{dz}{z} \left\{ P_{g \leftarrow q}^{(l)}(z) \sum_f \left[f_f(\frac{x}{z}, Q^2) + f_{\overline{f}}(\frac{x}{z}, Q^2) \right] + P_{g \leftarrow g}^{(l)}(z) f_g(\frac{x}{z}, Q^2) \right\}$$

$P_{i \leftarrow j}^{(l)}(z)$ are the splitting functions.

Heavy quark contribution: heavy quark Wilson coefficient, $H^{\rm S,NS}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)$

The **Renormalization Group Equations**[†] imply factorization for all non-power terms:

$$H^{\rm S,NS}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\rm S,NS}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–Wilson coefficients}}.$$

holds for polarized and unpolarized case in limit $Q^2 \gg m_Q^2$, which means $Q^2/m_Q^2 \geq 10$ for $F_2(x,Q^2)$.

Here $\langle i|A_l|j\rangle$ denote the partonic operator matrix elements,

OMEs obey expansion

$$A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\rm S,NS}_k|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A^{\rm S,NS,(l)}_{k,i}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[† Buza, Matiounine, Migneron, Smith, van Neerven, 1996;
Buza, Matiounine, Smith, van Neerven, 1997.]

Expansion up to $O(\alpha_s^2)$ of unpolarized Heavy Flavor Wilson Coefficient H_2:

$$
\begin{aligned}
H_{2,g}^S\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s\left[A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right)\right]\\
&+ a_s^2\left[A_{Qg}^{(2)}\left(\frac{m^2}{\mu^2}\right)+A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right)\otimes C_{2,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right)+\widehat{C}_{2,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right)\right],\\
H_{2,q}^{PS}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s^2\left[A_{Qq}^{\mathrm{PS},(2)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,q}^{\mathrm{PS},(2)}\left(\frac{Q^2}{\mu^2}\right)\right],\\
H_{2,q}^{\mathrm{NS}}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s^2\left[A_{qq,Q}^{\mathrm{NS},(2)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,q}^{\mathrm{NS},(2)}\left(\frac{Q^2}{\mu^2}\right)\right].
\end{aligned}
$$

- Polarized and longitudinal Heavy Wilson coefficients obey similar expansion.
- For H_L, $O(a_s^3)$ contributions have been derived recently.
  [J. Blümlein, A. De Freitas, W. van Neerven, S. Klein (2006)].

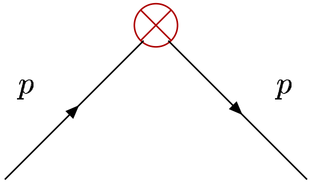
Gluonic Massive Operator Matrix Elements have the same structure in the polarized and unpolarized case. Up to $O(a_s^2)$ they are given by:

$$
\begin{aligned}
A_{Qg}^{(1)} &= -\frac{1}{2}\widehat{P}_{qg}^{(0)}\ln\left(\frac{m^2}{\mu^2}\right) \\
A_{Qg}^{(2)} &= \frac{1}{8}\left\{\widehat{P}_{qg}^{(0)}\otimes\left[P_{qq}^{(0)}-P_{gg}^{(0)}+2\beta_0\right]\right\}\ln^2\left(\frac{m^2}{\mu^2}\right)-\frac{1}{2}\widehat{P}_{qg}^{(1)}\ln\left(\frac{m^2}{\mu^2}\right) \\
&\quad +\overline{a}_{Qg}^{(1)}\left[P_{qq}^{(0)}-P_{gg}^{(0)}+2\beta_0\right]+a_{Qg}^{(2)} \\
A_{Qq}^{\mathrm{PS},(2)} &= -\frac{1}{8}\widehat{P}_{qg}^{(0)}\otimes P_{gq}^{(0)}\ln^2\left(\frac{m^2}{\mu^2}\right)-\frac{1}{2}\widehat{P}_{qq}^{\mathrm{PS},(1)}\ln\left(\frac{m^2}{\mu^2}\right)+a_{Qq}^{\mathrm{PS},(2)}+\overline{a}_{Qg}^{(1)}\otimes P_{gq}^{(0)} \\
A_{qq,Q}^{\mathrm{NS},(2)} &= -\frac{\beta_{0,Q}}{4}P_{qq}^{(0)}\ln^2\left(\frac{m^2}{\mu^2}\right)-\frac{1}{2}\widehat{P}_{qq}^{\mathrm{NS},(1)}\ln\left(\frac{m^2}{\mu^2}\right)+a_{qq,Q}^{\mathrm{NS},(2)}+\frac{1}{4}\beta_{0,Q}\zeta_2 P_{qq}^{(0)}\ .
\end{aligned}
$$

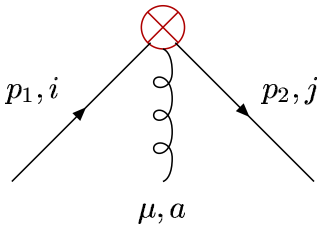
with

$$\widehat{f} = f(N_F+1) - f(N_F)\ .$$

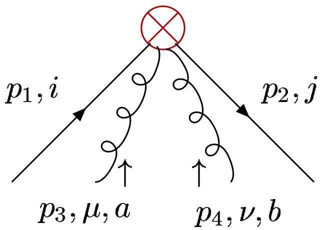
Operator insertions in light–cone expansion:



$$\not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} ,$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} ,$$



