Precision QCD: 50 Years Later

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Introduction

This talk is dedicated to our friend and colleague Harald Fritzsch.

Loops and Legs 2004, Zinnowitz

Introduction

QCD 2009, Berlin

We first met through preprints over the iron curtain. IfH evaluation $\rightarrow$ DESY (1990); Special Humboldt Fellowship 1990/91 LMU;
Met at many conferences; Harald came often to DESY Zeuthen; Common conference series: QCD@LHC, annually since 2010 (with M. Mangano).

Introduction

The Beginning

1972/73: SU_{3c}

H. Fritzsch, M. Gell-Mann and H. Leutwyler, Phys. Lett. B 47 (1973) 365.

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{k=1}^{N_F} \overline{\psi}_k (i\gamma^\mu D_\mu - m_k)\psi_k + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}$$

Yang-Mills:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \quad D_\mu = \partial_\mu - igT^a A^a_\mu$$

- Renormalizability, 't Hooft (1971)
- Anomaly Freedom of the SM, Bouchiat, Iliopoulos, Meyer (1972)
- Asymptotic freedom: $\beta(g(N_F)) < 1$ allowed perturbation theory to higher orders, Gross, Wilczek, Politzer (1973).
- **Precision analyzes of the strong sector are possible.**

What do we want to know ?

Fundamental quantities of the Standard Model

- $g_s = \sqrt{4\pi\alpha_s}$
- m_c, m_b, m_t
- The twist–2 parton densities: $q_i, \overline{q}_i, g,\ \Delta q_i, \Delta\overline{q}_i, \Delta g$.

- as precisely as possible.

Because of confinement: high virtualities Q^2 are needed to probe these quantities.

- Deep-inelastic scattering (fixed target, HERA, EIC)
- hard scattering processes at e^+e^- (LEP) and hadron colliders (LHC)

The QCD β function

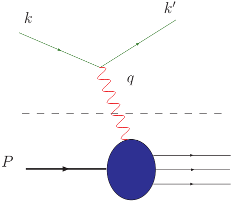
- 1973 : One loop: Gross, Wilczek; Politzer
- 1974 : Two loops: Caswell; Jones
- 1980 : Three loops: Tarasov, Vladimirov, Zharkov (conf.: Larin, Vermaseren, 1993; Chetyrkin, Misiak, Münz, 1997)
- 1997 : Four loops: Larin, van Ritbergen, Vermaseren (conf.: Czakon, 2005)
- 2016/17 : Five loops: Baikov, Chetyrkin, Kühn; Herzog, Ruijl, Ueda, Vermaseren, Vogt; Luthe, Maier, Marquard, Schröder

$$\frac{da_s}{d\ln(\mu^2)} = -\sum_{k=0}^{\infty} \beta_k a_s^{k+2}, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$\beta_0 = 11 - \frac{2}{3} N_F$$

$$\beta(\alpha_s, N_F = 5) = \beta_0[1 + 0.40135\alpha_s + 0.14943\alpha_s^2 + 0.31722\alpha_s^3 + 0.08092\alpha_s^4]$$

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.
At 3-Loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z) \ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x) \ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Moch, Vermaseren, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].

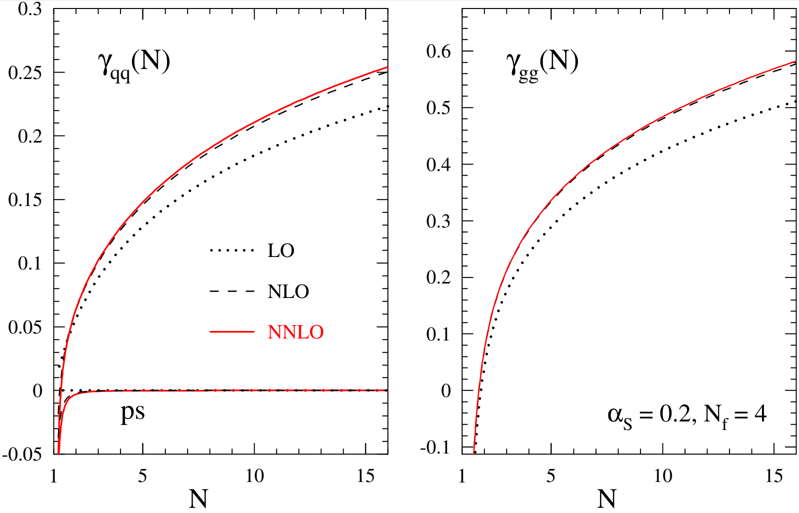
For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The anomalous dimensions

