Recent 3-Loop Heavy Flavor Corrections to Deep-Inelastic Scattering

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DESY

Based on:

- A. Behring, J.B., and K. Schönwald, The inverse Mellin transform via analytic continuation, DESY 20–053, JHEP (2023) in print.
- J. Ablinger et al., The unpolarized and polarized single-mass three-loop heavy flavor operator matrix elements $A_{gg}^{(3)}$ *and* $\Delta A_{gg}^{(3)}$, JHEP **12** (2022) 134.

In collaboration with:

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Outline

Introduction

- Massive OMEs allow to describe the massive DIS Wilson coefficients for $Q^2 \gg m_Q^2$.
- Furthermore, they form the transition elements in the variable flavor numer scheme (VFNS).
- The current state of art is 3-loop order, including two-mass corrections, because m_c/m_b is not small.
- After having calculated a series of moments in 2009 I. Bierenbaum, JB, S. Klein, Nucl. Phys B **820** (2009) 417, we started to calculate all OMEs for general values of the Mellin variable N.
- There are the following massive OMEs: $A_{qq,Q}^{\rm NS}, A_{qg,Q}, A_{qq,Q}^{\rm PS}, A_{gq,Q}, A_{Qq}^{\rm PS}, A_{gg,Q}, A_{Qg}$.
- To 2-loop order $A_{qq,Q}^{\rm NS}, A_{Qq}^{\rm PS}, A_{Qg}$, [2007] $A_{gq,Q}, A_{gg,Q}$ [2009] contribute. These quantities are represented by harmonic sums resp. harmonic polylogarithms. [Older work by van Neerven, et al.]
- The 3-loop contributions of $O(N_F)$ [2010] to all OMEs and the $A_{qq,Q}^{\rm NS}, A_{qg,Q}, A_{gq,Q}, A_{qq,Q}^{\rm PS}$ [2014] are also given by harmonic sums only. [Also all logarithmic terms of all OMEs.]
- For $A_{Qq}^{\rm PS}$ [2014] also generalized harmonic sums are necessary.
- $A_{gg,Q}$ [2022] requires finite binomial sums.
- Finally, A_{Qg} depends also on ${}_2F_1$-solutions [2017] (or modular forms).
- In the two-mass case to 3-loop order $A_{qq,Q}^{\rm NS}, A_{qg,Q}, A_{qq,Q}^{\rm PS}, A_{Qq}^{\rm PS}, A_{gq,Q}, A_{gg,Q}$ [2017-2020] can be solved analytically due to 1st order factorization of the respective differential equations. The solution for A_{Qg} is by far more involved.

I. The massive OME $A_{gg,Q}^{(3)}$

A 1st order factorizing, but involved case.

$$
\begin{aligned}
\hat{\hat{A}}_{gg,Q}^{(1)} &= \left(\frac{\hat{m}^2}{\mu^2}\right)^{\varepsilon/2}\left[\frac{\hat{\gamma}_{gg}^{(0)}}{\varepsilon} + a_{gg,Q}^{(1)} + \varepsilon \overline{a}_{gg,Q}^{(1)} + \varepsilon^2 \overline{\overline{a}}_{gg,Q}^{(1)}\right] + O(\varepsilon^3), \\
\hat{\hat{A}}_{gg,Q}^{(2)} &= \left(\frac{\hat{m}^2}{\mu^2}\right)^{\varepsilon}\left[\frac{1}{\varepsilon^2} c_{gg,Q,(2)}^{(-2)} + \frac{1}{\varepsilon} c_{gg,Q,(2)}^{(-1)} + c_{gg,Q,(2)}^{(0)} + \varepsilon c_{gg,Q,(2)}^{(1)}\right] + O(\varepsilon^2), \\
\hat{\hat{A}}_{gg,Q}^{(3)} &= \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2}\left[\frac{1}{\varepsilon^3} c_{gg,Q,(3)}^{(-3)} + \frac{1}{\varepsilon^2} c_{gg,Q,(3)}^{(-2)} + \frac{1}{\varepsilon} c_{gg,Q,(3)}^{(-1)} + a_{gg,Q}^{(3)}\right] + O(\varepsilon)\ .
\end{aligned}
$$

The alphabet:

$$
\mathfrak{A} = \{f_k(x)\}|_{k=1..6} = \left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{\sqrt{1-x}}{x}, \sqrt{x(1-x)}, \frac{1}{\sqrt{1-x}}\right\}.
$$

Principal computation steps

Chains of packages are used to perform the calculation:

- `QGRAF`, Nogueira, 1993 Diagram generation
- `FORM`, Vermaseren, 2001; Tentyukov, Vermaseren, 2010 Lorentz algebra
- `Color`, van Ritbergen, Schellekens and Vermaseren, 1999 Color algebra
- `Reduze 2` Studerus, von Manteuffel, 2009/12, `Crusher`, Marquard, Seidel IBPs
- Method of arbitrary high moments, JB, Schneider, 2017 Computing large numbers of Mellin moments
- `Guess`, Kauers et al. 2009/2015; JB, Kauers, Schneider, 2009 Computing the recurrences
- `Sigma`, `EvaluateMultiSums`, `SolveCoupledSystems`, Schneider, 2007/14 Solving the recurrences
- `OreSys`, Zürcher, 1994; Gerhold, 2002; Bostan et al., 2013 Decoupling differential and difference equations
- `Diffeq`, Ablinger et al, 2015, JB, Marquard, Rana, Schneider, 2018 Solving differential equations
- `HarmoncisSums`, Ablinger and Ablinger et al. 2010-2019 Simplifying nested sums and iterated integrals to basic building blocks, performing series and asymptotic expansions, Almkvist-Zeilberger algorithm etc.

Binomial Sums

$$\mathrm{BS}_0(N) = \frac{1}{2N-(2l+1)}, \quad l \in \mathbb{N}, \qquad \mathrm{BS}_1(N) = 4^N \frac{(N!)^2}{(2N)!},$$

$$\mathrm{BS}_2(N) = \frac{1}{4^N}\frac{(2N)!}{(N!)^2}, \qquad \mathrm{BS}_3(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)!}{(\tau_1!)^2 \tau_1},$$

$$\mathrm{BS}_4(N) = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}(\tau_1!)^2}{(2\tau_1)!\tau_1^2}, \qquad \mathrm{BS}_5(N) = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}(\tau_1!)^2}{(2\tau_1)!\tau_1^3},$$

$$\mathrm{BS}_6(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^2}}{(\tau_1!)^2\tau_1}, \qquad \mathrm{BS}_7(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^3}}{(\tau_1!)^2\tau_1},$$

$$\mathrm{BS}_8(N) = \sum_{\tau_1=1}^{N} \frac{\sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^2}}{\tau_1}, \qquad \mathrm{BS}_9(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2 \sum_{\tau_3=1}^{\tau_2}\frac{1}{\tau_3}}{(2\tau_2)!\tau_2^2}}{(\tau_1!)^2\tau_1},$$

$$\mathrm{BS}_{10}(N) = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}}{\binom{2\tau_1}{\tau_1}} \frac{1}{\tau_1^2} S_1(\tau_1).$$

Recursions and Asymptotic Representation

$$\mathrm{BS}_8(N) - \mathrm{BS}_8(N-1) = \frac{1}{N}\mathrm{BS}_4(N),$$
$$\mathrm{BS}_9(N) - \mathrm{BS}_9(N-1) = \frac{1}{N}\mathrm{BS}_3(N)\mathrm{BS}_{10}(N),$$
$$\mathrm{BS}_{10}(N) - \mathrm{BS}_{10}(N-1) = \frac{1}{N}\mathrm{BS}_1(N)S_1.$$

$$\mathrm{BS}_0(N) \propto \frac{1}{2N}\sum_{k=0}^{\infty}\left(\frac{2l+1}{2N}\right)^k,$$

$$\mathrm{BS}_8(N) \propto -7\zeta_3 + \left[+3(\ln(N)+\gamma_E) + \frac{3}{2N} - \frac{1}{4N^2} + \frac{1}{40N^4} - \frac{1}{84N^6} + \frac{1}{80N^8} - \frac{1}{44N^{10}}\right]\zeta_2$$
$$+\sqrt{\frac{\pi}{N}}\left[4 - \frac{23}{18N} + \frac{1163}{2400N^2} - \frac{64177}{564480N^3} - \frac{237829}{7741440N^4} + \frac{5982083}{166526976N^5}\right.$$
$$+\frac{5577806159}{438593126400N^6} - \frac{12013850977}{377864847360N^7} - \frac{1042694885077}{90766080737280N^8}$$
$$\left.+\frac{6663445693908281}{127863697547722752N^9} + \frac{23651830282693133}{1363413316298342400N^{10}}\right],$$