$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{0 \le j < l}^{N-2} \Big[(\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_4)^{l-j-1} (\Delta p_2)^j (t^a t^b)_{ji} + (\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_3)^{l-j-1} (\Delta p_2)^j (t^b t^a)_{ji} \Big] ,$$

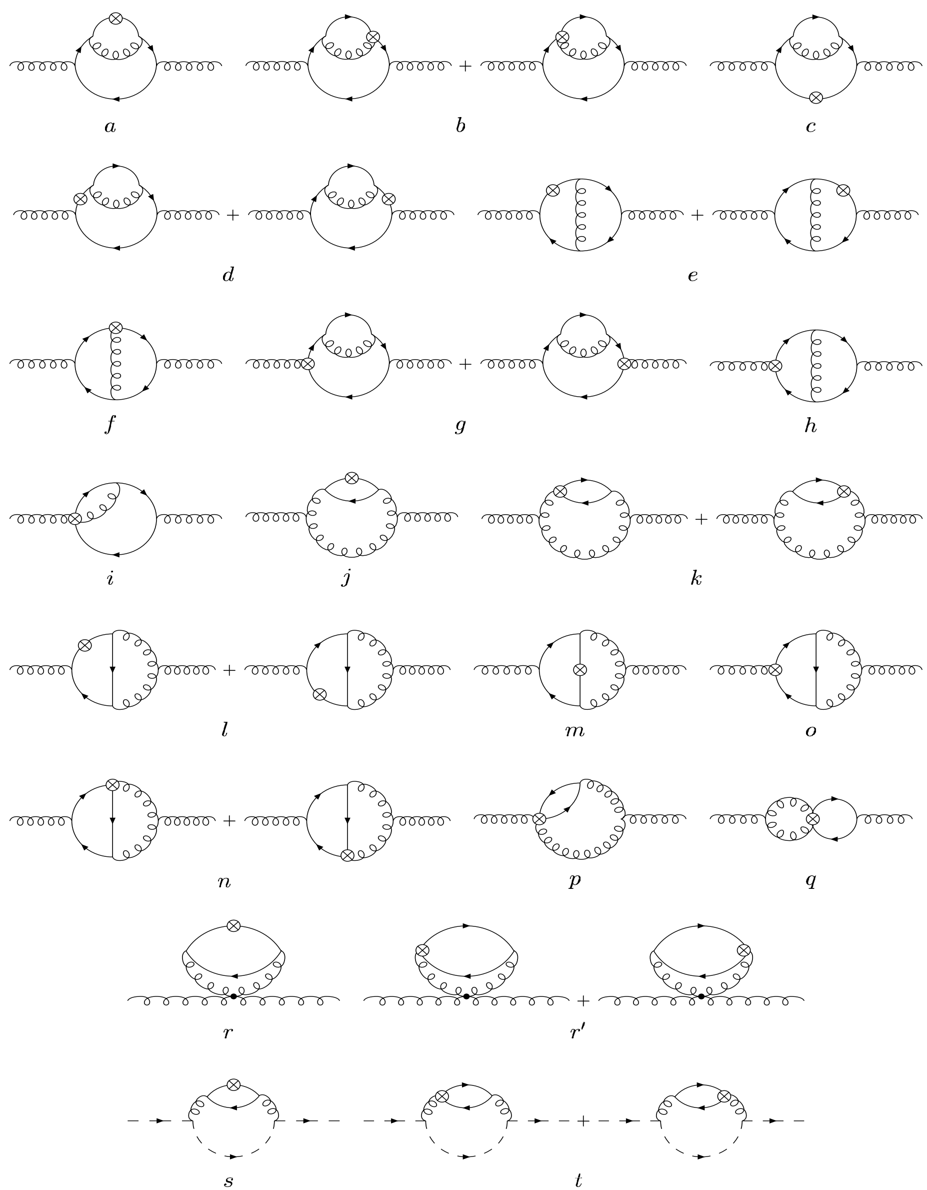
$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

Δ: light–like momentum, $\Delta^2 = 0$.

γ_5 was treated in the 't Hooft–Veltman–Scheme:

$$\not{\Delta} \gamma_5 = \frac{i}{6} \varepsilon_{\mu\nu\rho\sigma} \Delta^\mu \gamma^\nu \gamma^\rho \gamma^\sigma .$$

Diagrams contributing to the gluonic OME $\hat{A}_{Qg}^{(2)}$:



Calculation in Mellin-space: for space–like Q^2: $0 \le x \le 1$:

$$\Rightarrow \quad F(N) = \mathbf{M}[f, N] = \int_0^1 x^{N-1} f(x)\, dx$$

Convolution:

$$[f \otimes g](x) = \int_0^1 dx_1 \int_0^1 dx_2 \;\; \delta(x - x_1 x_2)\; f(x_1) g(x_2) \;,$$