- 1973 : One loop: Gross, Wilczek; Georgi, Politzer
- 1975 : One loop: Sasaki; Ahmed, Ross [pol.]
- 1977 : One loop: partonic splittig functions Altarelli, Parisi
- 1977/92 : Two loops: Floratos, Ross, Sachrajda, Gonzalez-Arroyo, Lopez, Yndurain, Curci, Furmanski, Petronzio, Kounnas, Floratos, Lacaze, Hamberg, van Neerven
- 1995 : Two loops: Mertig, van Neerven; Vogelsang [pol.]
- 2004 : Three loops: Moch, Vermaseren, Vogt; Ablinger et al. (2014, 2017); Anastasiou et al. (2015); Mistlberger (2018); Duhr et al. (2020); Luo et al. (2019); Ebert et al. (2020), JB, Marquard, Schneider, Schönwald (2021); Baranowski et al. (2022); Gehrmann et al. (2023)
- 2014 : Three loops: Moch, Vermaseren, Vogt; Behring et al. (2019); JB, Marquard, Schneider, Schönwald (2021) [pol.]
- 2006(16): Four loops: [Moments] Baikov, Chetyrkin, Kühn; Velizhanin; Davies, Falcioni, Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt
- > 1994 : Large N_F expansion to all orders Gracey et al.

Different methods: Forward Compton amplitude; on-shell massive OMEs; off-shell massless OMEs.

Anomalous dimensions

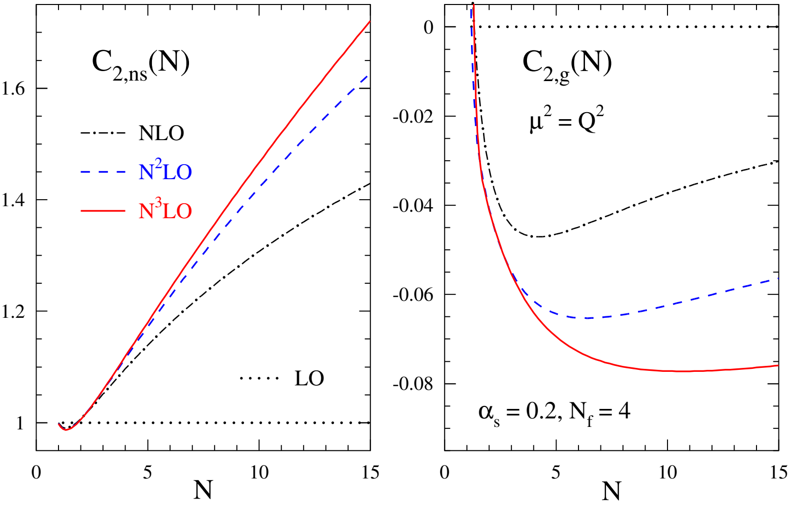


from: Moch, Vermaseren, Vogt, 2004

The massless DIS Wilson coefficients

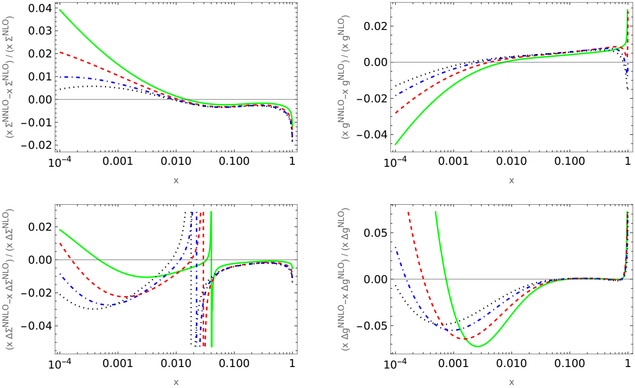
- 1979/80: One loop: Bardeen, Buras, Duke, Muta + various other authors. Consensus on correct results are reached in: Furmanski, Petronzio
- 1989 : One loop: Bodwin, Qiu [pol.]
- 1987/92: Two loops: Kazakov, Kotikov; Sanchez-Guillen et al., van Neerven, Zijlstra; Moch, Vermaseren (1999)
- 1993 : Two loops: van Neerven, Zijlstra [pol.]
- 2005/22: Three loops: Moch, Vermaseren, Vogt; JB, Marquard, Schneider, Schönwald
- 2022 : Three loops: JB, Marquard, Schneider, Schönwald [pol.]
- 2023 : Four loops: NS $O(N_F^2)$ Basdew-Sharma, Pelloni, Herzog, Vogt

