Mathematical Methods for Higher Loop Feynman Diagrams

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1. Introduction
2. Momentum Integrals
3. Methods to Calculate Feynman Parameter Integrals
4. Function Spaces
5. Some Recent Results

1. Introduction

Loops and Legs:

Feynman diagrams describe elementary scattering processes between bosons and fermions in Quantum Field Theory (QFT). We will thoroughly refer to renormalizable QFTs.

Where are these techniques important?

1. Perturbation Theory of the Standard Model and its renormalizable extensions.
2. String amplitude calculations
3. Perturbative calculations in Gravity
4. non-relativistic field theories in vacuum and at finite temperature and/or density

We will calculate Feynman diagrams. These are skeletons according to Feynman rules, connecting vertices with propagators.

They possess external lines: The Legs.

They possess internal closed lines: The Loops.

Introduction

Why are these calculations important ?

1. Precision extraction of coupling constants: $\alpha_s(M_Z)@1\%$
2. Do couplings unite at high scales and in which field theories?
3. Precision measurements of m_c, m_b, m_t at LHC and a future ILC
4. Precision understanding of the Higgs and top sector (at the LHC, ILC and possibly other machines)
5. Unravel the mathematical structure of microscopic processes analytically: get further with the Stueckelberg-Feynman programme as far as you can.

$\Longrightarrow$ Genetic Code of the Micro Cosmos

Plan of the Session

Session 9	Symbolic computation and elementary particle physics

Session 9.1	**Thursday, July 14, 10:30-12:10, Room 2006**	
10:30-11:00	Johannes Blümlein	The mathematical function spaces of higher loop Feynman integrals
11:05-11:35	Andreas v. Manteuffel	Reducing Feynman integrals with finite fields
11:40-12:10	Abilio De Freitas	Three-loop heavy flavor corrections to DIS structure functions

Session 9.2	**Thursday, July 14, 13.10-14:50, Room 2006**	
13:10-13:40	Stefan Weinzierl	Algorithms for all-order expansions
13:40-14:10	Mark Round	Summation techniques for Feynman diagrams via special functions
14:10-14:40	Erik Panzer	Conical sums and multiple polylogarithms

Session 9.3	**Thursday, July 14, 15:00-16:15, Room 2006**	
15:00-15:30	Dirk Kreimer	Motivating computational practice
15:30-16:00	Christian Bogner	MPL - a program for computations with multiple polylogarithms
16:00-16:15	Carsten Schneider	Symbolic summation packages for elementary particle physics

Plan of the Session

Computational Methods: J. Blümlein
IBP relations: modular image and reconstruction techniques A. von Manteuffel
Massive 3-loop integrals: A. De Freitas
All order expansions - which functions contribute? S. Weinzierl
Summation algorithms: M. Round, C. Schneider
Multiple Polylogarithms: E. Panzer, C. Bogner
Connection to Motivics: aspects of algebraic and arithmetic geometry D. Kreimer

A gigantic number of terms has to be processed analytically: **tera terms and more!**
Analytic results still have to be so compact, that they can be published.

2. Perform all Momentum Integrals

We outline an algorithm first perform all momentum integrals in D-space. $D = 4 + \varepsilon$.

Example:

$$\int d^D k \frac{1}{[(p-k)^2 - m^2][k^2 - m^2]}$$

How to integrate this and related loop integrals of more general kind ?

$$\int d^D k_1 \ldots \int d^D k_n \prod_{i=1}^{m} \frac{1}{(p_i[L(k_j; q_a)])^2 - M_i^2}; \quad L \text{ - linear function}$$

k_j loop momenta; q_a external momenta; $m > n$

Integrate Loop by Loop.

Most of the higher loop integrals are very difficult to compute analytically.

Perform all Momentum Integrals

- The task is algorithmically solved in complete form $\Longrightarrow$ Feynman parameter integrals.

$$\frac{1}{a^\alpha \cdot b^\beta} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xa+(1-x)b]^{\alpha+\beta}}; \quad \alpha, \beta \in \mathbb{R}_+$$

- Genuine representations exist $\Longrightarrow$ using Symanzik polynomials
- Extensions to diagrams carrying local operators are possible.
- Remaining task: calculate the Feynman parameter integrals.
- Because the former step is voluminous, first IBP reductions to master integrals are applied.