$\Rightarrow$ Product:

$$\mathbf{M}[f \otimes g, N] = \mathbf{M}[f, N]\, \mathbf{M}[g, N] \;\; = F(N)\, G(N).$$

$$\begin{aligned} F_2^{Q\overline{Q}} \;\; = \;\; & \sum_{k=1}^{n_f} e_k^2 \left[f_{k-\overline{k}}(N, \mu^2) H_{2,q}^{NS}\left(N, \frac{Q^2}{m^2}, \frac{Q^2}{\mu^2}\right)\right] \\ & + e_Q^2 \left[\Sigma(N, \mu^2) H_{2,q}^{PS}\left(N, \frac{Q^2}{m^2}, \frac{Q^2}{\mu^2}\right) + G(N, \mu^2) H_{2,g}^{S}\left(N, \frac{Q^2}{m^2}, \frac{Q^2}{\mu^2}\right)\right] \end{aligned}$$

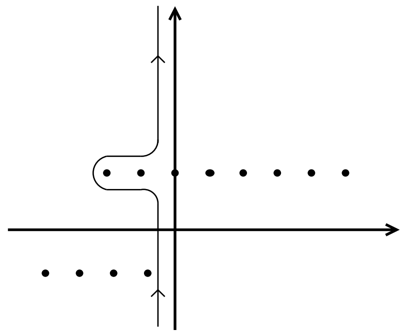
light-quark densities:

$$\begin{aligned} f_{k-\overline{k}}(N, \mu^2) \;\; &= \;\; f_k(N, \mu^2) - f_{\overline{k}}(N, \mu^2), \\ \Sigma(N, \mu^2) \;\; &= \;\; \sum_{k=1}^{n_f} f_{k+\overline{k}}(N, \mu^2). \end{aligned}$$

Our calculation:

- use of Mellin-Barnes integrals

$$\frac{1}{(A+B)^\nu} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\sigma A^\sigma B^{-\nu-\sigma}\frac{\Gamma(-\sigma)\Gamma(\nu+\sigma)}{\Gamma(\nu)}$$



$\rightsquigarrow$ numerical check & some analytic results

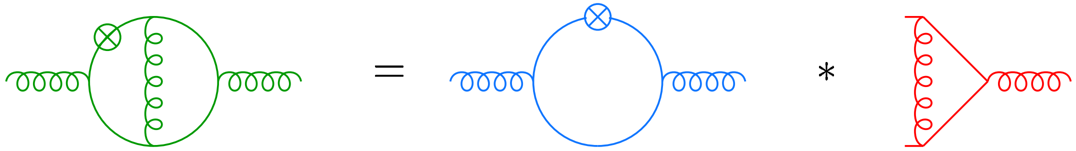
- use of hypergeometric functions for general analytic results

$$_PF_Q\left[\begin{matrix}(a_1)...(a_P)\\(b_1)...(b_Q)\end{matrix};z\right] = \sum_{i=0}^{\infty}\frac{(a_1)_i...(a_P)_i}{(b_1)_i...(b_Q)_i}\frac{z^i}{\Gamma(i+1)}, \quad (c)_i = \frac{\Gamma(c+i)}{\Gamma(c)}.$$

- Summation of (new) infinite one-parameter sums into harmonic sums.
- Algebraic and structural simplification of the harmonic sums [J. Blümlein, 2003]

Calculating scalar Feynman diagrams by Mellin-Barnes integrals:

[I.Bierenbaum, S. Weinzierl, 2003 (massless case); I. Bierenbaum, J. Blümlein and S. Klein, 2006]

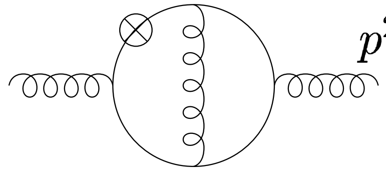


$$I_{e,\nu_1} = \frac{(\Delta p)^{N-1}}{(4\pi)^D (2\pi i)^2} \frac{(m^2)^{D-\nu_{12345}}(-1)^{\nu_{12345}+1}}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(D-\nu_{235})} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} d\sigma \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} d\tau \, \Gamma(-\sigma)\Gamma(\nu_3+\sigma)$$
$$\times \frac{\Gamma(-\sigma+\nu_4+N-1)}{\Gamma(-\sigma+\nu_4)} \Gamma(-\tau)\Gamma(\nu_2+\tau) \frac{\Gamma(\sigma+\tau+\nu_{235}-D/2)\Gamma(\sigma+\tau+\nu_5)}{\Gamma(\sigma+\tau+\nu_{23})}$$
$$\times \Gamma(-\sigma-\tau+D-\nu_{23}-2\nu_5) \frac{\Gamma(-\sigma-\tau+\nu_{14}-D/2)}{\Gamma(-\sigma-\tau+\nu_{14}+N-1)} \, ,$$

N	2	3	4	5
$\mathrm{I}_{e,1}$	+0.49999	+0.31018	+0.21527	+0.16007
$\mathrm{I}_{e,2}$	−0.09028	−0.04398	−0.02519	−0.01596

[package MB, M. Czakon, 2006]

Hypergeometric functions: Example, scalar Diagram e:



$$I_{e,1} := \iint \frac{d^Dq\; d^Dk}{(2\pi)^{2D}} \frac{(\Delta q)^{N-1}}{[q^2-m^2]^a[(q-p)^2-m^2][k^2-m^2][(k-p)^2-m^2][(k-q)^2]}$$