Wilson coefficients



from: Moch, Vermaseren, Vogt, 2005

The unpolarized and polarized NNLO evolution



$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines.

At the 1% level of DIS structure functions N^{2}LO is not yet enough at the theory side. High luminosity measurements at EIC and perhaps LHeC.

Why are Heavy Flavor Contributions important ?

- They form a significant contribution to F_2, F_L and g_1 particularly at small x and high Q^2.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching, including 2–mass corrections.

Heavy flavor Wilson coefficients

- 1976/80 : One loop: Witten; Babcock, Sivers, Wolfram; Shifman, Vainshtein, Zakharov; Leveille, Weiler; Glück, Hoffmann, Reya
- 1981/90 : One loop: Watson; Glück, Reya, Vogelsang [pol.]
- 1992 : Two loops: Laenen, Riemersma, Smith, van Neerven
- 1995 : Two loops: Buza, Matiounine, Smith, van Neerven; Hekhorn, Stratmann (2018) [pol.]
- 2010/now: Three loops: Ablinger, Behring, JB, De Freitas, Hasselhuhn, von Manteuffel, Raab, Round, Saragnese, Schneider, Schönwald, Wißbrock [unpol + pol.] + two-mass case.

The Wilson Coefficients at large Q^2

2014
$$L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) = a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\;\delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] + a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\;\delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]$$

2010
$$L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^3\Big[\;A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2 + A_{gq,Q}^{(2)}(N_F)\;\; N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F$$

2010
$$L_{g,(2,L)}^{\mathsf{S}}(N_F+1) = a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\;A_{qg,Q}^{(3)}(N_F+1)\;\delta_2 + A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F + + A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\;,$$

2014
$$H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^2\Big[\;A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;\delta_2 + \;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\;A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2 + \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\;,$$

$$H_{g,(2,L)}^{\mathsf{S}}(N_F+1) = a_s\Big[\;A_{Qg}^{(1)}(N_F+1)\;\delta_2 + \;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\;A_{Qg}^{(2)}(N_F+1)\;\delta_2 + A_{Qg}^{(1)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] + \;a_s^3\Big[\;A_{Qg}^{(3)}(N_F+1)\;\delta_2 + \;A_{Qg}^{(2)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F + + A_{gg,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \;A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} + \;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + \;\tilde{C}_{g,(2,L)}^{(3)}(N_F+1$$

All first order factorizable contributions to $H_{Qg}^{(3)}$ are known since 2017.
All logarithmic corrections are known since 2010.

Variable Flavor Number Scheme

$$f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2) = A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\bar{k}}(n_f,\mu^2)\right]$$

$$+ \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)$$

$$f_{Q+\bar{Q}}(n_f+1,\mu^2) = \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\ .$$

$$G(n_f+1,\mu^2) = A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\ .$$

$$\Sigma(n_f+1,\mu^2) = \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2)\right]$$

$$= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f\,\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right]$$

$$\otimes\Sigma(n_f,\mu^2)$$

$$+\left[n_f\,\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)$$

$A_{gq}^{(3)}$: 2014, $A_{gg}^{(3)}$: 2022

There are generalizations necessary in the 2-mass case.