Inverse Mellin Transform

$$\mathbf{M}^{-1}[\mathrm{BS}_8(N)](x) = \Biggl[-\frac{4\left(1-\sqrt{1-x}\right)}{1-x} + \left(\frac{2(1-\ln(2))}{1-x} + \frac{\mathrm{H}_0(x)}{\sqrt{1-x}}\right)\mathrm{H}_1(x) - \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}} + \frac{\mathrm{H}_1(x)\mathrm{G}(\{6,1\},x)}{2(1-x)} - \frac{\mathrm{G}(\{6,1,2\},x)}{2(1-x)}\Biggr]_+ ,$$

$$\mathbf{M}^{-1}[\mathrm{BS}_{10}(N)](x) = \Biggl[-\frac{1}{1-x}\left[-4-4\ln(2)\left(-1+\sqrt{1-x}\right)+4\sqrt{1-x}+\zeta_2\right] + 2(-1+\ln(2))\left(-1+\sqrt{1-x}+x\right)\frac{\mathrm{H}_0(x)}{(1-x)^{3/2}} - 2\frac{\mathrm{H}_1(x)}{\sqrt{1-x}} + \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}} - \frac{(-2+\ln(2))\mathrm{G}(\{6,1\},x)}{1-x} + \frac{\mathrm{G}(\{6,1,2\},x)}{2(1-x)} - \frac{\mathrm{G}(\{1,6,1\},x)}{2(1-x)}\Biggr]_+ .$$

Small x limits of $a_{gg,Q}^{(3)}$

$$a_{gg,Q}^{x\to 0}(x) \propto$$

$$\frac{1}{x}\Biggl\{\ln(x)\Biggl[C_A^2 T_F\left(-\frac{11488}{81}+\frac{224\zeta_2}{27}+\frac{256\zeta_3}{3}\right)+C_A C_F T_F\left(-\frac{15040}{243}-\frac{1408\zeta_2}{27}\right.$$

$$\left.-\frac{1088\zeta_3}{9}\right)\Biggr]+C_A T_F^2\Biggl[\frac{112016}{729}+\frac{1288}{27}\zeta_2+\frac{1120}{27}\zeta_3+\left(\frac{108256}{729}+\frac{368\zeta_2}{27}-\frac{448\zeta_3}{27}\right)$$

$$\times N_F\Biggr]+C_F\Biggl[T_F^2\left(-\frac{107488}{729}-\frac{656}{27}\zeta_2+\frac{3904}{27}\zeta_3+\left(\frac{116800}{729}+\frac{224\zeta_2}{27}-\frac{1792\zeta_3}{27}\right)N_F\right)$$

$$+C_A T_F\left(-\frac{5538448}{3645}+\frac{1664\mathrm{B}_4}{3}-\frac{43024\zeta_4}{9}+\frac{12208}{27}\zeta_2+\frac{211504}{45}\zeta_3\right)\Biggr]$$

$$+C_A^2 T_F\left(-\frac{4849484}{3645}-\frac{352\mathrm{B}_4}{3}+\frac{11056\zeta_4}{9}-\frac{1088}{81}\zeta_2-\frac{84764}{135}\zeta_3\right)$$

$$+C_F^2 T_F\left(\frac{10048}{5}-640\mathrm{B}_4+\frac{51104\zeta_4}{9}-\frac{10096}{9}\zeta_2-\frac{280016}{45}\zeta_3\right)\Biggr\}$$

Small x limits of $a_{gg,Q}^{(3)}$

$$
\begin{aligned}
&+\left[-\frac{4}{3}C_F C_A T_F+\frac{2}{15}C_F^2 T_F\right]\ln^5(x)+\Biggl[-\frac{40}{27}C_A^2 T_F+\frac{4}{9}C_F^2 T_F+C_F\Biggl(-\frac{296}{27}C_A T_F\\
&+\left(\frac{28}{27}+\frac{56}{27}N_F\right)T_F^2\Biggr)\Biggr]\ln^4(x)+\Biggl[\frac{112}{81}C_A(1+2N_F)T_F^2+C_F\Biggl(\left(\frac{1016}{81}+\frac{496}{81}N_F\right)T_F^2\\
&+C_A T_F\left(-\frac{10372}{81}-\frac{328\zeta_2}{9}\right)\Biggr)+C_F^2 T_F\left[-\frac{2}{3}+\frac{4\zeta_2}{9}\right]+C_A^2 T_F\left[-\frac{1672}{81}+8\zeta_2\right]\Biggr]\ln^3(x)\\
&+\Biggl[\frac{8}{81}C_A(155+118N_F)T_F^2+C_F\Biggl[T_F^2\left(-\frac{32}{81}+N_F\left(\frac{3872}{81}-\frac{16\zeta_2}{9}\right)+\frac{232\zeta_2}{9}\right]\\
&+C_A T_F\left(-\frac{70304}{81}-\frac{680\zeta_2}{9}+\frac{80\zeta_3}{3}\right)\Biggr)+C_A^2 T_F\left[\frac{4684}{81}+\frac{20\zeta_2}{3}\right]+C_F^2 T_F\Biggl[56\\
&+\frac{8\zeta_2}{3}-40\zeta_3\Biggr]\Biggr]\ln^2(x)+\Biggl[C_F\Biggl[T_F^2\left(\frac{140992}{243}+N_F\left(\frac{182528}{243}-\frac{400\zeta_2}{27}-\frac{640\zeta_3}{9}\right)\right.
\end{aligned}
$$