- introduce Feynman parameters
- do momentum integrations

$$I_{e,1} := \frac{(\Delta p)^{N-1}\Gamma(1-\varepsilon)}{N(N+1)(4\pi)^{4+\varepsilon}(m^2)^{1-\varepsilon}} \int_0^1 dz\, dw\, \frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2} z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}} \left[1 - w^{N+1} - (1-w)^{N+1}\right],$$

using $\Delta^2 = 0$.

$$_2F_1\left[\begin{matrix} a, b+1 \\ c+b+2 \end{matrix} ; z\right] = \frac{\Gamma(c+b+2)}{\Gamma(c+1)\Gamma(b+1)} \int_0^1 dx\, x^b (1-x)^c (1-zx)^{-a},$$

$$I_{e,1} = \frac{S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} \frac{(\Delta p)^{N-1}}{N(N+1)} \exp\left\{\sum_{i=2}^{\infty} \zeta_i \frac{\varepsilon^i}{i}\right\}\Bigg\{B(\varepsilon/2+1, 1-\varepsilon/2)B(1,-\varepsilon/2)\, {}_3F_2\left[\begin{matrix} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, 1-\varepsilon/2 \end{matrix}; 1\right]$$

$$- B(\varepsilon/2+1, 1-\varepsilon/2)B(1, N+1-\varepsilon/2)\, {}_3F_2\left[\begin{matrix} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{matrix}; 1\right]$$

$$- B(\varepsilon/2+1, 1-\varepsilon/2)B(N+2, -\varepsilon/2)\, {}_3F_2\left[\begin{matrix} 1-\varepsilon, N+2, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{matrix}; 1\right]\Bigg\}$$

with Beta–function:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}~, \qquad \Gamma(1-\varepsilon) = \exp(\varepsilon\gamma_E)\exp\left\{\sum_{i=2}^{\infty}\zeta_i \frac{\varepsilon^i}{i}\right\}~, \qquad |\varepsilon| < 1~.$$

$$\Psi(x) = \frac{1}{\Gamma(x)}\frac{d}{dx}\Gamma(x) \qquad \Psi(N+1) = S_1(N) - \gamma.$$

harmonic sums: [J. Blümlein and S. Kurth, 1999; J. Vermaseren, 1999]

$$S_{a_1,\ldots,a_m}(N) = \sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}}\frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}}\cdots\frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}}$$

$$N \in \mathbb{N},\ \forall\, \ell,\ a_\ell \in \mathbb{Z}\setminus\{0\}$$

$$I_{e,1} = \frac{-S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} \frac{(\Delta p)^{N-1}}{N(N+1)} \exp\left\{\sum_{i=2}^{\infty} \zeta_i \frac{\varepsilon^i}{i}\right\}$$
$$\times \sum_{s=0}^{\infty} \left\{ \frac{S_1(s) - S_1(1+N+s)}{(1+s)} + \frac{B(N+1, s+1)}{(1+s)} \right\} + O(\varepsilon)$$
$$= \frac{-S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} \frac{(\Delta p)^{N-1}}{N(N+1)} \sum_{s=1}^{\infty} \left\{ -\frac{1}{s^2} + \frac{S_1(s)}{s} - \frac{S_1(N+s)}{s} + \frac{B(N+1, s)}{s} \right\} + O(\varepsilon)$$

$$I_{e,1} = \frac{S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} (\Delta p)^{N-1} \left\{ \frac{S_1^2(N) + 3S_2(N)}{2N(N+1)} + \frac{S_1^3(N) + 3S_1(N)S_2(N) + 8S_3(N)}{12N(N+1)} \varepsilon \right\}$$

More complicated sums → solved both with combinations out of analytic and algebraic methods and also with package SIGMA [C. Schneider, 2007], [I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, `arXiv:0707.4659 [math-ph]`].

Unpolarized case, examples for individual diagrams – numeric:

Diagram	N	$1/\varepsilon^2$	$1/\varepsilon$	1	ε	ε^2
b	2	−8	4.66666	−8.82690	2.47728	−5.69523
	6	−7.73333	0.81936	−8.89777	−1.84111	−7.25674
c	2	−8	39.6	−7.23431	34.66217	6.52891
	6	−2.66666	16.53968	−2.68048	14.25224	2.77564
d	2	−8	7.86666	−6.34542	4.71236	−2.18586
	6	−2.66666	−0.69523	−2.60657	−1.74990	−2.37611
e	2	8.88889	−11.2593	9.82824	−12.8921	2.39145
	6	2.93878	−4.24257	3.39094	−4.3892	0.826978
f	2	5.33333	−9.77777	18.34139	−2.52360	16.20210
	6	3.31428	−6.87289	12.25672	−1.63790	10.86956
g	2	2.66666	−9.55555	4.59662	−8.92015	1.07313
	6	0.57142	−2.00204	1.04814	−1.89142	0.32219

Polarized:
Individual diagrams
– numeric:

Diagram	moment	$1/\varepsilon^2$	$1/\varepsilon$	1	ε	ε^2
a	N = 3	−0.44444	0.12962	−0.26687	−0.30734	−0.12416
	N = 7	−0.06122	0.00819	−0.03339	−0.03800	−0.01278
b	N = 3	4.44444	−1.07407	4.45579	0.515535	3.13754
	N = 7	5.46122	0.74491	6.09646	2.97092	5.35587
c	N = 3	2.66666	−16.28888	0.26606	−13.11030	−5.29203
	N = 7	1.71428	−10.24659	0.28684	−8.21536	−3.19052
d	N = 3	2.66666	−0.02222	2.19940	1.03927	1.69331
	N = 7	1.71428	0.85340	1.78773	1.56227	1.80130
e	N = 3	−2.66666	4.99999	−2.27718	4.89956	0.73208
	N = 7	−1.71428	2.97857	−1.34709	2.83548	0.44608
f	N = 3	0	1.55555	−11.60184	−5.27120	−13.14668
	N = 7	0	2.80210	−7.08455	−1.57130	−7.44933
l	N = 3	−9.33333	0.25000	−8.83933	−3.25228	−6.84460
	N = 7	−6.73877	−1.86855	−7.09938	−4.56050	−6.50099
m	N = 3	−0.44444	1.42592	−0.82397	1.39877	−0.23237
	N = 7	−0.06122	0.22649	−0.11722	0.23939	−0.02415
n	N = 3	−2.22222	1.26851	−1.37562	0.69748	−0.36030
	N = 7	−3.19183	−0.50674	−3.39831	−1.76669	−2.97338

Results to order O(1): [I. Bierenbaum, J. Blümlein, S. Klein, 2006 & 2007]

$$
\begin{aligned}
A_e^{Qg} = T_R \left[C_F - \frac{C_A}{2}\right] \Bigg\{ & \frac{1}{\varepsilon^2}\frac{16(N+3)}{(N+1)^2} + \frac{1}{\varepsilon}\left[- \frac{8(N+2)}{N(N+1)} S_1(N) - 8\frac{3N^3+9N^2+12N+4}{N(N+1)^3(N+2)}\right] \\
& + \Bigg[- 2\frac{9N^4+40N^3+71N^2-12N-36}{N(N+1)^2(N+2)(N+3)} S_2(N) - 2\frac{N^3-N^2-8N-36}{N(N+1)(N+2)(N+3)} S_1^2(N) + 4\frac{(N+3)}{(N+1)^2}\zeta_2 \\
& + 4\frac{4N^5+19N^4+31N^3-30N^2-44N-24}{N^2(N+1)^2(N+2)(N+3)} S_1(N) + \frac{4P_4(N)}{N^2(N+1)^4(N+2)^2(N+3)}\Bigg] \\
& + \varepsilon\Bigg[- 2\frac{N+2}{N(N+1)}\Big(2S_{2,1}(N) + S_1(N)\zeta_2\Big) - \frac{2}{3}\frac{13N^4+60N^3+111N^2+4N-36}{N(N+1)^2(N+2)(N+3)} S_3(N) \\
& - \frac{1}{3}\frac{N^3-N^2-8N-36}{N(N+1)(N+2)(N+3)}\Big(3S_2(N)S_1(N) + S_1^3(N)\Big) - 2\frac{3N^3+9N^2+12N+4}{N(N+1)^3(N+2)}\zeta_2 \\
& + \frac{P_{e1}}{N^2(N+1)^3(N+2)(N+3)} S_2(N) + \frac{4N^5+11N^4+15N^3-86N^2-92N-24}{N^2(N+1)^2(N+2)(N+3)} S_1^2(N) \\
& - 2\frac{P_{e2}}{N^2(N+1)^3(N+2)^2(N+3)} S_1(N) - 2\frac{P_{e3}}{N^3(N+1)^5(N+2)^3(N+3)} + \frac{4}{3}\frac{N+3}{(N+1)^2}\zeta_3\Bigg]\Bigg\}
\end{aligned}
$$

Unpolarized case, Singlet O(1)

$$
\begin{aligned}
a_{Qg}^{(2)}(N) = 4C_F T_R \Bigg\{ & \frac{N^2+N+2}{N(N+1)(N+2)} \Bigg[-\frac{1}{3} S_1^3(N-1) + \frac{4}{3} S_3(N-1) - S_1(N-1) S_2(N-1) \\
& - 2\zeta_2 S_1(N-1) \Bigg] + \frac{N^4+16N^3+15N^2-8N-4}{N^2(N+1)^2(N+2)} S_2(N-1) + \frac{3N^4+2N^3+3N^2-4N-4}{2N^2(N+1)^2(N+2)} \zeta_2 \\
& + \frac{2}{N(N+1)} S_1^2(N-1) + \frac{N^4-N^3-16N^2+2N+4}{N^2(N+1)^2(N+2)} S_1(N-1) + \frac{P_1(N)}{2N^4(N+1)^4(N+2)} \Bigg\} \\
& + 4C_A T_R \Bigg\{ \frac{N^2+N+2}{N(N+1)(N+2)} \Bigg[4\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1) + \frac{1}{3} S_1^3(N) + 3 S_2(N) S_1(N) \\
& + \frac{8}{3} S_3(N) + \beta''(N+1) - 4\beta'(N+1) S_1(N) - 4\beta(N+1)\zeta_2 + \zeta_3 \Bigg] - \frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2} S_1^2(N) \\
& - 2\frac{N^4-2N^3+5N^2+2N+2}{(N-1)N^2(N+1)^2(N+2)} \zeta_2 - \frac{7N^5+21N^4+13N^3+21N^2+18N+16}{(N-1)N^2(N+1)^2(N+2)^2} S_2(N) \\
& - \frac{N^6+8N^5+23N^4+54N^3+94N^2+72N+8}{N(N+1)^3(N+2)^3} S_1(N) - 4\frac{N^2-N-4}{(N+1)^2(N+2)^2} \beta'(N+1) \\
& + \frac{P_2(N)}{(N-1)N^4(N+1)^4(N+2)^4} \Bigg\} .
\end{aligned}
$$