3. Methods to Evaluate Feynman Parameter Integrals

1. Integration by parts technique
2. Mellin-Barnes techniques
3. PSLQ: zero-dimensional integrals
4. Guessing: one-dimensional integrals
5. Generalized hypergeometric functions (and extensions)
6. Hyperlogarithms
7. Risch algorithms
8. Solution of master-integrals using difference and differential equations
9. Summation techniques: construction of difference and product fields
10. (multivalued) Almkvist-Zeilberger algorithm ... and others.

Integration by Parts Technique

Obtain a homogeneous difference equation for a Feynman-integral by using Gauß' Law.

$$\int d^D k \frac{\partial}{\partial k_\mu} p_\mu f(k,p) = 0$$

Example:

$$F(a) = \int d^D k \frac{1}{(k^2 - m^2)^a} \quad Operator: \frac{\partial}{\partial k} k$$

$$0 = \int d^D k \left[\frac{D}{(k^2-m^2)^a} - \frac{2a(k^2 - m^2 + m^2)}{(k^2-m^2)^{a+1}}\right]$$

$$(D - 2a)F(a) = 2am^2 F(a+1)$$

$$F(a) = \frac{D - 2a + 2}{2(a-1)m^2} F(a-1) = \frac{(-1)^a (1 - D/2)_{a-1}}{\Gamma(a)\,(m^2)^{a-1}} I_1$$

$$I_1 = -i\pi^{D/2}\Gamma\left(1 - \frac{D}{2}\right)\left(m^2\right)^{D/2-1} \quad \text{[Master-Integral]}$$

$$(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha) \quad \text{[Pochhammer symbol]}$$

$\Longrightarrow$ Difference or differential equations to be solved.

The PSLQ-Method

Seek an Integer Relation over a basis of special numbers out of a special class.

Example:

$$I = \int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x}$$

The integral is of "transcendentality" $\tau = 4$.

The expected HPL(1) basis is spanned by:

$\ln^4(2), \; \ln(2)\zeta_3, \; \ln^2(2)\zeta_2, \; \zeta_2^2, \; \mathrm{Li}_4(1/2)$.

Calculate this integral numerically to high number of digits, e.g. 40 digits.

$$I \approx 0.3395454690873598695906678484608602061388$$

The PSLQ algorithm yields:

$$\begin{aligned} I &= -\frac{1}{12}\ln^4(2) + \frac{\pi^4}{60} + \frac{3}{4}\ln(2)\zeta_3 + \frac{1}{12}\ln^2(2)\pi^2 - 2\mathrm{Li}_4\left(\frac{1}{2}\right) \\ \zeta_{2k} &= (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}; \quad B_n \quad \text{[Bernoulli number]} \end{aligned}$$

Guessing Difference Equations

It is often easier to calculate Mellin moments for a quantity for fixed values of N than to derive the relation for general values of N in the first place. If the quantity under consideration is known to be recurrent than its difference equation is of finite order and degree.

$$\exists \quad \sum_{k=0}^{O} P_k^{(l)}(N) F(k+N) = 0; \qquad \text{max}\{l\} \text{ - degree; } O \text{ - order}$$

Example:

$$-(N+1)^3 F(N) - (3N^2 - 9N - 7) F(N+1) + (N+2)^3 F(N+2) = 0$$

$$F(1) = 1; \quad F(2) = \frac{1}{8}$$

Solution: $$F(N) = \sum_{k=1}^{N} \frac{1}{k^3} = S_3(N)$$

Guessing Difference Equations

Solution of large problems

Assume you would like to calculate the massless 3-loop Wilson coefficients in deep-inelastic scattering using this method. How many moments would you need and how do they look like ?

About 5200 moments are needed. The largest ones are ratios of #13000/#13000 digits. They are calculable within 15 min.

After 3 weeks you will find a difference equation of degree ~ 1000 and order 35, if you have a reasonable computer (100 Gbyte RAM).

After another week you have the solution as function of N.

Problem: It is sophisticated to obtain the input a priori. Combined solution-methods do work, however, to $O(1500)$ moments.

Recent result: a 3-loop anomalous dimension computed from scratch.

Generalized Hypergeometric Functions

At lower number of legs and/or loops Feynman integrals happen to be represented by these functions.

After suitable mappings these functions have compact representations in infinite (multiple) absolutely convergent sums.

