RENORMALIZATION OF TWIST 8 OPERATORS IN ϕ^3 THEORY IN SIX DIMENSIONS

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1. EXTRAPOLATION OF THE AP EQUATIONS TO SMALL x
2. OPE IN ϕ^3_6
3. CONSTRUCTION OF AN OPERATOR BASIS
4. ANOMALOUS DIMENSIONS
5. OUTLOOK

AP EQUATIONS:

$$\frac{d f^{a}(x, Q^2)}{d \ln Q^2} = P\left(x, \frac{\alpha_s(Q^2)}{2\pi}\right)_{ab} \otimes f_b(x, Q^2)$$

$$P\left(x, \frac{\alpha_s}{2\pi}\right)_{ab} = \frac{\alpha_s}{2\pi}\left\{ P^{0}_{ab}(x) + \frac{\alpha_s}{2\pi} P^{1}_{ab}(x) + \dots \right\}$$

$$x \ll 1$$

	1ST ORDER	2nd ORDER
FF	$C_F \frac{1+x^2}{1-x}$	$\frac{1}{x} 2N_f T_R C_F \left(+\frac{20}{9}\right)$
FG	$2N_f T_R \left[x^2 + (1-x)^2\right]$	$\frac{1}{x} 2N_f T_R C_G \left(+\frac{20}{9}\right)$
GF	$C_F \frac{1}{x}\left[1 + (1-x)^2\right]$	$\frac{1}{x} 2N_f T_R \left(-\frac{20}{9}\right) + C_F C_G$
GG	$2C_G\left[\frac{1}{x} + \frac{1}{1-x} - 2 + x - x^2\right]$	$\frac{1}{x} 2N_f T_R \left(-\frac{23}{9} C_G + \frac{2}{3} C_F\right)$

$\uparrow$

WU-KI TUNG

1. EXTRAPOLATION OF THE AP EQUATIONS TO SMALL x

CONSIDER THE GLUON DISTRIBUTION: $q_s(x) \ll G(x)$

$$\mathcal{G}(x,Q^2) := x\,G(x,Q^2)$$

$$\frac{d\mathcal{G}(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dx'}{x'}\left[6 - \frac{61}{9}N_f\frac{\alpha_s}{2\pi}\right]\frac{x^2}{x'^2}\,\mathcal{G}(x',Q^2)$$

↑ LO ↑ NTLO

DF.: $y = \frac{8N_c}{\beta_0}\ln\frac{1}{x}$, $\xi = \ln\ln\frac{Q^2}{\Lambda^2}$

$$\boxed{\begin{array}{ll} \dfrac{\partial^2\mathcal{G}(y,\xi)}{\partial y\,\partial\xi} = \dfrac{1}{2}\,\mathcal{G}(y,\xi) & \text{LO} \\[2ex] \dfrac{\partial\,\mathcal{G}(y,\hat{\xi})}{\partial y\,\partial\hat{\xi}} = \dfrac{1}{2}\,\mathcal{G}(y,\hat{\xi}) & \text{NTLO} \end{array}}$$

$$\hat{\xi} = \xi + f(\xi) \quad ; \quad f'(\xi) = -\left[\frac{\beta_1}{\beta_0}\xi e^{-\xi} + \frac{61}{63}\frac{2N_f}{\beta_0}e^{-\xi}\left(1 - \frac{\beta_1}{\beta_0}\xi e^{-\xi}\right)^2\right]$$

SOLUTION:

$$\mathcal{G}(y,\overset{(\wedge)}{\xi}) = \sum_{\nu=0}^{\infty}\left\{A_\nu\left(\frac{2\overset{(\wedge)}{\xi}}{y}\right)^{\nu/2} + B_\nu\left(\frac{y}{2\overset{(\wedge)}{\xi}}\right)^{\nu/2}\right\}$$