Small and large x limits of $a_{gg,Q}^{(3)}$

$$
-\frac{728}{27}\zeta_2 - \frac{224}{9}\zeta_3\Bigg) + C_A T_F\Bigg(-\frac{514952}{243} + \frac{152\zeta_4}{3} - \frac{21140\zeta_2}{27} - \frac{2576\zeta_3}{9}\Bigg)\Bigg]
$$

$$
+C_A T_F^2\Bigg[\frac{184}{27} + N_F\Bigg(\frac{656}{27} - \frac{32\zeta_2}{27}\Bigg) + \frac{464\zeta_2}{27}\Bigg] + C_A^2 T_F\Bigg[-\frac{42476}{81} - 92\zeta_4 + \frac{4504\zeta_2}{27}
$$

$$
+\frac{64\zeta_3}{3}\Bigg] + C_F^2 T_F\Bigg[-\frac{1036}{3} - \frac{976\zeta_4}{3} - \frac{58\zeta_2}{3} + \frac{416\zeta_3}{3}\Bigg]\Bigg]\ln(x),
$$

$$
a_{gg,Q}^{(3),x\to 1}(x) \propto a_{gg,Q,\delta}^{(3)}\delta(1-x) + a_{gg,Q,\mathrm{plus}}^{(3)}(x) + \Bigg[-\frac{32}{27}C_A T_F^2(17+12N_F) + C_A C_F T_F\Bigg(56 - \frac{32\zeta_2}{3}\Bigg)
$$

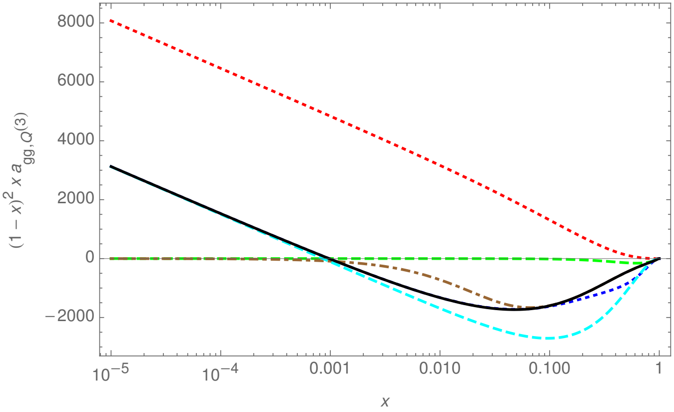
$$
+C_A^2 T_F\Bigg(\frac{9238}{81} - \frac{104\zeta_2}{9} + 16\zeta_3\Bigg)\Bigg]\ln(1-x) + \Bigg[-\frac{8}{27}C_A T_F^2(7+8N_F)
$$

$$
+C_A^2 T_F\Bigg(\frac{314}{27} - \frac{4\zeta_2}{3}\Bigg)\Bigg]\ln^2(1-x) + \frac{32}{27}C_A^2 T_F\ln^3(1-x).
$$

Representations of the OME

- The logarithmic parts of $(\Delta)A_{Qg}^{(3)}$ were computed in [Behring et al., (2014)], [JB et al. (2021)].
- We did not spent efforts to choose the MI basis such that the needed ε-expansion is minimal, which we could afford in all first order factorizing cases.
- N space
  - Recursions available for all building blocks: $N \to N + 1$.
  - Asymptotic representations available.
  - Contour integral around the singularities of the problem at the non-positive real axis.
- x space
  - All constants occurring in the transition $t \to x$ can be calculated in terms of ζ-values.
  - This can be proven analytically by first rationalizing and then calculating the obtained cyclotomic G-functions.
  - Separate the $\delta(1 - x)$ and $+$-function terms first.
  - Series representations to 50 terms around $x = 0$ and $x = 1$ can be derived for the regular part analytically (12 digits).
  - The accuracy can be easily enlarged, if needed.