J. Ablinger et al., Nucl.Phys. B921 (2017) 585 [arXiv:1705.07030]

Computer-algebraic framework: Principal computation steps

- QGRAF, Nogueira, 1993 Diagram generation
- FORM, Vermaseren, 2001; Tentyukov, Vermaseren, 2010 Lorentz algebra
- Color, van Ritbergen, Schellekens and Vermaseren, 1999 Color algebra
- Reduze 2 Studerus, von Manteuffel, 2009/12, Crusher, Marquard, Seidel IBPs
- Method of arbitrary high moments, JB, Schneider, 2017 Computing large numbers of Mellin moments
- Guess, Kauers et al. 2009/2015; JB, Kauers, Schneider, 2009 Computing the recurrences
- Sigma, EvaluateMultiSums, SolveCoupledSystems, Schneider, 2007/14 Solving the recurrences
- OreSys, Zürcher, 1994; Gerhold, 2002; Bostan et al., 2013 Decoupling differential and difference equations
- Diffeq, Ablinger et al, 2015, JB, Marquard, Rana, Schneider, 2018 Solving differential equations
- HarmoncisSums, Ablinger and Ablinger et al. 2010-2019 Simplifying nested sums and iterated integrals to basic building blocks, performing series and asymptotic expansions, Almkvist-Zeilberger algorithm etc.

How to integrate analytically ?

In the massive case the simple view of just harmonic sums, like widely in the massless case, fails.

Anti-Differentiation:

- Risch-like algorithms: define the proper function space, in which your problem can be solved
- obtain the basis of this function space over which the integrals are expressed
- shuffle algebras reduce the original integrals

Iterative integrals over $\mathfrak{A}$:

$$G_{a,\vec{b}}(x) = \int_0^x dy f_b(y) G_{\vec{b}}(y)$$

Alphabet:

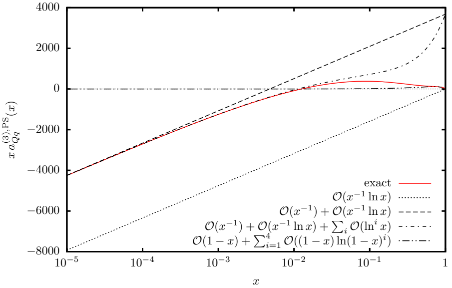
$$\mathfrak{A} = \left\{ \frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, ..., g(x), ... \right\}$$

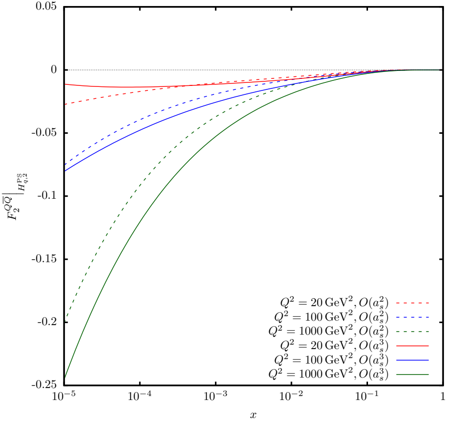
Here $g(x)$ can be a higher transcendental function, which is no iterative integral.

For more details on the different methods see:

JB and C. Schneider, Int. J. Mod. Phys. **A 33** (2018) 17, 1830015.

$H_{q,2}^{\rm PS}$



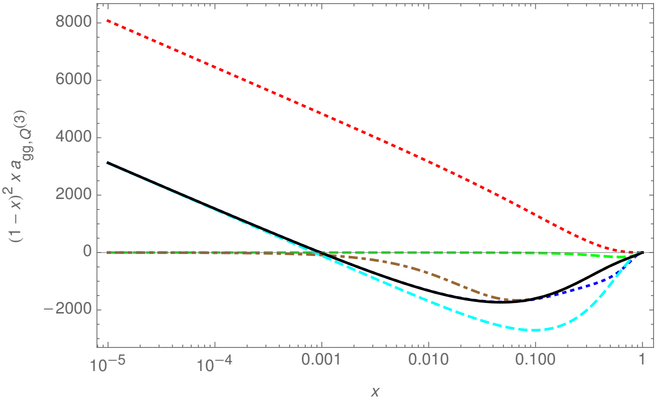


$a_{Qq}^{(3),\rm PS}$

Contribution to $F_2(x, Q^2)$

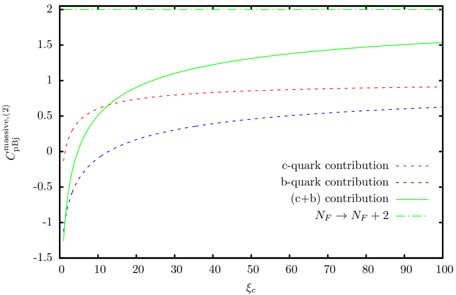
The leading small x approximation corresponding to High-energy factorization and small x heavy flavor production S. Catani, M. Ciafaloni, F. Hautmann, Nucl.Phys. B366 (1991) 135 departs from the physical result everywhere except for $x = 1$ (dotted line).

