# 3-Loop Heavy Flavor Corrections to DIS: an Update

2024 $\alpha_s$-workshop, Trento, I, February 5-9, 2024

Johannes Blümlein | February 3, 2024

DESY

Based on:

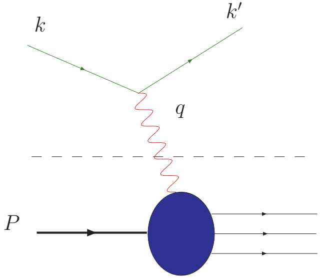
- A. Behring, J.B., and K. Schönwald, The inverse Mellin transform via analytic continuation, JHEP **06** (2023) 62.
- J. Ablinger et al., The first-order factorizable contributions to the three-loop massive operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$, 2311.00644 [hep-ph]

In collaboration with:

J. Ablinger, A. Behring, A. De Freitas, A. von Manteuffel, C. Schneider, K. Schönwald

# Outline

# Unpolarized Deep–Inelastic Scattering (DIS):




$\longrightarrow L_{\mu\nu}$

$\longrightarrow W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\,.$$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions.
At 3-Loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: $c$ and $b$ quarks in one graph.

# Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \, .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \, .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996]

factorizes into the light flavor Wilson coefficients $C$ and the massive operator matrix elements (OMEs) of local operators $O_i$ between partonic states $j$

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j \rangle .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Vermaseren, Moch, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].

For $F_2(x,Q^2)$ : at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

# Introduction

- Massive OMEs allow to describe the massive DIS Wilson coefficients for $Q^2 \gg m_Q^2$.
- Furthermore, they form the transition elements in the variable flavor number scheme (VFNS).
- **What is known:**
  Single mass: $A_{qq,Q}^{\rm NS}, A_{qg,Q}, A_{qq,Q}^{\rm PS}, A_{gq,Q}, A_{Qq}^{\rm PS}, A_{gg,Q}, A_{Qg}$ to 3-loop order; $A_{Qg}$ to 2-loop order;
  Two-mass case to 3-loop order $A_{qq,Q}^{\rm NS}, A_{Qq}^{\rm PS}, A_{gq,Q}, A_{gg,Q}$; $A_{Qg}$ to 2-loop order.
- The same OMEs are also known in the polarized case.
- Objective of this talk: First non-logarithmic results in calculating $A_{Qg}$.
- $\Longrightarrow$ The necessary master integrals
- $\Longrightarrow$ The first-order factorizable contributions to $(\Delta)A_{Qg}$
- I will discuss the developed and used technologies to some extent, to give you a glimpse of information on the relatively deep technical problems at hand. Many of the algorithms needed could not even be envisaged at the beginning of the project.
- Also needed: massless WCs to 3-loops in the unpolarized and polarized case
  Vermaseren, Moch, Vogt [2005]; JB, Marquard, Schneider, Schönwald [2022].

# Results in $N$ space and non-first order factorizing recurrences

- The method of arbitrary high moments [JB, Schneider 2017] allows to determine all results for complete color-$\zeta$ factors in the first-order factorizing case [$<$ 2000 moments].
- Other recurrences, obtained for a higher number of moments, are not first order factorizing.
- Complete color-$\zeta$ factors: Example polarized case $\Longrightarrow$