$a_{gg,Q}^{(3)}$



The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$, BFKL limit; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution.

II. Inverse Mellin transform via analytic continuation: $a_{Qg}^{(3)}$

Resumming Mellin N into a continuous variable t, observing crossing relations. Ablinger et al. 2014

$$\sum_{k=0}^{\infty} t^k (\Delta.p)^k \frac{1}{2}[1 \pm (-1)^k] = \frac{1}{2}\left[\frac{1}{1-t\Delta.p} \pm \frac{1}{1+t\Delta.p}\right]$$

$$\mathfrak{A} = \{f_1(t), \ldots, f_m(t)\}, \quad \mathrm{G}(b, \vec{a}; t) = \int_0^t dx_1 f_b(x_1) \mathrm{G}(\vec{a}; x_1), \quad \left[\frac{d}{dt}\frac{1}{f_{a_{k-1}}(t)}\frac{d}{dt}\cdots\frac{1}{f_{a_1}(t)}\frac{d}{dt}\right]\mathrm{G}(\vec{a}; t) = f_{a_k}(t).$$

Regularization for $t \to 0$ needed.

$$\begin{aligned} F(N) &= \int_0^1 dx x^{N-1}[f(x) + (-1)^{N-1} g(x)] \\ \tilde{F}(t) &= \sum_{N=1}^{\infty} t^N F(N) \\ f(x) + (-1)^{N-1} g(x) &= \frac{1}{2\pi i}\left[\mathrm{Disc}_x \tilde{F}\left(\frac{1}{x}\right) + (-1)^{N-1}\mathrm{Disc}_x \tilde{F}\left(-\frac{1}{x}\right)\right]. \end{aligned} \tag{3}$$

t-space is still Mellin space. One needs closed expressions to perform the analytic continuation (3). Continuation is needed to calculate the small x behaviour analytically.

Harmonic polylogarithms

$$\mathfrak{A}_{\mathrm{HPL}} = \{f_0, f_1, f_{-1}\} \left\{\frac{1}{t}, \frac{1}{1-t}, \frac{1}{1+t}\right\}$$

$$\mathrm{H}_{b,\vec{a}}(x) = \int_0^x dy f_b(y) \mathrm{H}_{\vec{a}}(y), \quad f_c \in \mathfrak{A}_{\mathrm{HPL}}, \quad \mathrm{H}_{\underbrace{0,\ldots,0}_{k}}(x) := \frac{1}{k!} \ln^k(x).$$

A finite monodromy at $x = 1$ requires at least one letter $f_1(t)$.

Example:

$$\tilde{F}_1(t) = \mathrm{H}_{0,0,1}(t)$$

$$F_1(x) = \frac{1}{2}\mathrm{H}_0^2(x)$$

$$\mathbf{M}[F_1(x)](n-1) = \frac{1}{n^3}$$

$$\tilde{F}_1(t) = t + \frac{t^2}{8} + \frac{t^3}{27} + \frac{t^4}{64} + \frac{t^5}{125} + \frac{t^6}{216} + \frac{t^7}{343} + \frac{t^8}{512} + \frac{t^9}{729} + \frac{t^{10}}{1000} + O(t^{11})$$

Cyclotomic harmonic polylogarithms

Also here the index set has to contain $f_{\pm}1(t)$.

$$\mathfrak{A}_{\rm cycl} = \left\{\frac{1}{x}\right\} \cup \left\{\frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1+x+x^2}, \frac{x}{1+x+x^2}\frac{1}{1+x^2}, \frac{x}{1+x^2}, \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2}, \dots\right\}.$$

Example:

$$\tilde{F}_3(t) = \frac{1}{3(1-t)t^{1/3}} \mathrm{G}\left[\frac{\xi^{1/3}}{1-\xi}; t\right]$$

$$= \frac{1}{1-t}\left(-1 + \frac{t^{-1/3}}{3}\left(\mathrm{H}_1(t^{1/3}) + 2\mathrm{H}_{\{3,0\}}(t^{1/3}) + \mathrm{H}_{\{3,1\}}(t^{1/3})\right)\right).$$

$$F_3(x) = -\frac{1}{3}\left[\frac{1}{1-x}\right]_+ + \frac{1}{18}\left[\sqrt{3}\pi + 9(-2+\ln(3))\right]\delta(1-x) + \frac{1-x^{4/3}}{3(1-x)}$$