This allows for the Laurent-expansion in ε under the summation operator.

Important Examples:

1. $B(a, b)$
2. ${}_pF_q(a_i; b_j; x)$; always single sums
3. Appell functions; double sums
4. Kampé de Feriet functions, Horn functions and higher; more sums

The sums may be expanded and summed using algorithms like nestedsums, xsummer, HarmonicSums, Sigma, EvaluateMultiSums

Generalized Hypergeometric Functions

Example:

Integrals of the following type emerge:

$$\begin{aligned} I_1(z) &= \int_0^1 dy y^\delta (1-y)^\eta \int_0^1 dx x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xyz)^{-\alpha} \\ &= B(\beta,\gamma-\beta) \int_0^1 dy y^\delta (1-y)^\eta \; {}_2F_1(\alpha,\beta;\gamma;yz) \\ &= B(\beta,\gamma-\beta)B(\delta,\eta-\delta) \; {}_3F_2(\delta,\alpha,\beta;\eta,\gamma;z) \end{aligned}$$

All ${}_pF_q$'s have single series representations. One series counts as one integral.

$${}_pF_q(a_1,...,a_p;b_1...b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \frac{z^k}{k!}$$

Mellin-Barnes Integrals

Used to resolve sums in denominators. In a way a counterpart to the binomial expansion of numerators.

$$\frac{1}{(A+B)^c} = \oint_{-i\infty+\gamma}^{i\infty+\gamma} d\sigma \frac{\Gamma(-\sigma)\Gamma(\sigma+c)}{\Gamma(c)} A^\sigma B^{-\sigma+c}$$

The contour integral covers the residues and closes either left or right.

One may use Beta-functions and (generalized) hypergeometric functions to perform the integrals further.

Important: One has to be able to undo the Mellin-Barnes integral(s).

1. Barnes Lemmas; map to Γ-functions, (in the most simple cases).
2. Residue theorem $\Longrightarrow$ leads to nested sums $\Longrightarrow$ use SIGMA.

The Method of Hyperlogarithms

1. Assume that a Feynman parameterization exists, which is multilinear in all parameters.
2. Assume that the integration procedure maintains this property in one order of integrations [Fubini sequence].
3. Assume, the integral has no poles in ε; or find a method to deal with it.
4. Then: the integral can be organized fully in Hyperlogarithms.
5. Hyperlogarithms are Kummer-Poincaré-Lappo-Danielevsky-Chen-Goncharov iterated integrals over an alphabet, the letters of which contain further integration variables in the multilinear sense.
6. Very many of them have coefficient zero in the final result.
7. In various cases the multi-linearity may not persist, but a solution can be found as well in extended function spaces.

$$L_{a_1,\dots a_k}(x) = \int_0^x \frac{dx_1}{x_1 - a_1} \int_0^{x_1} \frac{dx_2}{x_2 - a_2} \dots \int_0^{x_{k-1}} \frac{dx_k}{x_k - a_k}, a_l \in \mathbb{C}$$

Summation Techniques

The integrals can usually be traded for a lower number of sums (finite or infinite).

Solve these sums for N and/or in terms of special constants.

Principal Idea:

1. Sums may be considered to form vector spaces, algebras, and finally fields.
2. Consider difference and product fields.
3. Implement relations due to difference equations
4. Telescoping, creative telescoping, and other principles.
5. Try to solve the recurrences; possible for most sums occurring from Feynman integrals.

Telescoping: Find a function $g(k)$ such

$$\begin{aligned} f(k) &= g(k+1) - g(k) \\ F(N) &= \sum_{k=1}^{N} f(k) = g(N+1) - g(1) \end{aligned}$$

$\Longrightarrow$ nested sums algebras $\Longrightarrow$ bases

Differential Equations

The IBPs deliver a vast amount of differential equations forming systems, which are nested hierarchically.

Provide boundary conditions [usually using other methods]

Perform uncoupling of these systems

- In case of complete 1st order uncoupling: $\exists$ complete solution algorithm in case of any basis choice for 1 parameter systems
All solutions are iterative integrals over whatsoever alphabet:
$\int_0^x dy f_a(y) H_{\vec{b}}(y)$

- Irreducible nth order systems $(n \geq 2)$: present target of research even in mathematics; good prospects in case of 2nd order systems
[convergent near integer power series (CIS)]
At least one function is given by a definite integral, others iterate on.