- $\mathcal{G}(y,\xi)$ GROWS FASTER THAN A POWER OF $\ln\frac{1}{x}$ FOR $x \to 0$
- $\mathcal{G}(y,\hat{\xi}) < \mathcal{G}(y,\xi)$

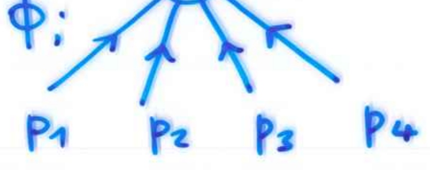
- UNITARITY VIOLATION AT TWIST 2 (LO, NTLO) ... ?
- HIGHER TWIST TERMS ARE LIKELY TO BE IMPORTANT AT SMALL x

→ START AT A CONVENTIONAL APPROACH

- OPE WORKS WELL SO FAR (→ DATA).
- ONLY WAY TO GET 'FULL x' RESULTS (→ 'SUBLEADING' x TERMS → WHICH ARE OFTEN <u>AS</u> IMPORTANT)
- NO USE OF ARGUMENTS OUTSIDE PT QCD. (AS E.G. AGK CUTTING RULES ...)

→ SYSTEMATIC STUDY OF GLUODYNAMICS IN TWIST 4
TECHNICALLY NOT SIMPLE

→ 0^{th} APPROACH TO SEE FIRST STRUCTURES:
ϕ^3_6 → SAME TOPOLOGIES SELECTED WHICH WILL YIELD QCD_g – TWIST 4 TERMS.



$D = 6$, $\dim[\phi_i] = 2$ ↷ TWIST = 8.

GLUODYNAMICS : · ALSO 5 GLUON INSERTIONS (D_μ)
· TENSORIAL NUMERATOR STRUCT.

2 Operator Product Expansion

DIM 6:

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + \frac{g}{3!}\Phi^3$$

RENORMALIZATION:

$$\begin{aligned}\partial^\mu\Phi_r\partial_\mu\Phi_r &= \frac{1}{Z_3}\partial^\mu\Phi\partial_\mu\Phi\\ g_r\Phi_r^3 &= \frac{1}{Z_1}g_0\Phi^3\end{aligned}$$

$$\begin{aligned}\gamma_\Phi(g) &= \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_3 = -\frac{1}{24}\frac{g^3}{(4\pi)^3}\\ \beta(g) &= \mu\frac{\partial g}{\partial\mu} = -\frac{3}{8}\frac{g^3}{(4\pi)^3}\end{aligned}$$

OPERATORS:

$$\mathcal{O}^0_{0,0,0} = \Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_4(x)$$

$$\mathcal{O}^n_{n_1,n_2,n_3} = \Phi_1(x)\overleftarrow{\partial}_{\lambda_1}\ldots\overleftarrow{\partial}_{\lambda_{n_1}}\Phi_2(x)\overleftrightarrow{\partial}_{\mu_1}\ldots\overleftrightarrow{\partial}_{\mu_{n_2}}\Phi_3(x)\overrightarrow{\partial}_{\nu_1}\ldots\overrightarrow{\partial}_{\nu_{n_3}}\Phi_4(x)$$

$$\langle\Psi|\mathcal{O}^n_{n_1,n_2,n_3}|\Psi\rangle = \sum_i c^{i,n}_{n_1,n_2,n_3}\hat{O}^{(n)}_i$$

REN. OP. MATRIXELEMENT

$$\widehat{O}_a^{(n)} = Z_3^2 \sum_b \left(Z_n^{-1}\right)_{ab} \widehat{O}_0^{b,(n)}$$

ANOM. DIMENSION

$$\left(\gamma^{(n)}\right)_{ab} = \left(\mu \frac{\partial}{\partial \mu} \ln Z_n\right)_{ab}$$