Generalized harmonic polylogarithms

$$\mathfrak{A}_{\rm gHPL} = \left\{\frac{1}{x-a}\right\}, \quad a \in \mathbb{C}.$$

$$F_5(x) = \frac{1}{\pi}\mathrm{Im}\frac{t}{t-1}\left[\mathrm{H}_{0,0,0,1}(t) + 2\mathrm{G}(\gamma_1,0,0,1;t)\right] = -\frac{1}{1-x}\Biggl\{\theta(1-x)\Biggl[\frac{1}{24}\left(4\ln^3(2) - 2\ln(2)\pi^2 + 21\zeta_3\right)$$
$$-\mathrm{H}_{2,0,0}(x)\Biggr] - \theta(2-x)\frac{1}{24}\left(4\ln^3(2) - 2\ln(2)\pi^2 + 21\zeta_3\right)\Biggr\},$$

In intermediary steps Heaviside functions occur and the support of the x-space functions is here [0,2].

$$\tilde{\mathbf{M}}_a^{+,b}[g(x)](N) = \int_0^a dx(x^N - b^N)f(x), \quad a, b \in \mathbb{R},$$

$$\tilde{\mathbf{M}}_2^{+,1}[F_5(x)](N) = -S_{1,3}\left(2, \frac{1}{2}\right)(N-1),$$

$$S_{b,\vec{a}}(c,\vec{d})(N) = \sum_{k=1}^{N}\frac{c^k}{k^b}S_{\vec{a}}(\vec{d})(k), \quad b, a_i \in \mathbb{N}\backslash\{0\}, \quad c, d_i \in \mathbb{C}\backslash\{0\}.$$

Square root valued alphabets

$$\mathfrak{A}_{\rm sqrt} = \left\{ f_4, f_5, f_6 \dots \right\}$$
$$= \left\{ \frac{\sqrt{1-x}}{x}, \sqrt{x(1-x)}, \frac{1}{\sqrt{1-x}}, \frac{1}{\sqrt{x}\sqrt{1\pm x}}, \frac{1}{x\sqrt{1\pm x}}, \frac{1}{\sqrt{1\pm x}\sqrt{2\pm x}}, \frac{1}{x\sqrt{1\pm x/4}}, \dots \right\},$$

Monodromy also through:

$$(1-t)^\alpha, \quad \alpha \in \mathbb{R},$$

$$F_7(x) = \frac{1}{\pi} \mathrm{Im} \frac{1}{t} \mathrm{G}\left(4; \frac{1}{t}\right) = 1 - \frac{2(1-x)(1+2x)}{\pi} \sqrt{\frac{1-x}{x}} - \frac{8}{\pi} \mathrm{G}(5; x),$$

$$F_8(x) = \frac{1}{\pi} \mathrm{Im} \frac{1}{t} \mathrm{G}\left(4, 2; \frac{1}{t}\right) = -\frac{1}{\pi} \Bigg[4 \frac{(1-x)^{3/2}}{\sqrt{x}} + 2(1-x)(1+2x) \sqrt{\frac{1-x}{x}} [\mathrm{H}_0(x) + \mathrm{H}_1(x)]$$
$$+8[\mathrm{G}(5,2;x) + \mathrm{G}(5,1;x)] \Bigg],$$

Iterative non-iterative Integrals

- Master integrals, solving differential equations not factorizing to 1st order
- ${}_2F_1$ solutions Ablinger et al. [2017]
- Mapping to complete elliptic integrals: duplication of the higher transcendental letters.
- Complete elliptic integrals, modular forms Sabry, Broadhurst, Weinzierl, Remiddi, Duhr, Broedel et al. and many more
- Abel integrals
- K3 surfaces Brown, Schnetz [2012]
- Calabi-Yau motives Klemm, Duhr, Weinzierl et al. [2022]

Refer to as few as possible higher transcendental functions, the properties of which are known in full detail.

- $A_{Qg}^{(3)}$: effectively only one 3×3 system of this kind.
- The system is connected to that occurring in the case of ρ parameter. Ablinger et al. [2017], JB et al. [2018], Abreu et al. [2019]
- Most simple solution: two ${}_2F_1$ functions.

Iterative non-iterative Integrals

$$\frac{d}{dt}\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{t} & -\frac{1}{1-t} & 0 \\ 0 & -\frac{1}{t(1-t)} & -\frac{2}{1-t} \\ 0 & \frac{2}{t(8+t)} & \frac{1}{8+t} \end{bmatrix} \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} + \begin{bmatrix} R_1(t,\varepsilon) \\ R_2(t,\varepsilon) \\ R_3(t,\varepsilon) \end{bmatrix} + O(\varepsilon),$$

It is very important to which function $F_i(t)$ the system is decoupled.