Unpolarized case, Singlet O(ε)

$$\begin{aligned}
\overline{a}_{Qg}^{(2)} = T_F C_F \Biggl\{ & \frac{2}{3} \frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)} S_1^2 \\
& + \frac{N^2+N+2}{N(N+1)(N+2)} \Bigl(16 S_{2,1,1} - 8 S_{3,1} - 8 S_{2,1} S_1 + 3 S_4 - \frac{4}{3} S_3 S_1 - \frac{1}{2} S_2^2 - S_2 S_1^2 - \frac{1}{6} S_1^4 + 2\zeta_2 S_2 - 2\zeta_2 S_1^2 - \frac{8}{3} \zeta_3 S_1 \Bigr) \\
& - 8 \frac{N^2-3N-2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 + \frac{2}{3} \frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2 S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
& + \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3} \zeta_2 - 2 \frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \Biggr\} \\
+ T_F C_A \Biggl\{ & \frac{N^2+N+2}{N(N+1)(N+2)} \Bigl(16 S_{-2,1,1} - 4 S_{2,1,1} - 8 S_{-3,1} - 8 S_{-2,2} - 4 S_{3,1} - \frac{2}{3} \beta''' + 9 S_4 - 16 S_{-2,1} S_1 \\
& + \frac{40}{3} S_1 S_3 + 4\beta'' S_1 - 8\beta' S_2 + \frac{1}{2} S_2^2 - 8\beta' S_1^2 + 5 S_1^2 S_2 + \frac{1}{6} S_1^4 - \frac{10}{3} S_1 \zeta_3 - 2 S_2 \zeta_2 - 2 S_1^2 \zeta_2 - 4\beta' \zeta_2 - \frac{17}{5} \zeta_2^2 \Bigr) \\
& + \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2} \Bigl(-4 S_{-2,1} + \beta'' - 4\beta' S_1 \Bigr) - \frac{2}{3} \frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2} S_1^3 + 8 \frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3} \beta' \\
& + 2 \frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2} S_2 S_1 - \frac{16}{3} \frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2} S_3 - 8 \frac{N^2+N-1}{(N+1)^2(N+2)^2} \zeta_2 S_1 \\
& - \frac{2}{3} \frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3} S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2} \zeta_2 \\
& - \frac{P_5}{N(N+1)^3(N+2)^3} S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4} S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5} \Biggr\} .
\end{aligned}$$

Unpolarized case, pure-singlet and non-singlet

$$a_{Qq}^{\mathrm{PS},(2)} = C_F T_R \Biggl\{ \Biggl[-4 \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \Bigl(2S_2(N) + \zeta_2 \Bigr) + \frac{4P_5(N)}{(N-1)N^4(N+1)^4(N+2)^3} \Biggr]$$
$$+ \varepsilon \Biggl[-2 \frac{(5N^3+7N^2+4N+4)(N^2+5N+2)}{(N-1)N^3(N+1)^3(N+2)^2} \Bigl(2S_2(N) + \zeta_2 \Bigr)$$
$$- \frac{4}{3} \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \Bigl(3S_3(N) + \zeta_3 \Bigr) + 2 \frac{P_9}{(N-1)N^5(N+1)^5(N+2)^4} \Biggr] \Biggr\} .$$

$$a_{qq,Q}^{\mathrm{NS},(2)} = C_F T_R \Biggl\{ \Biggl[-\frac{8}{3} S_3(N) - \frac{8}{3} \zeta_2 S_1(N) + \frac{40}{9} S_2(N) + 2 \frac{3N^2+3N+2}{3N(N+1)} \zeta_2 - \frac{224}{27} S_1(N)$$
$$+ \frac{219N^6+657N^5+1193N^4+763N^3-40N^2-48N+72}{54N^3(N+1)^3} \Biggr]$$
$$+ \varepsilon \Biggl[\frac{4}{3} S_4(N) + \frac{4}{3} S_2(N) \zeta_2 - \frac{8}{9} S_1(N) \zeta_3 - \frac{20}{9} S_3(N) - \frac{20}{9} S_1(N) \zeta_2 + 2 \frac{3N^2+3N+2}{9N(N+1)} \zeta_3 + \frac{112}{27} S_2(N)$$
$$+ \frac{3N^4+6N^3+47N^2+20N-12}{18N^2(N+1)^2} \zeta_2 - \frac{656}{81} S_1(N) + \frac{P_8}{648N^4(N+1)^4} \Biggr] \Biggr\} .$$