RGE'S

$$\begin{aligned}
\sum_j \left\{[\mathcal{D} - 4\gamma_\Phi(g)]\,\delta_{ij} + \gamma_{ij}^{(n)}(g)\right\} \widehat{O}^{j,(n)} &= 0 \\
\sum_j \left\{\mathcal{D}\delta_{ij} + \gamma_{ji}^{(n)}(g)\right\} \widehat{C}^{j,(n)} &= 0.
\end{aligned}$$

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}$$

EVOLUTION EQ. FOR FIXED SPIN.

$$\left(\langle E_i(Q^2)\rangle_n\right) = \left(\exp[-e_j^n/(2\beta_0)\hat{s}]\right) \mathcal{M}^{(n)} \left(\langle E_i(Q_0^2)\rangle_n\right)$$

$$\hat{s} = \ln[\ln(Q^2/\Lambda^2)/\ln(Q_0^2/\Lambda^2)]$$

3 Construction of an Operator Basis

$$
\begin{aligned}
O_{n_1,n_2,n_3} &= (\Delta p_3)^{n_1}(\Delta p_3+\Delta p_2)^{n_2}(\Delta p_1)^{n_3} + (\Delta p_3)^{n_1}(\Delta p_3+\Delta p_1)^{n_2}(\Delta p_2)^{n_3}\\
&+ (\Delta p_2)^{n_1}(\Delta p_2+\Delta p_3)^{n_2}(\Delta p_1)^{n_3} + (\Delta p_1)^{n_1}(\Delta p_1+\Delta p_3)^{n_2}(\Delta p_2)^{n_3}\\
&+ (\Delta p_1)^{n_1}(\Delta p_1+\Delta p_2)^{n_2}(\Delta p_3)^{n_3} + (\Delta p_2)^{n_1}(\Delta p_2+\Delta p_1)^{n_2}(\Delta p_3)^{n_3}\\
&+ (\Delta p_4)^{n_1}(\Delta p_4+\Delta p_2)^{n_2}(\Delta p_1)^{n_3} + (\Delta p_4)^{n_1}(\Delta p_4+\Delta p_1)^{n_2}(\Delta p_2)^{n_3}\\
&+ (\Delta p_4)^{n_1}(\Delta p_4+\Delta p_3)^{n_2}(\Delta p_1)^{n_3} + (\Delta p_4)^{n_1}(\Delta p_4+\Delta p_3)^{n_2}(\Delta p_2)^{n_3}\\
&+ (\Delta p_4)^{n_1}(\Delta p_4+\Delta p_1)^{n_2}(\Delta p_3)^{n_3} + (\Delta p_4)^{n_1}(\Delta p_4+\Delta p_2)^{n_2}(\Delta p_3)^{n_3}\\
&+ (\Delta p_2)^{n_1}(\Delta p_2+\Delta p_4)^{n_2}(\Delta p_1)^{n_3} + (\Delta p_1)^{n_1}(\Delta p_1+\Delta p_4)^{n_2}(\Delta p_2)^{n_3}\\
&+ (\Delta p_3)^{n_1}(\Delta p_3+\Delta p_4)^{n_2}(\Delta p_1)^{n_3} + (\Delta p_3)^{n_1}(\Delta p_3+\Delta p_4)^{n_2}(\Delta p_2)^{n_3}\\
&+ (\Delta p_1)^{n_1}(\Delta p_1+\Delta p_4)^{n_2}(\Delta p_3)^{n_3} + (\Delta p_2)^{n_1}(\Delta p_2+\Delta p_4)^{n_2}(\Delta p_3)^{n_3}\\
&+ (\Delta p_2)^{n_1}(\Delta p_2+\Delta p_1)^{n_2}(\Delta p_4)^{n_3} + (\Delta p_1)^{n_1}(\Delta p_1+\Delta p_2)^{n_2}(\Delta p_4)^{n_3}\\
&+ (\Delta p_3)^{n_1}(\Delta p_3+\Delta p_1)^{n_2}(\Delta p_4)^{n_3} + (\Delta p_3)^{n_1}(\Delta p_3+\Delta p_2)^{n_2}(\Delta p_4)^{n_3}\\
&+ (\Delta p_1)^{n_1}(\Delta p_1+\Delta p_3)^{n_2}(\Delta p_4)^{n_3} + (\Delta p_2)^{n_1}(\Delta p_2+\Delta p_3)^{n_2}(\Delta p_4)^{n_3}
\end{aligned}
$$

THE OP. MATRIXELEMENT FOR FIXED SPIN IS A SYMMETRIC FUNCTION IN Δp_i (POLYNOMIAL).