# Results in $N$ space and non-first order factorizing recurrences

$$
\begin{aligned}
\Delta a_{Qg}^{(3)} = {} & \frac{1}{2}[1-(-1)^N]\Big\{ C_F \Big[ T_F^2 \Big[ N_F \Big[ \frac{8 S_2 Q_{18}}{27N^4(1+N)^4(2+N)} - \frac{16 S_3 Q_{12}}{81 N^3(1+N)^3(2+N)} \\
& + \frac{Q_{20}}{243N^6(1+N)^6(2+N)} + \Big(\frac{32(10N^3+49N^2+19N-24)S_2}{27N^2(1+N)(2+N)} - \frac{16Q_5}{243N^2(1+N)^2(2+N)} \\
& - \frac{256}{27}\Delta p_{qg} S_3 - \frac{128}{3}\Delta p_{qg} S_{2,1}\Big) S_1 - \Big(\frac{32(651+442N+175N^2+10N^3)}{81N^2(1+N)(2+N)} + \frac{64}{9}\Delta p_{qg} S_2\Big) S_1^2 \\
& + \frac{32(-24+19N+49N^2+10N^3)S_1^3}{81N^2(1+N)(2+N)} + \Delta p_{qg}\Big(-\frac{32}{27}S_1^4 - \frac{128}{9}S_2^2 - \frac{128}{3}S_{3,1} + \frac{256}{3}S_{2,1,1} \\
& + \frac{256}{9}S_4\Big) - \frac{128 S_{2,1}}{3N^2} + \Delta p_{qg}\Big(\frac{2Q_{13}}{9N^3(1+N)^3} + \frac{16(6+5N)S_1}{9N} - \frac{16}{3}S_1^2\Big)\zeta_2 \\
& + \Delta p_{qg}\Big(-\frac{56Q_1}{9N^2(1+N)^2} + \frac{224S_1}{9}\Big)\zeta_3\Big] + \Delta p_{qg}\Big(\frac{2Q_{15}}{9N^3(1+N)^3} + \frac{80(3+N)S_1}{9N} \\
& - \frac{40}{3}S_1^2 + 8S_2\Big)\zeta_2\Big] + C_A T_F\Big[ \mathsf{B}_4 \Delta p_{qg}\Big(\frac{32(-5+3N+3N^2)}{N(1+N)} + 32S_1\Big) + \Big(-\frac{4S_1Q_9}{9N^3(1+N)^3} \\
& + \frac{Q_{16}}{18N^4(1+N)^4} + \frac{8(27-40N-12N^2+N^3)S_1^2}{3N^2(1+N)^2} + 32\Delta p_{qg}S_1^3 - 12{\Delta p_{qg}}^2(2+N)S_2 \\
& -8\Delta p_{qg}S_3 + \Delta p_{qg}\Big(-\frac{8(1+3N+3N^2)}{N(1+N)} + 16S_1\Big)S_{-2} - 8\Delta p_{qg}S_{-3} + 16\Delta p_{qg}S_{-2,1}\Big)\zeta_2 \\
& + \Big(-\frac{288\Delta p_{qg}(-5+3N+3N^2)}{5N(1+N)} - \frac{288}{5}\Delta p_{qg}S_1\Big)\zeta_2^2\Big]\Big] \\
& + C_A T_F^2\Big[N_F\Big[-\frac{32S_3Q_4}{81N^2(1+N)^2(2+N)} + \frac{8S_2Q_{11}}{81N^3(1+N)^3(2+N)} - \frac{8Q_{19}}{243N^5(1+N)^5(2+N)} \\
& + \Big(\frac{16Q_{10}}{243N(1+N)^4(2+N)} - \frac{16(139+38N+71N^2+40N^3)S_2}{27N(1+N)^2(2+N)} + \frac{1888}{27}\Delta p_{qg}S_3 \\
& + \frac{224}{9}\Delta p_{qg}S_{2,1} - 64\Delta p_{qg}S_{-2,1}\Big)S_1 + \Big(\frac{8Q_3}{81N(1+N)^3(2+N)} + \frac{176}{9}\Delta p_{qg}S_2\Big)S_1^2 \\
& - \frac{16(83+82N+67N^2+20N^3)S_1^3}{81N(1+N)^2(2+N)} + \Delta p_{qg}\Big(\frac{32}{27}S_1^4 + \frac{80}{9}S_2^2 + \frac{640}{9}S_4\Big) + \Big(32\Delta p_{qg}S_1^2 \\
& + \frac{160}{3}\Delta p_{qg}S_2 + \frac{32(-296+49N-40N^2+47N^3)}{81N(1+N)^3} - \frac{64(7-6N+5N^2)S_1}{9N(1+N)^2}\Big)S_{-2}
\end{aligned}
$$