$\Longrightarrow$ iterated integral algebras $\Longrightarrow$ bases

The Almkvist-Zeilberger Algorithm

- Given a multiple integral over hyperexponential terms:

$F(n) = \int_0^1 dx_1 ... dx_j \prod_{k=1}^{l} (P(x_i, n))^{r_k, \epsilon}$, $r_k \in \mathbb{R}$ and $n \in \mathbb{N}$ a parameter.

- Find a recurrence: $\sum_{k=0}^{m} p_k(n, \epsilon) F(n+k) = H(n, \epsilon)$ with some inhomogeneity $H(n, \epsilon)$.

- Correspondingly $n \to x$, a differential equation:

$\sum_{k=0}^{m} p_k(x, \epsilon) \frac{d^k}{dx^k} F(x) = K(x, \epsilon)$ with some inhomogeneity $K(x, \epsilon)$.

Either the inhomogeneities can be forced to vanish, or a hierarchy of equations has to be solved using summation techniques and DEQ-solvers (which may also be summation techniques).

4. Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals on CIS fct.

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

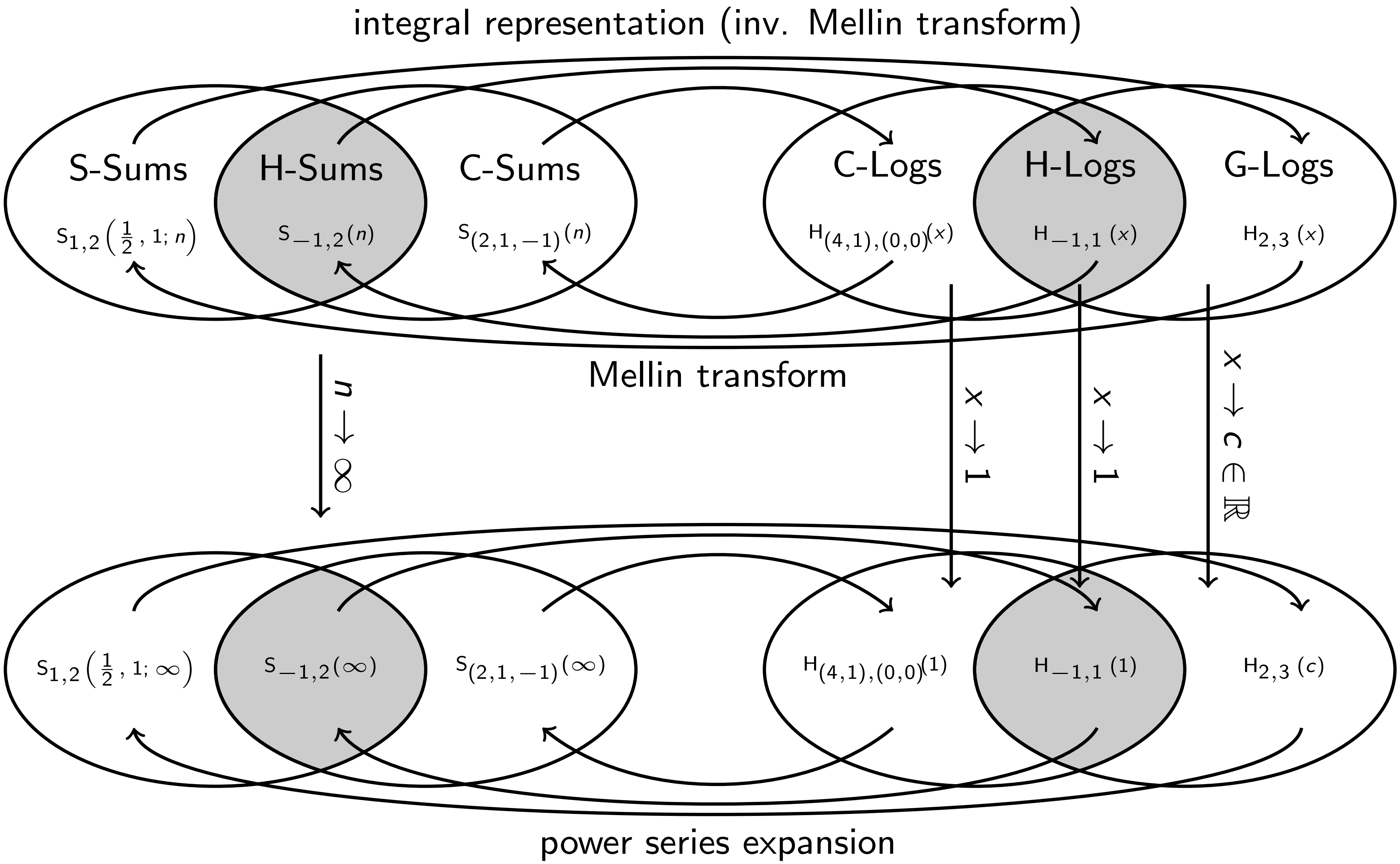
associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx\ {}_2F_1\left[\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras



square-root valued letters $\Longleftrightarrow$ nested binomial sums $\binom{2i}{i}$

non-iterative iterative integrals $\Longrightarrow$ Iterate on CIS ${}_2F_1$'s: rat. argument

(special cases: complete elliptic integrals)

5. A Few Recent Results

Massless 3-loop results: anomalous dimensions & Wilson coefficients: harmonic sums only

Massive 2-loop results: Wilson coefficients: harmonic sums only

Massive 3-loop results: DIS Wilson coefficients and OMEs
all other structures above; also expected for $pp \rightarrow t\bar{t}$ and similar reactions at NNLO.

Status: 7 of 8 OMEs $\surd$; 4 of 5 Wilson Coefficients $\surd$.

$\delta m_c^{\rm DIS}$: 6.5% $\Longrightarrow 3\%$ after NNLO corrections are finished

Only then: NNLO DIS QCD analyses at larger scales Q^2 are consistently possible. Improved value of $\alpha_s(M_Z^2)$ and of $xG(x,Q^2)$

The relevance of small x predictions can only be judged then.

PS-case: leading term by CCH correct, but nowhere dominant
[found 23 years after]

Outlook

Lots of new technologies and mathematical insight for QFT as a whole.

- Proceed to higher loops at zero and single scale problems. Massless and massive.
- Proceed to 2-loop $2 \to 2$ and $2 \to 3$ problems to be solved analytically. Various scales and masses.Massless.
- Unravel the corresponding mathematical structures being associated, their mutual relations and efficient implementations.
- Not to forget about final numerical solutions @ high efficiency.

References

1. **Recent reviews:** J. Ablinger and J. Blümlein, arXiv:1304.7071 [math-ph];
J. Ablinger, J. Blümlein and C. Schneider, J. Phys. Conf. Ser. **523** (2014) 012060 [arXiv:1310.5645 [math-ph]].
2. **Graph Polynomials:** N. Nakanishi, Suppl. Progr. Theor. Phys. **18** (1961) 1–125; *Graph Theory and Feynman Integrals*, (Gordon and Breach, New York, 1970);
C. Bogner and S. Weinzierl, Int. J. Mod. Phys. A **25** (2010) 2585–2618 [arXiv:1002.3458 [hep-ph]];
S. Weinzierl, arXiv:1301.6918 [hep-ph].
3. **Harmonic Sums:** J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037–2076 [hep-ph/9806280];
J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [hep-ph/9810241].
4. **Shuffle- Algebraic Relations:** M.E. Hoffman, J. Algebraic Combin., **11** (2000) 49–68, [arXiv:math/9907173 [math.QA]]; Nucl. Phys. (Proc. Suppl.) **135** (2004) 215 [arXiv:math/0406589];
J. Blümlein, Comput. Phys. Commun. **159** (2004) 19–54, [hep-ph/0311046];
J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Trans. Am. Math. Soc. **353** (2001) 907–941, [math/9910045 [math-ca]].
5. **Hopf Algebras:** H. Hopf, Ann. of Math. **42** (1941) 22–52;
J. Milner and J. Moore, Ann. of Math. **81** (1965) 211–264;
M.E. Sweedler, *Hopf Algebras*, (Benjamin, New York, 1969).
6. **Structural Relastions of Harmonic Sums:** J. Blümlein, Comput. Phys. Commun. **180** (2009) 2218–2249, [arXiv:0901.3106 [hep-ph]].
7. **Sum-Function Bases in QFT:** J. Blümlein and V. Ravindran, Nucl. Phys. B **716** (2005) 128–172 [hep-ph/0501178], Nucl. Phys. B **749** (2006) 1–24 [hep-ph/0604019];
J. Blümlein and S. Klein, PoS ACAT (2007) 084 [arXiv:0706.2426 [hep-ph]].
8. **Lyndon Words and Basis Counting:** R.C. Lyndon, Trans. Amer. Math. Soc. **77** (1954) 202–215; **78** (1955) 329–332;
C. Reutenauer, *Free Lie algebras*. (Oxford, University Press, 1993);
D.E. Radford, J. Algebra, **58** (1979) 432–454;
E. Witt, Journ. Reine und Angew. Mathematik, **177** (1937) 152–160; Math. Zeitschr. **64** (1956) 195–216.
9. **Factorial Series:** N. Nielsen, *Handbuch der Theorie der Gammafunktion*, (Teubner, Leipzig, 1906); reprinted by (Chelsea Publishing Company, Bronx, New York, 1965);
E. Landau, *Über die Grundlagen der Theorie der Fakultätenreihen*, S.-Ber. math.-naturw. Kl. Bayerische Akad. Wiss. München, **36** (1906) 151–218.
10. **Analytic Continuation of Harmonic Sums:** J. Blümlein, Comput. Phys. Commun. **133** (2000) 76–104, [hep-ph/0003100];
J. Blümlein and S.-O. Moch, Phys. Lett. B **614** (2005) 53–61, [hep-ph/0503188];
see also the References und Structural Relations.
A. V. Kotikov and V. N. Velizhanin, hep-ph/0501274.
11. **Di- and Polylogarithms:** L. Lewin, *Dilogarithms and associated functions*, (Macdonald, London, 1958);
A. N. Kirillov, Prog. Theor. Phys. Suppl. **118** (1995) 61–142, [hep-th/9408113];
L.C. Maximon, Proc. R. Soc. **A459** (2003) 2807–2819;
D. Zagier, Journal of Mathematical and Physical Sciences **22** (1988) 131–145;