Iterative non-iterative Integrals

- Decoupling for F_1 first leads to a very involved solution: ${}_2F_1$-terms seemingly enter at $O(1/\varepsilon)$ already.
- However, these terms are actually not there.
- Furthermore, there is also a singularity at $x = 1/4$.
- All this can be seen, when decoupling for F_3 first.

Homogeneous solutions:

$$F_3'(t) + \frac{1}{t} F_3(t) = 0, \quad g_0 = \frac{1}{t}$$

$$F_1''(t) + \frac{(2-t)}{(1-t)t} F_1'(t) + \frac{2+t}{(1-t)t(8+t)} F_1(t) = 0,$$

with

$$\begin{aligned} g_1(t) &= \frac{2}{(1-t)^{2/3}(8+t)^{1/3}} \, {}_2F_1\left[\begin{matrix} \frac{1}{3}, \frac{4}{3} \\ 2 \end{matrix}; -\frac{27t}{(1-t)^2(8+t)}\right], \\ g_2(t) &= \frac{2}{(1-t)^{2/3}(8+t)^{1/3}} \, {}_2F_1\left[\begin{matrix} \frac{1}{3}, \frac{4}{3} \\ \frac{2}{3} \end{matrix}; 1 + \frac{27t}{(1-t)^2(8+t)}\right], \end{aligned}$$

Iterative non-iterative Integrals

Alphabet:

$$\mathfrak{A}_2 = \left\{ \frac{1}{t}, \frac{1}{1-t}, \frac{1}{8+t}, g_1, g_2, \frac{g_1}{t}, \frac{g_1}{1-t}, \frac{g_1}{8+t}, \frac{g_1'}{t}, \frac{g_1'}{1-t}, \frac{g_1'}{8+t}, \frac{g_2}{t}, \frac{g_2}{1-t}, \frac{g_2}{8+t}, \frac{g_2'}{t}, \frac{g_2'}{1-t}, \frac{g_2'}{8+t}, tg_1, tg_2 \right\}$$

$$F_1(t) = \frac{8}{\varepsilon^3}\left[1 + \frac{1}{t}\mathrm{H}_1(t)\right] - \frac{1}{\varepsilon^2}\left[\frac{1}{6}(106+t) + \frac{(9+2t)}{t}\mathrm{H}_1(t) + \frac{4}{t}\mathrm{H}_{0,1}(t)\right]$$
$$+\frac{1}{\varepsilon}\left\{\frac{1}{12}(271+9t) + \left[\frac{71+32t+2t^2}{12t} + \frac{3\zeta_2}{t}\right]\mathrm{H}_1(t) + \frac{(9+2t)}{2t}\mathrm{H}_{0,1}(t) + \frac{2}{t}\mathrm{H}_{0,0,1}(t)\right.$$
$$\left.+3\zeta_2\right\} + \frac{1}{t}\left\{\frac{6696-22680t-16278t^2-255t^3-62t^4}{864t} + (9+9t+t^2)g_1(t)\left[\frac{31\ln(2)}{16}\right.\right.$$
$$\left.+\frac{1}{144}(265+31\pi(-3i+\sqrt{3})) + \frac{3}{8}\ln(2)\zeta_2 + \frac{1}{24}(10+\pi(-3i+\sqrt{3}))\zeta_2 - \frac{7}{4}\zeta_3\right]$$