$$
\begin{aligned}
& + \Big(-\frac{160(7-6N+5N^2)}{27N(1+N)^2} + \frac{160}{3}\Delta p_{qg}S_1\Big)S_{-3} + \frac{352}{9}\Delta p_{qg}S_{-4} - \frac{64(7-9N+5N^2)S_{2,1}}{27N(1+N)^2} \\
& - \frac{32}{9}\Delta p_{qg}S_{3,1} + \frac{64(7-6N+5N^2)S_{-2,1}}{9N(1+N)^2} + \Delta p_{qg}\Big(-\frac{128}{3}S_{-2,2} - \frac{160}{3}S_{-3,1} - \frac{416}{9}S_{2,1,1} + 64S_{-2,1,1}\Big) \\
& + \Big(-\frac{4Q_6}{9N^3(1+N)^3} - \frac{16(7+5N^2)S_1}{9N(1+N)^2} + \Delta p_{qg}\big(\frac{16}{3}S_1^2 + \frac{16}{3}S_2 + \frac{32}{3}S_{-2}\big)\Big)\zeta_2 \\
& + \Delta p_{qg}\Big(\frac{448}{9N(1+N)} - \frac{224S_1}{9}\Big)\zeta_3\Big] + \Big(-\frac{4Q_7}{9N^3(1+N)^3} + \frac{160(-2+N)(2+N)S_1}{9N(1+N)^2} \\
& + \Delta p_{qg}\Big(\frac{40}{3}S_1^2 + \frac{40}{3}S_2 + \frac{80}{3}S_{-2}\Big)\Big)\zeta_2\Big] + C_F^2 T_F\Big[-\frac{16 \mathsf{B}_4 \Delta p_{qg}(-2+3N+3N^2)}{N(1+N)} \\
& + \Delta p_{qg}\Big(-\frac{Q_{14}}{2N^3(1+N)^3} + \Big(\frac{8Q_2}{N^2(1+N)^2} + 32S_2\Big)S_1 + \frac{4(2+N)(5+3N)S_1^2}{N(1+N)} - 16S_1^3 \\
& - \frac{8(2+3N+3N^2)S_2}{N(1+N)} + 16S_3 + \Big(-\frac{16}{N(1+N)} + 32S_1\Big)S_{-2} + 16S_{-3} - 32S_{-2,1}\Big)\zeta_2 \\
& + \frac{144\Delta p_{qg}(-2+3N+3N^2)}{5N(1+N)}\zeta_2^2\Big] + C_A^2 T_F\Big[\mathsf{B}_4\Big(-36{\Delta p_{qg}}^2(2+N) - 16\Delta p_{qg}S_1\Big) \\
& + \Big(\frac{2Q_{17}}{9N^4(1+N)^4} + \Big(\frac{4Q_8}{9N^3(1+N)^3} - 32\Delta p_{qg}S_2\Big)S_1 - \frac{4(24-83N+11N^3)S_1^2}{3N^2(1+N)^2} \\
& -16\Delta p_{qg}S_1^3 - \frac{4\Delta p_{qg}(-48+11N+11N^2)S_2}{3N(1+N)} - 8\Delta p_{qg}S_3 - 8\Delta p_{qg}S_{-3} \\
& + \Big(-\frac{8\Delta p_{qg}(-36+11N+11N^2)}{3N(1+N)} - 48\Delta p_{qg}S_1\Big)S_{-2} + 16\Delta p_{qg}S_{-2,1}\Big)\zeta_2 \\
& + \Delta p_{qg}\Big(\frac{144(-8+3N+3N^2)}{5N(1+N)} + \frac{288}{5}S_1\Big)\zeta_2^2\Big] - \frac{64}{9}\Delta p_{qg}T_F^3\zeta_3\Big\} + O(\mathrm{rat}) + O(\zeta_3),
\end{aligned}
$$

# Results in $N$ space and non-first order factorizing recurrences

- Characteristics of recurrences:
  - Calculate up to 15000 Mellin moments.
  - Largest rational numbers 31k digits/26.6k digits.
  - Largest recurrence size obtained: 0.7 Gb.
  - These are not all recurrences needed. I.e. the N space computational efforts are even more demanding than $O(1 year)$ of CPU time.
  - These large recurrences cannot be solved currently. The few general factorization algorithms are by far not efficient enough and sometimes do not terminate. Methods to do this are, however, under development.
  - One can compute the asymptotic representations for these recurrences in principle. The requested time investment is, however, large.
- Use $N \to t$ and differential eq. technologies there to solve the whole project.