FUNDAMENTAL THEOREM ON SYMM. POLYNOMIALS:

EVERY SYMMETRIC POLYNOMIAL IN N VARIABLES CAN BE REPRESENTED UNIQUELY AS A POLYNOMIAL OF THE ASSOCIATED N ELEMENTARY SYMMETRIC POLYNOMIALS σ_i :

N = 4 :

$$
\begin{aligned}
\sigma_1 &= \sum_{i=1}^{4} \Delta p_i\\
\sigma_2 &= \sum_{i<j}^{4} \Delta p_i \Delta p_j\\
\sigma_3 &= \sum_{i<j<k}^{4} \Delta p_i \Delta p_j \Delta p_k\\
\sigma_4 &= \Delta p_1 \Delta p_2 \Delta p_3 \Delta p_4
\end{aligned}
$$

The functions σ_i are uniquely related to the power sums

$$P_i = \sum_{l=1}^{4} (\Delta p_l)^i$$

by NEWTON'S relations [5]

$$\begin{aligned} P_1 &= \sigma_1 \\ P_2 &= \sigma_1 P_1 - 2\sigma_2 \\ P_3 &= \sigma_1 P_2 - \sigma_2 P_1 + 3\sigma_3 \\ P_4 &= \sigma_1 P_3 - \sigma_2 P_2 + \sigma_3 P_1 - 4\sigma_4. \end{aligned}$$

Due to the zero momentum insertion

$$P_1 = \sigma_1 \equiv 0$$

$$\begin{aligned} P_2 &= -2\sigma_2 \\ P_3 &= 3\sigma_3 \\ P_4 &= 2\sigma_2^2 - 4\sigma_4. \end{aligned}$$

OPERATOR REPRESENTATION:

$$O^{(n)}_{n_1,n_2,n_3} = \sum_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1,\alpha_2,\alpha_3} \underbrace{P_2^{\alpha_2} P_3^{\alpha_3} P_4^{\alpha_4}}_{\text{BASIC VECTORS.}} \delta(n - 2\alpha_1 - 3\alpha_2 - 4\alpha_3)\delta(n - n_1 - n_2 - n_3)$$

NUMBER OF BASIC OPERATOS AT A GIVEN SPIN:

≡ RANK OF THE ANOM. DIM. MATRIX (GROWS WITH SPIN!)

$$n_{op}^{odd}(n) = n_{op}^{even}(n-3) \quad \text{for} \quad n \geq 3$$

$$n_{op}^{even}(n = 12m + 2l) = 3(m+1)^2 + (l-3)(m+1) + \delta_{0l} \quad \text{for} \quad l \in N \cap [0,5].$$

REPRESENTATION OF THE OPERATORS OF GIVEN SPIN:

n=0

$$O_{0,0,0} = 1.$$

n=1

$$O_{0,2n+1,0} = 0, \quad O_{1,2n,0} = O_{0,2n,1} = 0$$

MORE GENERALLY:

$$\begin{aligned}
O_{1,n_2,n_3} &= \frac{1}{2} O_{0,n_2+1,n_3} \\
O_{n_1,n_2,1} &= -\frac{1}{2} O_{n_1,n_2+1,0} \\
O_{n_1,1,n_3} &= \frac{1}{2} \left(O_{n_1+1,0,n_3} - O_{n_1,0,n_3+1} \right) \\
O_{1,1,n_3} &= \frac{1}{2} O_{0,2,n_3} \\
O_{1,n_2,1} &= -\frac{1}{4} O_{0,n_2+2,0} \\
O_{n_1,1,1} &= -\frac{1}{2} O_{n_1,2,0}.
\end{aligned}$$