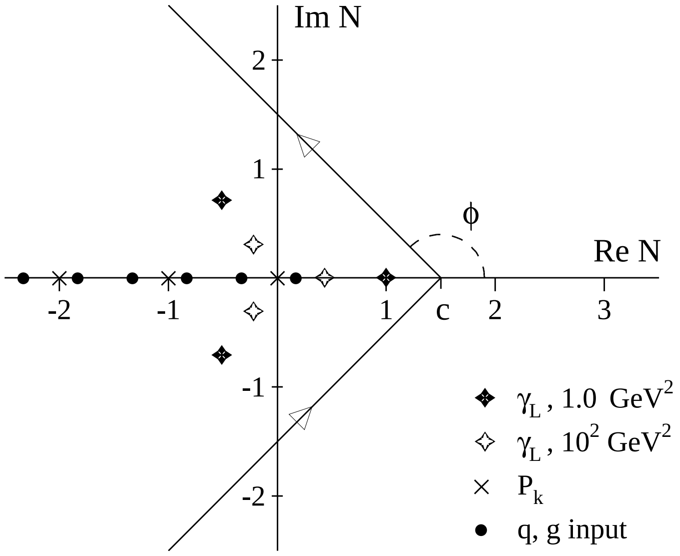
Polarized case, Singlet

$$
\begin{aligned}
a_{Qg}^{(2)} =& C_F T_R \Biggl\{ 4\frac{N-1}{3N(N+1)}\Bigl(-4S_3(N)+S_1^3(N)+3S_1(N)S_2(N)+6S_1(N)\zeta_2\Bigr) \\
& -4\frac{N^4+17N^3+43N^2+33N+2}{N^2(N+1)^2(N+2)}S_2(N)-4\frac{3N^2+3N-2}{N^2(N+1)(N+2)}S_1^2(N) \\
& -2\frac{(N-1)(3N^2+3N+2)}{N^2(N+1)^2}\zeta_2-4\frac{N^3-2N^2-22N-36}{N^2(N+1)(N+2)}S_1(N)-\frac{2P_3(N)}{N^4(N+1)^4(N+2)}\Biggr\} \\
& +C_A T_R \Biggl\{ 4\frac{N-1}{3N(N+1)}\Bigl(12\mathbf{M}\Bigl[\frac{\mathrm{Li}_2(x)}{1+x}\Bigr](N+1)+3\beta''(N+1)-8S_3(N)-S_1^3(N) \\
& -9S_1(N)S_2(N)-12S_1(N)\beta'(N+1)-12\beta(N+1)\zeta_2-3\zeta_3\Bigr)-16\frac{N-1}{N(N+1)^2}\beta'(N+1) \\
& +4\frac{N^2+4N+5}{N(N+1)^2(N+2)}S_1^2(N)+4\frac{7N^3+24N^2+15N-16}{N^2(N+1)^2(N+2)}S_2(N)+8\frac{(N-1)(N+2)}{N^2(N+1)^2}\zeta_2 \\
& +4\frac{N^4+4N^3-N^2-10N+2}{N(N+1)^3(N+2)}S_1(N)-\frac{4P_4(N)}{N^4(N+1)^4(N+2)}\Biggr\} .
\end{aligned}
$$

[J. Blümlein and S. Klein, 2007]

Heavy Flavor Wilson Coefficient for experimental use :

Inversion from Mellin-space to z-space: [J. Blümlein, ANCONT, 2000]



Continuation of harmonic sums:

$S_1(N) = \Psi(N+1) + \gamma$,
etc.

$$F_2^{Q\bar{Q}}(x, Q^2) = \int_0^\infty dz\, \mathrm{Im}\left[\, e^{i\Phi} x^{-c(z)}\, F_2^{Q\bar{Q}}(c(z), Q^2)\,\right],$$
$$c(z) = c_0 + z e^{i\Phi}$$

First Calculation to $O(\alpha_S^2)$: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996]

$\rightsquigarrow$ Integration-by-parts method

$\rightsquigarrow$ direct integration of individual Feynman-parameter integrals in z-space

$\Rightarrow$ combinations of Nielsen integrals: $S_{p,n}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx)$