# Inverse Mellin transform via analytic continuation: $a_{Qg}^{(3)}$

Resumming Mellin $N$ into a continuous variable $t$, observing crossing relations. Ablinger et al. 2014

$$\sum_{k=0}^{\infty} t^k (\Delta.p)^k \frac{1}{2}[1 \pm (-1)^k] = \frac{1}{2}\left[\frac{1}{1 - t\Delta.p} \pm \frac{1}{1 + t\Delta.p}\right]$$

$$\mathfrak{A} = \{f_1(t), ..., f_m(t)\}, \quad \mathrm{G}(b, \vec{a}; t) = \int_0^t dx_1 f_b(x_1) \mathrm{G}(\vec{a}; x_1), \quad \left[\frac{d}{dt} \frac{1}{f_{a_{k-1}}(t)} \frac{d}{dt} \cdots \frac{1}{f_{a_1}(t)} \frac{d}{dt}\right] \mathrm{G}(\vec{a}; t) = f_{a_k}(t).$$

Regularization for $t \to 0$ needed.

$$\begin{aligned} F(N) &= \int_0^1 dx x^{N-1} [f(x) + (-1)^{N-1} g(x)] \\ \tilde{F}(t) &= \sum_{N=1}^{\infty} t^N F(N) \\ f(x) + (-1)^{N-1} g(x) &= \frac{1}{2\pi i}\left[\mathrm{Disc}_x \tilde{F}\left(\frac{1}{x}\right) + (-1)^{N-1} \mathrm{Disc}_x \tilde{F}\left(-\frac{1}{x}\right)\right]. \end{aligned} \tag{1}$$

t-space is still Mellin space. One needs closed expressions to perform the analytic continuation (1). Continuation is needed to calculate the small $x$ behaviour analytically.

# Harmonic polylogarithms

$$\mathfrak{A}_{\rm HPL} = \{f_0, f_1, f_{-1}\}\left\{\frac{1}{t}, \frac{1}{1-t}, \frac{1}{1+t}\right\}$$

$$\mathrm{H}_{b,\vec{a}}(x) = \int_0^x dy f_b(y) \mathrm{H}_{\vec{a}}(y), \quad f_c \in \mathfrak{A}_{\rm HPL}, \quad \mathrm{H}_{\underbrace{0,\dots,0}_{k}}(x) := \frac{1}{k!}\ln^k(x).$$

A finite monodromy at $x = 1$ requires at least one letter $f_1(t)$.

Example:

$$\tilde{F}_1(t) = \mathrm{H}_{0,0,1}(t)$$

$$F_1(x) = \frac{1}{2}\mathrm{H}_0^2(x)$$

$$\mathbf{M}[F_1(x)](n-1) = \frac{1}{n^3}$$

$$\tilde{F}_1(t) = t + \frac{t^2}{8} + \frac{t^3}{27} + \frac{t^4}{64} + \frac{t^5}{125} + \frac{t^6}{216} + \frac{t^7}{343} + \frac{t^8}{512} + \frac{t^9}{729} + \frac{t^{10}}{1000} + O(t^{11})$$

# Square root valued alphabets

$$
\begin{aligned}
\mathfrak{A}_{\rm sqrt} &= \left\{ f_4, f_5, f_6 \ldots \right\} \\
&= \left\{ \frac{\sqrt{1-x}}{x}, \sqrt{x(1-x)}, \frac{1}{\sqrt{1-x}}, \frac{1}{\sqrt{x}\sqrt{1 \pm x}}, \frac{1}{x\sqrt{1 \pm x}}, \frac{1}{\sqrt{1 \pm x}\sqrt{2 \pm x}}, \frac{1}{x\sqrt{1 \pm x/4}}, \ldots \right\},
\end{aligned}
$$