n=2

$$(O_{2,0,0}, O_{0,2,0}, O_{0,0,2}, O_{1,1,0}, O_{1,0,1}, O_{0,1,1}) = (6, 8, 6, 4, -2, -4) P_2$$

n=3

$$(O_{3,0,0}, O_{0,3,0}, O_{0,0,3}, O_{2,1,0}, O_{2,0,1}, O_{1,2,0}, O_{1,0,2}, O_{0,1,2}, O_{0,2,1}, O_{1,1,1}) = (6, 0, 6, 4, -2, 0, -2, 4, 0, 0) P_3$$

n=4

$$\begin{aligned}
&(O_{4,0,0}, O_{0,4,0}, O_{0,0,4}, O_{3,1,0}, O_{3,0,1}, O_{1,3,0}, O_{1,0,3}, O_{0,3,1}, O_{0,1,3}, O_{2,2,0}, O_{2,0,2}, O_{0,2,2}) \\
&= (0, 12, 0, 0, 0, -6, 0, -6, 0, 2, 2, 2) \underline{P_2^2}. + (6, -12, 6, 4, -2, 6, -2, 6, 4, 0, -2, 0) \underline{P_4}
\end{aligned}$$

n=5

$$\begin{aligned}
&(O_{5,0,0}, O_{0,5,0}, O_{0,0,5}, O_{4,1,0}, O_{4,0,1}, O_{1,0,4}, O_{1,4,0}, O_{0,1,4}, O_{0,4,1}, O_{2,3,0}, O_{2,0,3}, O_{0,2,3}, \\
&O_{0,3,2}, O_{3,0,2}, O_{3,2,0}, O_{2,2,1}, O_{2,1,2}, O_{1,2,2}) \\
&= \left(5, 0, 5, \frac{10}{3}, -\frac{5}{3}, -\frac{5}{3}, 0, -\frac{10}{3}, 0, \frac{4}{3}, \frac{1}{3}, 2, -\frac{4}{3}, \frac{1}{3}, -\frac{4}{3}, -\frac{2}{3}, 0, -\frac{2}{3}\right) \underline{P_2 P_3}
\end{aligned}$$

etc etc

4 Anomalous Dimensions

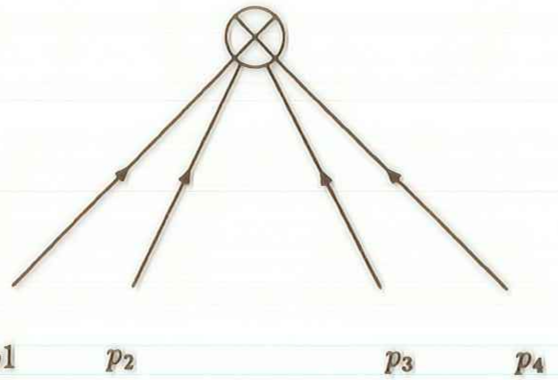


Figure 1: Lowest order term for the twist 8 operator. All momenta are ingoing.