References

11. **Di- and Polylogarithms:** P. Cartier, B. Julia, B.; P. Moussa et al.,Eds., *Frontiers in Number Theory, Physics, and Geometry II - On Conformal Field Theories, Discrete Groups and Renormalization*, (Springer, Berlin, 2007) 3–65;
L. Lewin, *Polylogarithms and associated functions*, (North Holland, New York, 1981);
A. Devoto and D. W. Duke Riv. Nuovo Cim. **7N6** (1984) 1–39;
N. Nielsen, Nova Acta Leopold. **XC** Nr. 3 (1909) 125–211;
K. S. Kölbig, J. A. Mignoco, E. Remiddi and , BIT **10** (1970) 38–74;
K. S. Kölbig, SIAM J. Math. Anal. **17** (1986) 1232–1258.
12. **Iterated Integrals:** H. Poincaré, Acta Math. **4** (1884) 201–312;
J.A. Lappo-Danilevsky, *Mémoirs sur la Théorie des Systèmes Différentielles Linéaires*, (Chelsea Publ. Co, New York, NY, 1953);
K.T. Chen, Trans. A.M.S. **156** (3) (1971) 359–379;
A. Jonquière, Bihang till Kongl. Svenska Vetenskaps-Akademiens Handlingar **15** (1889) 1–50;
E. Remiddi, J. A. M. Vermaseren and , Int. J. Mod. Phys. A **15** (2000) 725–754 [hep-ph/9905237];
T. Gehrmann and E. Remiddi, Comput. Phys. Commun. **141** (2001) 296–312 [arXiv:hep-ph/0107173];
J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005) 177–194 [arXiv:hep-ph/0410259].
13. **Generalized Harmonic Sums:** A. B. Goncharov. Mathematical Research Letters, **5** (1998) 497–516, [arXiv:1105.2076 [math.AG]];
S.-O. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363–3386, [hep-ph/0110083];
J. Ablinger, J. Blümlein and C. Schneider, arXiv:1302.0378 [math-ph].
14. **Cyclotomic Harmonic Sums, Polylogarithms, and Constants:** J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **52** (2011) 102301, [arXiv:1105.6063 [math-ph]].
15. **Binomial-weighted Nested Generalized Harmonic Sums:** J. Ablinger, J. Blümlein, C. Raab, C. Schneider, and F. Wißbrock, DESY 13–063.
16. **Periods:** M. Kontsevich and D. Zagier, IMHS/M/01/22, in B. Engquist and W. Schmid, Eds., *Mathematics unlimited - 2001 and beyond*, (Springer, Berlin, 2011), pp. 771–808;
C. Bogner and S. Weinzierl, J. Math. Phys. **50** (2009) 042302 [arXiv:0711.4863 [hep-th]].
17. **Zeta Values:** D. Zagier, in : First European Congress of Mathematics, Vol. II, (Paris, 1992), Progr. Math., **120**, (Birkhäuser, Basel–Boston, 1994), 497–512; J. Blümlein, D. J. Broadhurst and J. A. M. Vermaseren, Comput. Phys. Commun. **181** (2010) 582–625, [arXiv:0907.2557 [math-ph]] and references therein;
J.Kuipers and J.A.M.Vermaseren arXiv:1105.1884 [math-ph];
M.E. Hoffman's page `http://www.usna.edu/Users/math/~meh/biblio.html`;
S. Fischler, Sém. Bourbaki, Novembre 2002, exp. no. **910**, Asterisque **294** (2004) 27–62, `http://www.math.u-psud.fr/~fischler/publi.html`;
P. Colmez, in: Journées X-UPS 2002. La fontion zêta. Editions de l'Ecole polytechnique, Paris, 2002, 37–164, `http://www.math.polytechnique.fr/xups/vol02.html`;
M. Waldschmidt, Number Theory and Discrete Mathematics, Editors: A.K. Agarwal, B.C. Berndt, C.F. Krattenthaler, G.L. Mullen, K. Ramachandra and M. Waldschmidt, (Hindustan Book Agency, 2002), 1–12.