Monodromy also through:

$$
(1-t)^\alpha, \quad \alpha \in \mathbb{R},
$$

$$
\begin{aligned}
F_7(x) &= \frac{1}{\pi} \mathrm{Im} \frac{1}{t} \mathrm{G}\left(4; \frac{1}{t}\right) = 1 - \frac{2(1-x)(1+2x)}{\pi}\sqrt{\frac{1-x}{x}} - \frac{8}{\pi}\mathrm{G}(5; x), \\
F_8(x) &= \frac{1}{\pi} \mathrm{Im} \frac{1}{t} \mathrm{G}\left(4, 2; \frac{1}{t}\right) = -\frac{1}{\pi}\Bigg[4\frac{(1-x)^{3/2}}{\sqrt{x}} + 2(1-x)(1+2x)\sqrt{\frac{1-x}{x}}[\mathrm{H}_0(x) + \mathrm{H}_1(x)] \\
&\quad +8[\mathrm{G}(5, 2; x) + \mathrm{G}(5, 1; x)]\Bigg],
\end{aligned}
$$

# Iterative non-iterative Integrals

- Master integrals, solving differential equations not factorizing to 1st order
- ${}_2F_1$ solutions Ablinger et al. [2017]
- Mapping to complete elliptic integrals: duplication of the higher transcendental letters.
- Complete elliptic integrals, modular forms Sabry, Broadhurst, Weinzierl, Remiddi, Tancredi, Duhr, Broedel et al. and many more
- Abel integrals
- K3 surfaces Brown, Schnetz [2012]
- Calabi-Yau motives Klemm, Duhr, Weinzierl et al. [2022]

Refer to as few as possible higher transcendental functions, the properties of which are known in full detail.

- $A_{Qg}^{(3)}$: effectively only one $3 \times 3$ system of this kind.
- The system is connected to that occurring in the case of $\rho$ parameter. Ablinger et al. [2017], JB et al. [2018], Abreu et al. [2019]
- Most simple solution: two ${}_2F_1$ functions.

# Iterative non-iterative Integrals

$$\frac{d}{dt}\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & -\frac{1}{1-t} & 0 \\ 0 & -\frac{1}{t(1-t)} & -\frac{2}{1-t} \\ 0 & \frac{2}{t(8+t)} & \frac{1}{8+t} \end{bmatrix} \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} + \begin{bmatrix} R_1(t,\varepsilon) \\ R_2(t,\varepsilon) \\ R_3(t,\varepsilon) \end{bmatrix} + O(\varepsilon),$$

It is very important to which function $F_i(t)$ the system is decoupled.

# Iterative non-iterative Integrals

- Decoupling for $F_1$ first leads to a very involved solution: ${}_2F_1$-terms seemingly enter at $O(1/\varepsilon)$ already.
- However, these terms are actually not there.
- Furthermore, there is also a singularity at $x = 1/4$.
- All this can be seen, when decoupling for $F_3$ first.

Homogeneous solutions:

$$F_3'(t) + \frac{1}{t} F_3(t) = 0, \quad g_0 = \frac{1}{t}$$

$$F_1''(t) + \frac{(2-t)}{(1-t)t} F_1'(t) + \frac{2+t}{(1-t)t(8+t)} F_1(t) = 0,$$

with

$$\begin{aligned}
g_1(t) &= \frac{2}{(1-t)^{2/3}(8+t)^{1/3}}\,{}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{4}{3} \\ 2\end{matrix}; -\frac{27t}{(1-t)^2(8+t)}\right], \\
g_2(t) &= \frac{2}{(1-t)^{2/3}(8+t)^{1/3}}\,{}_2F_1\left[\begin{matrix}\frac{1}{3}, \frac{4}{3} \\ \frac{2}{3}\end{matrix}; 1+\frac{27t}{(1-t)^2(8+t)}\right],
\end{aligned}$$