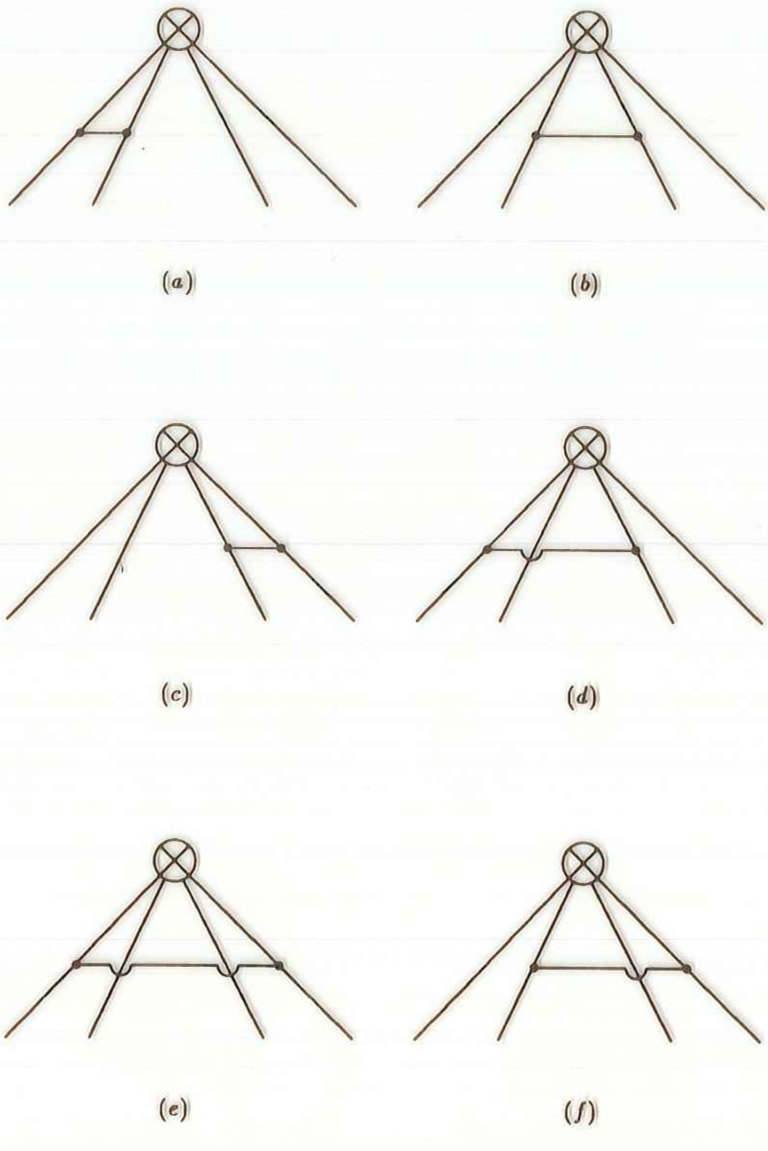
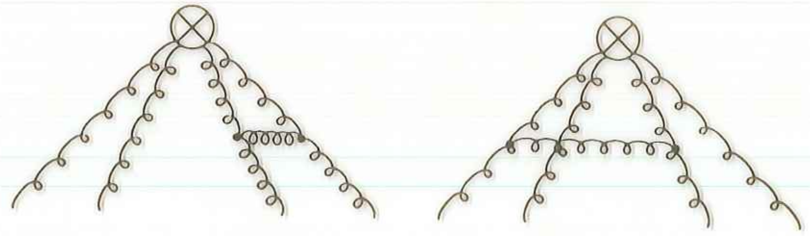
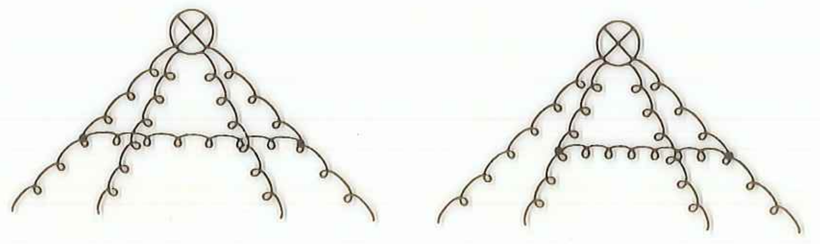


Figure 2: Diagrams for the $O(g^2)$ vertex corrections to the operator insertion.