References

17. **Zeta Values:** M. Waldschmidt, Journal de théorie des nombres de Bordeaux, **12** (2) (2000) 581–595;
P. Cartier, Sém. Bourbaki, Mars 2001, 53*e* année, exp. no. **885**, Asterisque **282** (2002) 137–173;
M. Huttner and M. Petitot, *Arithmeétique des fonctions d'zetas et Associateur de Drinfel'd*, (UFR de Mathématiques, Lille, 2005);
C. Hertling, AG Mannheim-Heidelberg, SS2007;
P. Cartier, Sém. Bourbaki, Mars 2001, 53*e* année, exp. no. **885**, Asterisque **282** (2002) 137–173.
18. **Number of Zeta Values:** A.B. Goncharov, *Multiple polylogarithms and mixed Tate motives*,
`arxiv:math.AG/0103059`;
T. Terasoma, *Mixed Tate Motives and Multiple Zeta Values*, Invent. Math. **149** (2) (2002) 339–369, `arxiv:math.AG/010423`;
P. Deligne and A.B. Goncharov, *Groupes fondamentaux motiviques de Tate mixtes*, Ann. Sci. Ecole Norm. Sup., Série IV **38** (1) (2005) 1–56;
D. J. Broadhurst, arXiv:hep-th/9604128;
D. J. Broadhurst and D. Kreimer, Phys. Lett. B **393** (1997) 403–412 [arXiv:hep-th/9609128];
F. Brown, Annals of Math. **175** (1) (2012) 949–976;
V.V. Zudilin, Uspekhi Mat. Nauk **58** (1) 3–22.
19. **Hypergeometric Functions and Generalizations:** P. Appell, *Sur Les Fonctions Hypérgéometriques de Plusieurs Variables*, (Gauthier-Villars, Paris, 1925);
P. Appell and J. Kampé de Fériet, *Fonctions Hypérgéometriques; Polynômes d'Hermite*, (Gauthier-Villars, Paris, 1926);
W.N. Bailey, *Generalized Hypergeometric Series*, (Cambridge University Press, Cambridge, 1935);
A. Erdélyi (ed.), *Higher Transcendental Functions*, Bateman Manuscript Project, Vol. **I**, (McGraw-Hill, New York, 1953);
H. Exton, *Multiple Hypergeometric Functions and Applications*, (Ellis Horwood Limited, Chichester, 1976); Integrals, (Ellis Horwood Limited, Chichester, 1978);
L.J. Slater, *Generalized Hypergeometric Functions*, (Cambridge University Press, Cambridge, 1966).
20. **Mellin-Barnes Integrals:** E.W. Barnes, Proc. Lond. Math. Soc. (2) **6** (1908) 141; Quart. Journ. Math. **41** (1910) 136–140;
H. Mellin, Math. Ann. **68** (1910) 305–337;
J. Gluza, K. Kajda and T. Riemann, Comput. Phys. Commun. **177** (2007) 879–893 [arXiv:0704.2423 [hep-ph]].
21. **Summation in Difference and Productfields:** C. Schneider, Discrete Math. Theor. Comput. Sci. **6** (2004) 365–386; J. Differ. Equations Appl., **11**(9) (2005) 799–821; Advances in Applied Math., **34**(4) (2005) (4) 740–767; Annals of Combinatorics, **9**(1) (2005) 75–99; Sem. Lothar. Combin., **56** (2007) 1–36; J. Symb. Comp., **43**(9) (2008) 611–644 arXiv:0808.2543 [cs.SC]; Ann. Comb., **14**(4) (2010) 533–552 arXiv:0808.2596 [cs.SC]; Appl. Algebra Engrg. Comm. Comput., **21**(1) (2010) 1–32; In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, **12** of *Clay Mathematics Proceedings*, pages 285–308. Amer. Math. Soc., (2010), arXiv:0904.2323 [cs.SC].