# Iterative non-iterative Integrals

Alphabet:

$$\mathfrak{A}_2 = \Biggl\{\frac{1}{t}, \frac{1}{1-t}, \frac{1}{8+t}, g_1, g_2, \frac{g_1}{t}, \frac{g_1}{1-t}, \frac{g_1}{8+t}, \frac{g_1'}{t}, \frac{g_1'}{1-t}, \frac{g_1'}{8+t}, \frac{g_2}{t}, \frac{g_2}{1-t}, \frac{g_2}{8+t}, \frac{g_2'}{t}, \frac{g_2'}{1-t}, \frac{g_2'}{8+t}, tg_1, tg_2\Biggr\}$$

$$F_1(t) = \frac{8}{\varepsilon^3}\left[1+\frac{1}{t}\mathrm{H}_1(t)\right] - \frac{1}{\varepsilon^2}\left[\frac{1}{6}(106+t) + \frac{(9+2t)}{t}\mathrm{H}_1(t) + \frac{4}{t}\mathrm{H}_{0,1}(t)\right]$$
$$+\frac{1}{\varepsilon}\Biggl\{\frac{1}{12}(271+9t) + \left[\frac{71+32t+2t^2}{12t} + \frac{3\zeta_2}{t}\right]\mathrm{H}_1(t) + \frac{(9+2t)}{2t}\mathrm{H}_{0,1}(t) + \frac{2}{t}\mathrm{H}_{0,0,1}(t)$$
$$+3\zeta_2\Biggr\} + \frac{1}{t}\Biggl\{\frac{6696-22680t-16278t^2-255t^3-62t^4}{864t} + (9+9t+t^2)g_1(t)\Biggl[\frac{31\ln(2)}{16}$$
$$+\frac{1}{144}(265+31\pi(-3i+\sqrt{3})) + \frac{3}{8}\ln(2)\zeta_2 + \frac{1}{24}(10+\pi(-3i+\sqrt{3}))\zeta_2 - \frac{7}{4}\zeta_3\Biggr]$$

$$
\begin{aligned}
&+\mathrm{G}(18,t)\Bigg[-\frac{93\ln(2)}{16}+\frac{1}{48}\big(-265-31\pi(-3i+\sqrt{3})\big)+\Bigg(-\frac{9\ln(2)}{8} \\
&+\frac{1}{8}\big(-10-\pi\big(-3i+\sqrt{3}\big)\big)\Bigg)\zeta_2+\frac{21}{4}\zeta_3\Bigg]\ldots \\
&+\frac{5}{2}[\mathrm{G}(4,14,1,2;t)-\mathrm{G}(5,8,1,2;t)]+\frac{1}{4}[\mathrm{G}(13,8,1,2;t)-\mathrm{G}(7,14,1,2;t)] \\
&+\frac{9}{4}[\mathrm{G}(10,14,1,2;t)-\mathrm{G}(16,8,1,2;t)]+\frac{3}{4}[\mathrm{G}(19,14,1,2;t)-\mathrm{G}(19,8,1,2;t)]\Bigg\}+O(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
F_2(t) \quad = \quad & \frac{8}{\varepsilon^3}+\frac{1}{\varepsilon^2}\Bigg[-\frac{1}{3}(34+t)+\frac{2(1-t)}{t}\mathrm{H}_1(t)\Bigg]+\frac{1}{\varepsilon}\Bigg[\frac{116+15t}{12}+3\zeta_2-\frac{(1-t)(8+t)}{3t}\mathrm{H}_1(t) \\
&-\frac{1-t}{t}\mathrm{H}_{0,1}(t)\Bigg]+\frac{992-368t+75t^2-27t^3}{144t}+(1-t)\Bigg(\frac{\big(43+10t+t^2\big)}{12t}\mathrm{H}_1(t)+\frac{(4-t)}{4t} \\
&\times\mathrm{H}_{0,1}(t)+\frac{3\zeta_2}{4t}\mathrm{H}_1(t)\Bigg)+(1-t)g_1(t)\Bigg(\frac{31\ln(2)}{16}+\frac{1}{144}\big(265+31\pi\big(-3i+\sqrt{3}\big)\big)\ldots
\end{aligned}
$$

Essential step for calculating $a_{Qg}^{(3)}$ completely.