$a_{gg,Q}^{(3)}$

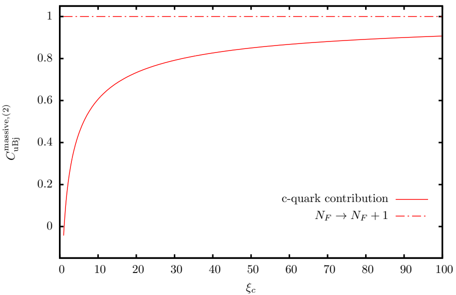


The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$, BFKL limit; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution.

$O(\alpha_s^2)$ Complete NS corrections



pol BJ sum rule



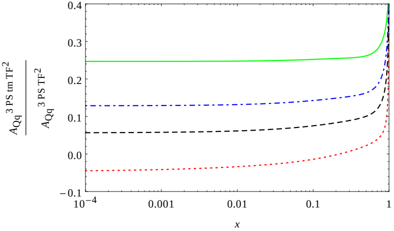
unp. BJ sum rule

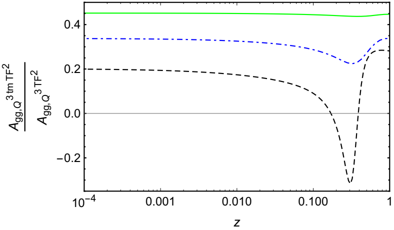
Note the negative corrections at low Q^2!

Already for charm it takes quite a while to become massless.

JB, G. Falcioni, A. De Freitas, Nucl. Phys. B910 (2016) 568.

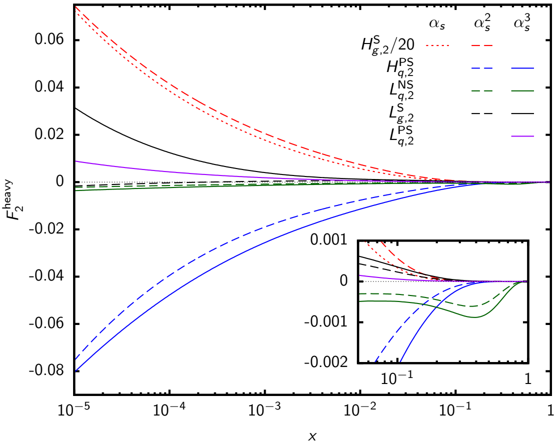
3-Loop 2 Mass contributions: PS and gg





The 2-mass contributions are a significant part of the the T_F^2 terms (also in the other channels).

The present NC corrections to $F_2(x, Q^2)$



$Q^2 = 100\text{GeV}^2$ [$H_{g,2}^S$ scaled down by a factor 20.]

We have calculated 18 of 28 color and ζ-factors of $A_{Qg}^{(3)}$, as well as 2000 moments [15000 in the T_F^2 case] analytically. (MATAD, 2009: $N \leq 10$).
Here the method of arbitrary high moments proved to be crucial.

Conclusions

- After 50 years we know a lot more on strong interactions at very small distances.
- QCD is very well established and reached a very good level of precision.
- We have learnt much more than believed on the structure of Feynman integrals and what QFTs are at the quantitative side.
- Theoretical physics and mathematics inspired each other.
- This fantastic development started in 1972/73.
- **As for the Future:** Fcc_ee and Fcc_pp are needed to probe and explore even higher scales and finer structures.
- One shall start to construct and build these machines as soon as possible.
- Associated theory developments have to proceed with even more and new efforts, along with the development of further computeralgebraic and mathematical technologies.

"Wir werden wissen, wir müssen wissen!"
D. Hilbert