References

22. **Hyperlogarithms:** F. Brown, Commun. Math. Phys. **287** (2009) 925 [arXiv:0804.1660 [math.AG]];
J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider and F. Wißbrock, Nucl. Phys. B **864** (2012) 52 [arXiv:1206.2252 [hep-ph]];
E. Panzer, Comput. Phys. Commun. **188** (2015) 148 [arXiv:1403.3385 [hep-th]].
23. **Risch algorithms:** M. Bronstein, *Symbolic Integration I: Transcendental Functions*, (Springer, Berlin, 1997);
C. Raab, *Definite Integration in Differential Fields*, PhD Tesis, August 2012, Johannes Kepler University Linz.
24. **Hopf Algebras and Renormalization:** A. Connes and D. Kreimer, Commun. Math. Phys. **210** (2000) 249 [hep-th/9912092]; Commun. Math. Phys. **216** (2001) 215 [hep-th/0003188].
25. **Integration by parts techniques.** J. Lagrange, Nouvelles recherches sur la nature et la propagation du son, Miscellanea Taurinensis, t. II, 1760-61; Oeuvres t. I, p. 263;
C.F. Gauß, Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractate, Commentationes societas scientiarum Gottingensis recentiores, Vol III, 1813, Werke Bd. **V** pp. 5-7;
G. Green, Essay on the Mathematical Theory of Electricity and Magnetism, Nottingham, 1828 [Green Papers, pp. 1-115];
M. Ostrogradski, Mem. Ac. Sci. St. Peters., **6**, (1831) 39;
K. G. Chetyrkin, A. L. Kataev and F. V. Tkachov, Nucl. Phys. B **174** (1980) 345;
S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087 [hep-ph/0102033];
C. Studerus, Comput. Phys. Commun. **181** (2010) 1293 [arXiv:0912.2546 [physics.comp-ph]];
A. von Manteuffel and C. Studerus, arXiv:1201.4330 [hep-ph].
26. **Use of Differential Equations:** A. V. Kotikov, Phys. Lett. B **254** (1991) 158;
E. Remiddi, Nuovo Cim. A **110** (1997) 1435 [hep-th/9711188];
M. Caffo, H. Czyz, S. Laporta and E. Remiddi, Acta Phys. Polon. B **29** (1998) 2627 [hep-th/9807119]; Nuovo Cim. A **111** (1998) 365 [hep-th/9805118];
T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329];
J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, Comput. Phys. Commun. **202** (2016) 33 [arXiv:1509.08324 [hep-ph]];
J.M. Henn, Phys. Rev. Lett. **110** (2013) 251601 [arXiv:1304.1806 [hep-th]].
27. **Irreducible Systems of Order larger than one, Elliptic Integrals :** S. Laporta and E. Remiddi, Nucl. Phys. B **704** (2005) 349 [hep-ph/0406160];
L. Adams, C. Bogner and S. Weinzierl, J. Math. Phys. **54** (2013) 052303 [arXiv:1302.7004 [hep-ph]];
L. Adams, C. Bogner and S. Weinzierl, J. Math. Phys. **56** (2015) no.7, 072303 [arXiv:1504.03255 [hep-ph]].