# 1st order factorizing contributions: $a_{Qg}^{(3)}$

- 1009 of 1233 contributing Feynman diagrams
- Solved: $N_F$-terms, $\zeta_2, \zeta_4$ and $B_4$ terms, unpolarized and polarized.
- Contributions to the rational and $\zeta_3$ terms:
  - The sum of the contributions vanishes for $N \rightarrow \infty$, while the individual terms $\propto 1$ and $\propto \zeta_3$ do strongly diverge.
  - Dynamical generation of a factor of $\zeta_3$.
  - Calculated asymptotic expansions in $N$ space: harmonic sums, generalized harmonic sums, binomial sums
  - Appearance of a large set of special numbers given as G-functions at $x = 1$
  - individually divergent contributions for $N \rightarrow \infty$: $\propto 2^N, 4^N$ cancel between the different terms
- Calculated inverse Mellin transforms: requires the use of the $t$-variable method in the most involved cases for nested binomial sums.

# Structure in $x$ space of the 1st order reducible terms

Expansion around $x = 1$:

$$\sum_{k=0}^{\infty} \sum_{l=0}^{L} \hat{a}_{k,l} (1-x)^k \ln^l(1-x).$$

Expansion around $x = 0$:

$$\frac{1}{x} \sum_{k=0}^{\infty} \sum_{l=0}^{S} \hat{b}_{k,l} x^k \ln^l(x).$$

Expansion around $x = 1/2$:

$$\sum_{k=0}^{\infty} \hat{c}_k \left(x - \frac{1}{2}\right)^k.$$

Wide double precision overlaps of the expansions around $x = 2/10$ and $x = 7/10$ by using 100 expansion terms.

These results are still analytic since the coefficients are given in terms of rational numbers, weighted by MZVs. As already in the case of HPLs, one finally needs a numerical representation.

# Structure in $x$ space

- The analytic results on the expansion coefficients contain iterated integrals over up to root-valued letters at main argument $x = 1$.
- One may rationalize the letters of these constants and switch to linear representations.
- This results into enormous numbers of Kummer-Poincaré integrals, which are calculated to 100 digits.
- **Checks:**
  - Mellin moments even diagram by diagram
  - Parallel solution of 1st order DEQs with high precision numerical matching at the (pseudo) thresholds.

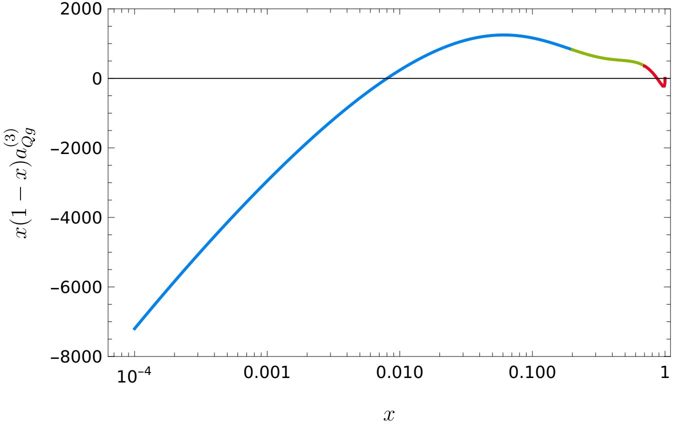
Unpolarized case:

- $\zeta_2$ term of the predicted $\ln(x)/x$ small $x$ expansion confirmed Catani, Ciafaloni, Hautmann, 1991

Polarized case:

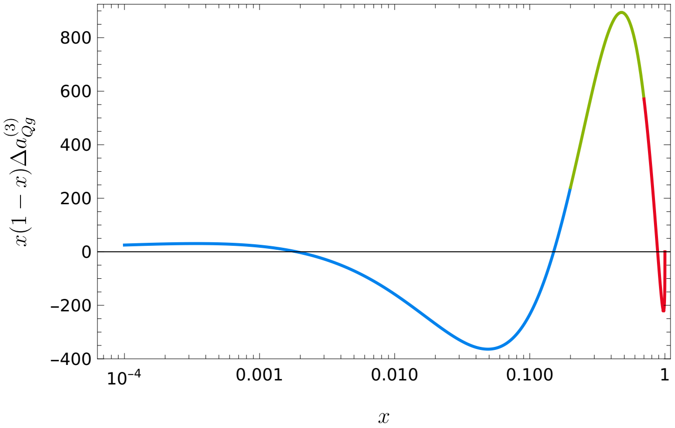
- Evanescent $\ln(x)/x$ and $1/x$ terms occur.
- One has to show their cancellation. Many special constants are involved here.
- New $\ln^5(x)$ term $\propto N_F$ found.