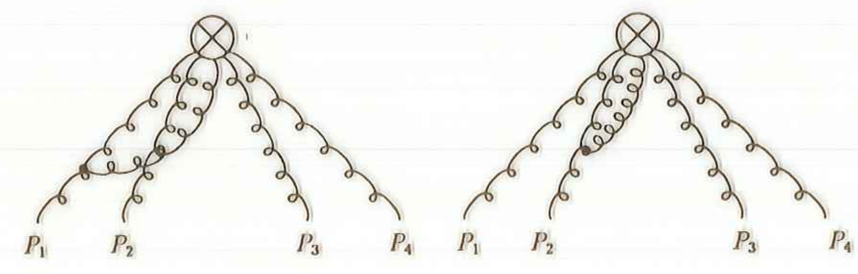
(a) (b)

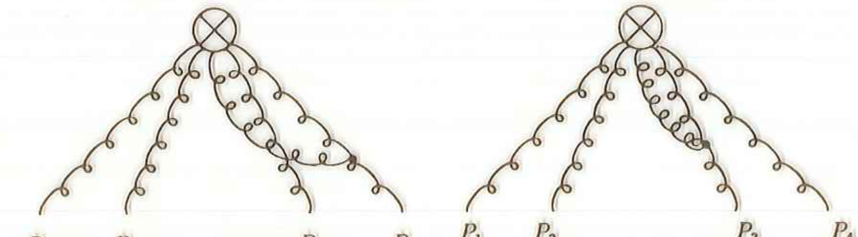


(c) (d)



(e) (f)





$+$ *permutations*

USEFUL, FIRST TO FORM: P_i's & TO SUBSTITUTE THEM AFTERWARDS.

$$\begin{aligned}
P_5 &= \frac{5}{6}P_2P_3 \\
P_6 &= -\frac{1}{8}P_2^2 + \frac{1}{3}P_3^2 + \frac{3}{4}P_2P_4 \\
P_7 &= \frac{7}{24}P_2^2P_3 + \frac{7}{12}P_3P_4 \\
P_8 &= -\frac{1}{16}P_2^4 + \frac{4}{9}P_2P_3^2 + \frac{1}{4}P_2^2P_4 + \frac{1}{4}P_4^2 \\
P_9 &= \frac{3}{4}P_2P_3P_4 + \frac{1}{9}P_3^3 \\
P_{10} &= -\frac{1}{64}P_2^5 + \frac{5}{18}P_2^2P_3^2 + \frac{5}{16}P_2P_4^2 + \frac{5}{18}P_3^2P_4
\end{aligned}$$

(EXPLIC. ILLUSTR. OF THE FUND. THEOREM).

ANOMALOUS DIMENSIONS:

$$\hat{\gamma}^{(0)} = 1$$

$$\hat{\gamma}^{(1)} = 0$$

$$\hat{\gamma}^{(2)} = \frac{9}{4}$$

$$\hat{\gamma}^{(3)} = \frac{7}{4}$$

$$\hat{\gamma}^{(4)}_{ij} = \begin{pmatrix} 1/3 & 19/15 \\ 11/6 & 1/5 \end{pmatrix}$$

$$\hat{\gamma}^{(5)} = \frac{14}{5}$$

$$\hat{\gamma}^{(6)}_{ij} = \begin{pmatrix} 993/560 & 46/105 & -39/280 \\ 0 & 5/4 & 3/20 \\ 1103/3360 & 83/315 & 1759/1680 \end{pmatrix}$$

$$\hat{\gamma}^{(7)}_{ij} = \begin{pmatrix} 313/240 & 23/40 \\ 39/160 & 281/240 \end{pmatrix}$$

etc. ⋮

HIGHER SPIN ANOM. DIMENSIONS CAN BE CALCULATED BY AN EXISTING ALGORITHM.

spin n	number of basic operators	eigenvalues $\eta_i(n)$	basis vectors
0	1	1	$1 \equiv P_0$
1	0	–	$0 \equiv P_1$
2	1	9/4	P_2
3	1	7/4	P_3
4	2	7/6 29/15	(1,-10/3); P_2^2, P_4 (2,1); P_2^2, P_4
5	1	14/5	P_2P_3
6	3	1.767001520 1.062444463 1.240792107	P_2^3, P_2P_4, P_3^2
7	2	0.857236020 1.617763980	(1,-1.2865531), $P_2^2P_3, P_3P_4$ (1.83356163,1), $P_2^2P_3, P_3P_4$

MORE & MORE SUBST. NEEDED.