$$
\begin{aligned}
&+\mathrm{G}(18,t)\Bigg[-\frac{93\ln(2)}{16}+\frac{1}{48}\big(-265-31\pi(-3i+\sqrt{3})\big)+\Bigg(-\frac{9\ln(2)}{8}\\
&+\frac{1}{8}\big(-10-\pi\big(-3i+\sqrt{3}\big)\big)\Bigg)\zeta_2+\frac{21}{4}\zeta_3\Bigg]\dots\\
&+\frac{5}{2}[\mathrm{G}(4,14,1,2;t)-\mathrm{G}(5,8,1,2;t)]+\frac{1}{4}[\mathrm{G}(13,8,1,2;t)-\mathrm{G}(7,14,1,2;t)]\\
&+\frac{9}{4}[\mathrm{G}(10,14,1,2;t)-\mathrm{G}(16,8,1,2;t)]+\frac{3}{4}[\mathrm{G}(19,14,1,2;t)-\mathrm{G}(19,8,1,2;t)]\Bigg\}+O(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
F_2(t) \;=\;& \frac{8}{\varepsilon^3}+\frac{1}{\varepsilon^2}\Bigg[-\frac{1}{3}(34+t)+\frac{2(1-t)}{t}\mathrm{H}_1(t)\Bigg]+\frac{1}{\varepsilon}\Bigg[\frac{116+15t}{12}+3\zeta_2-\frac{(1-t)(8+t)}{3t}\mathrm{H}_1(t)\\
&-\frac{1-t}{t}\mathrm{H}_{0,1}(t)\Bigg]+\frac{992-368t+75t^2-27t^3}{144t}+(1-t)\Bigg(\frac{\big(43+10t+t^2\big)}{12t}\mathrm{H}_1(t)+\frac{(4-t)}{4t}\\
&\times\mathrm{H}_{0,1}(t)+\frac{3\zeta_2}{4t}\mathrm{H}_1(t)\Bigg)+(1-t)g_1(t)\Bigg(\frac{31\ln(2)}{16}+\frac{1}{144}\big(265+31\pi\big(-3i+\sqrt{3}\big)\big)\dots
\end{aligned}
$$

Structure in x space

Expansion around $x = 1$:

$$\sum_{k=0}^{\infty} \sum_{l=0}^{L} \hat{a}_{k,l} (1-x)^k \ln^l(1-x).$$

Expansion around $x = 0$:

$$\frac{1}{x} \sum_{k=0}^{\infty} \sum_{l=0}^{S} \hat{b}_{k,l} x^k \ln^l(x).$$

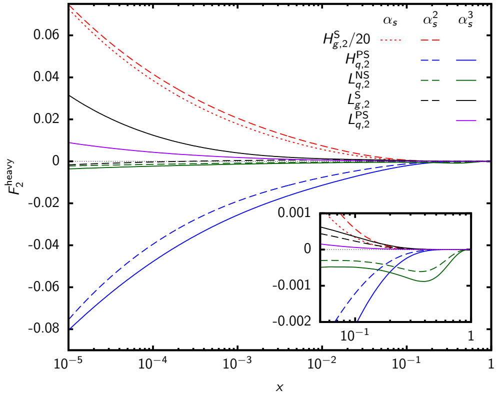
Expansion around $x = 1/2$:

$$\sum_{k=0}^{\infty} \hat{c}_k \left(x - \frac{1}{2}\right)^k.$$

The occurring constants $\mathrm{G}(...;1)$ are calculated numerically. [At most double integrals.]

Current summary on F_2^{charm}

An example to show numerical effects: the charm quark contributions to the structure function $F_2(x, Q^2)$



Allows to strongly reduce the current theory error on m_c.

Started $\sim$ 2009; might be completed this year.

Lots of new algorithms had to be designed; different new function spaces; new analytic calculation techniques ...

Conclusions

- Contributions to massless & massive OMEs and Wilson coefficients factorizing at 1st order can be computed in Mellin N space using difference ring techniques as implemented in the package `Sigma`.
- N-space methods also applicable in the case of non-1st order factorization are more involved and need further study.
- x-space representations are needed also to determine the small x behaviour, since it cannot be obtained by the N-space methods, because they are related to integer values in N not covered.
- The t-resummation of the original N-space expressions is already necessary to perform the IBP reduction.
- The transformation from the continuous variable t to the continuous variable x is possible trough the optical theorem.
- This applies to all 1st order factorizing cases and also to non-1st order factorizing situations, provided one can derive a closed form solution of the respective equations and perform the analytic continuation.
- This includes also the calculation of various new constants, which might open up a new field for special numbers, unless these quantities finally reduce to what is known already.
- The moments of the master integrals depend on ζ-values only.

Conclusions

- It is most efficient to work with ${}_2F_1$-solutions in the present examples, because they are most compact and since everything is known about them.
- For numerical representations analytic expansions around $x = 0$, $x = 1/2$ and $x = 1$ suffice, with ~ 50 terms, (Example: $a_{Qg}^{(3)}$). In some cases further overlapping series expansions have to be performed.
- $A_{gg,Q}^{(3)}$ has contributions from finite central binomial sums or square-root valued alphabets, factorizing at 1st order.
- Both efficient N- and x-space solutions can be derived which are very fast numerically. $\Longrightarrow$ QCD analysis.
- BFKL-like approaches are shown to utterly fail in describing these quantities.
- Polarized and unpolarized massless Wilson coefficients are available since 2022.