# $a_{Qg}^{(fact.,3)}$




The first order factorizable contributions to $a_{Qg}^{(3)}(N)$. Full line (blue): $x < 0.2$; Full line (green): $0.2 < x < 0.7$; Full line (blue): $0.7 < x < 1$ for $m_c = 1.59$ GeV and $N_F = 3$.

# $\Delta a_{Qg}^{(fact.,3)}$



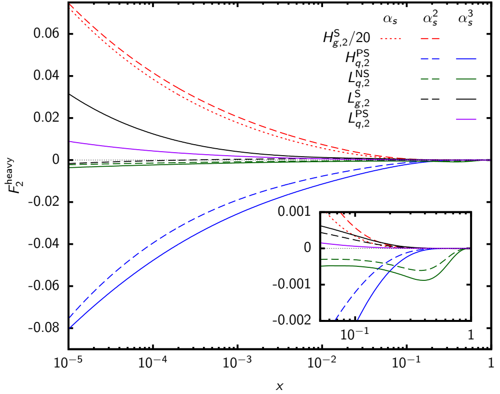

The first order factorizable contributions to $\Delta a_{Qg}^{(3)}(N)$. Full line (blue): $x < 0.2$; Full line (green): $0.2 < x < 0.7$; Full line (blue): $0.7 < x < 1$ for $m_c = 1.59$ GeV and $N_F = 3$.

# Some remarks about the constants

- Let $\mathfrak{A} = \{f_1, ...., f_m\}$ be the alphabet of the problem, including higher transcendental letters.
- The constants are $G(\{h_1, ..., h_k\}, 1)$ with $h_k \in \mathfrak{A}$.
- The constants up to square-root valued letters in the present project could all be rationalized concerning their letters, which can be partial fractioned to Kummer-Poincaré letters. The latter ones can be calculated numerically [Weinzierl, Vollinga, 2004]. These mappings lead to a large proliferation concerning the number of terms.
- In the case of ${}_2F_1$ contributions, these letters occur next to each other, probably with no suffix in the $G$-words.
- The prefix of the $G$-word is then a $G$-function of $x \in [0, 1]$ with up to square-root valued letters.
- The latter functions can be given very precise series representations, as outlined on page 19.
- The constants are then at most two-dimensional integrals.
- All of these constants have to be computed to very high precision.

# Current summary on $F_2^{charm}$

An example to show numerical effects: the charm quark contributions to the structure function $F_2(x, Q^2)$




for $Q^2 = 100$ GeV$^2$.

Allows to strongly reduce the current theory error on $m_c$.

Started $\sim$ 2009; will be completed very soon.

Lots of new algorithms had to be designed; different new function spaces; new analytic calculation techniques ...

# Conclusions

- All unpolarized and polarized single and two-mass OMEs, except the ones for $A_{Qg}^{(3)}$, and the associated massive Wilson coefficients for $Q^2 \gg m_Q^2$ have been calculated, including also all logarithmic contributions.
- Various new mathematical and technological steps were performed to prepare the calculation of $(\Delta)A_{Qg}^{(3)}$.
- Recently all elliptic base master integrals necessary to complete the calculation for $(\Delta)A_{Qg}^{(3)}$ were computed analytically.
- We have calculated already all the first-order factorizing contributions to $(\Delta)A_{Qg}^{(3)}$.
- The completion of $(\Delta)A_{Qg}^{(3)}$ is underway and will allow new precision analyses of the world DIS-data to measure $\alpha_s(M_Z)$ and $m_c$ at higher precision.
- In the small $x$ region BFKL approaches fail to present the physical result due to quite a lot of missing subleading series, substantially correcting the LO behaviour. The growth of $F_2$ at small $x$ is a consequence of the shape of the non-perturbative PDFs and complete fixed order evolution at twist 2.