NO CHAR. OF THE MIN EV. OBSERVED SO FAR.

$\eta_i(n) > 0$ SO FAR.

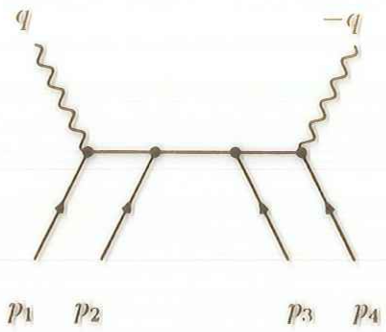
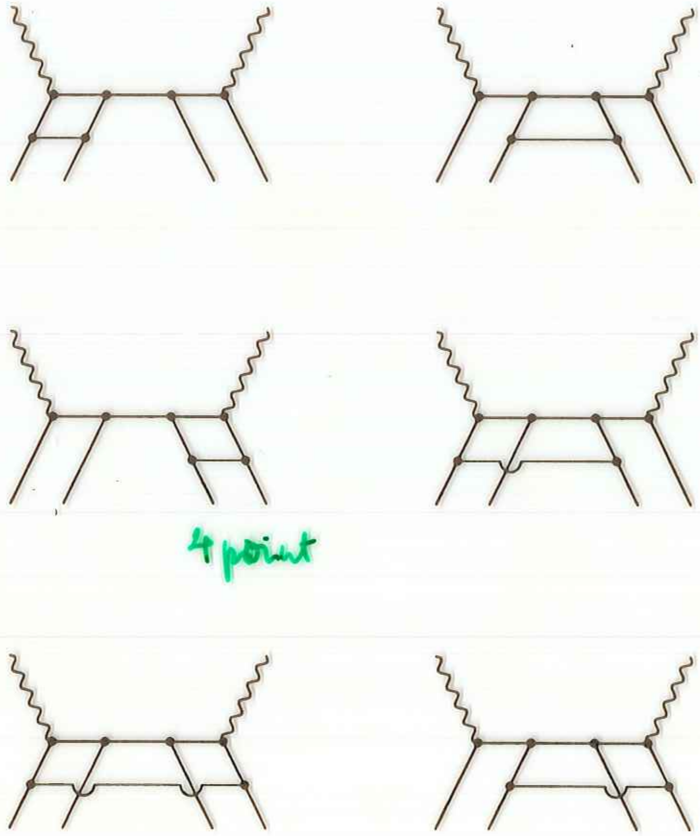


Figure 3: Diagram of the forward Compton amplitude.



4 point

6 point

5 point

+ 12 OTHER DIAGR.

Figure 4: Diagrams for the $O(g^2)$ vertex corrections to the forward Compton amplitude.

OUTLOOK

- STUDY 1 LOOP CORRECTIONS TO THE FORWARD COMPTON AMPLITUDE (18 DIAGRAMS, 1 6 POINT FCT, ... 5 PT. etc.)

- THE POLE TERMS CAN BE CALCULATED ANALYTICALLY IRRESP. OF THE KINEMATICAL LIMIT

→ BJORKEN LIMIT ↔ OPE RESULT.

→ REGGE LIMIT

POWER OF THE LOGS 'LOW x "ANOM. DIM." '

↔ $\gamma_i'^{s}|_{BJ.}$

THIS RELATION (MAPPING) MAY BE FOUND.

- BJ. LIMIT : FULL x RESULT.

→ INDICATION OF SCREENING ? IS THERE A MINUS SIGN |TWIST 2| − |TWIST 4| ?