$\delta(1-x)$	1	$\ln(x)$	$\ln^2(x)$	$\ln^3(x)$	$\ln(1-x)$
$\ln^2(1-x)$	$\ln^3(1-x)$	$\ln(x)\ln(1-x)$	$\ln(x)\ln^2(1-x)$	$\ln^2(x)\ln(1-x)$	$\ln(1+x)$
$\ln(x)\ln(1+x)$	$\ln^2(x)\ln(1+x)$	$\mathrm{Li}_2(1-x)$	$\ln(x)\mathrm{Li}_2(1-x)$	$\ln(1-x)\mathrm{Li}_2(1-x)$	$\mathrm{Li}_3(1-x)$
$S_{1,2}(1-x)$	$S_{1,2}(-x)$	$\frac{1}{1-x}$	$\frac{1}{1+x}$	$\frac{\ln(x)}{1-x}$	$\frac{\ln^2(x)}{1-x}$
$\frac{\ln^3(x)}{1-x}$	$\frac{\ln(x)}{1+x}$	$\frac{\ln^2(x)}{1+x}$	$\frac{\ln^3(x)}{1+x}$	$\frac{\ln(1+x)}{1+x}$	$\frac{\ln(x)\ln(1+x)}{1+x}$
$\frac{\ln(x)\ln^2(1+x)}{1+x}$	$\frac{\ln^2(x)\ln(1+x)}{1+x}$	$\frac{\ln(x)\ln(1-x)}{1-x}$	$\frac{\ln(x)\ln^2(1-x)}{1-x}$	$\frac{\ln(1-x)\mathrm{Li}_2(x)}{1-x}$	$\frac{\mathrm{Li}_2(1-x)}{1-x}$
$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x}$	$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1+x}$	$\frac{\ln(1+x)\mathrm{Li}_2(-x)}{1+x}$	$\ln(1+x)\mathrm{Li}_2(-x)$	$\mathrm{Li}_2(-x)$	$\frac{\mathrm{Li}_2(-x)}{1+x}$
$\frac{\ln(x)\mathrm{Li}_2(-x)}{1+x}$	$\frac{\mathrm{Li}_3(1-x)}{1-x}$	$\frac{\mathrm{Li}_3(-x)}{1+x}$	$\frac{S_{1,2}(1-x)}{1-x}$	$\frac{S_{1,2}(1-x)}{1+x}$	$\frac{S_{1,2}(-x)}{1+x}$

Complexity of the results in Mellin space, unpolarized case to order $O(\varepsilon)$:

Diag	S_1	S_2	S_3	S_4	S_{-2}	S_{-3}	S_{-4}	$S_{2,1}$	$S_{-2,1}$	$S_{-2,2}$	$S_{3,1}$	$S_{-3,1}$	$S_{2,1,1}$	$S_{-2,1,1}$
a		++	+											
b	++	++	++	+				++			+		+	
c		++	+											
d	++	++	+					+						
e	++	++	+					+						
f	++	++	++	+				++					+	
g	++	++	+					+						
h	++	++	+					+						
i	++	++	++	+	++	++	+	++	++	+	+	+	+	+
j		++	+											
k		++	+											
l	++	++	++	+				++			+		+	
m		++	+											
n	++	++	++	+	++	++	+	++	++	+	+	+	+	+
o	++	++	++	+				++			+		+	
p	++	++	++	+				++			+		+	
s		++	+											
t		++	+											
PS_a		++	+											
PS_b		++	+											
NS_a														
NS_b	++	++	++	+										
Σ	++	++	++	+	++	++	+	+	++	+	+	+	+	+

van Neerven et al. to O(1): unpolarized: 48 basic functions; polarized: 24 basic functions.

O(1): $\{S_1,\ S_2,\ S_3,\ S_{-2}, S_{-3}\}$, $S_{-2,1} \Longrightarrow$ 2 basic objects.

O(ε): $\{S_1,\ S_2,\ S_3,\ S_4,\ S_{-2}, S_{-3}, S_{-4}\}$, $S_{2,1}$, $S_{-2,1}$, $S_{-3,1}$, $S_{2,1,1}$, $S_{-2,1,1}$

$S_{-2,2}$ depends on $S_{-2,1}$, $S_{-3,1}$

$S_{3,1}$ depends on $S_{2,1}$

$\Longrightarrow$ 6 basic objects

These objects are in common to all single scale higher order processes.

Str. Functions, DIS HQ, Fragm. Functions, DY, Hadr. Higgs-Prod., s+v contr. to Bhabha scatt., ...

- Structure of expression is given by

$$\begin{aligned}
\beta(N+1) &= (-1)^N[S_{-1}(N) + \ln(2)]\ , \\
\beta^{(k)}(N+1) &= \Gamma(k+1)(-1)^{N+k}[S_{-k-1}(N) + (1-2^{-k})\zeta_{k+1}]\ ,\ k \geq 2\ , \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1) - \zeta_2\beta(N+1) &= (-1)^{N+1}[S_{-2,1}(N) + \frac{5}{8}\zeta_3]
\end{aligned}$$

- $\Longrightarrow$ harmonic sums with index $\{-1\}$ cancel (holds even for each diagram)

  [cf. J.B., 2004; J.B.and V. Ravindran, 2005,2006; J.B. and S. Klein, `arXiv: 0706.2426 [hep-ph]`, J.B. and S. Moch in preparation.]

Calculation of quark–mass effects in QCD Wilson–coefficients in asymptotic regime $Q^2 \gg m^2$

- Calculation in Mellin space, no use of the IBP-Method
  → essential for simplification of calculation
- Use of Mellin–Barnes integrals (mainly numerical checks) and generalized hypergeometric functions, new summation techniques
- Results in term of nested harmonic sums
  → use of algebraic relations of harmonic sums for simplification of results
  → up to $O(\varepsilon)$ the usual six basic harmonic sums contribute
- Calculation of the constant term of the Operator Matrix Elements
  → full agreement with results of van Neerven et al. (in a certain scheme).
- **New:** Calculation of the $\mathrm{O}(\varepsilon)$ term of the two-loop OMEs a_{Qg}, a_{qq} complete, necessary for the calculation of the Heavy Wilson coefficients up to $\mathrm{O}(\alpha_s^3)$