Heavy flavor corrections to DIS at 2- and 3-loop order

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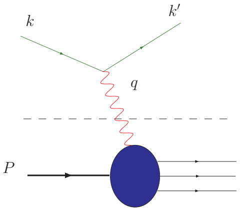
Johannes Blümlein | May 18, 2021

DESY

in collaboration with: J. Ablinger, A. Behring, A. De Freitas, M. Saragnese, C. Schneider, K. Schönwald

Outline

Theory of deep inelastic scattering



$\longrightarrow L_{\mu\nu}$

$\longrightarrow W_{\mu\nu}$

- Kinematic invariants:

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P.q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[J_\mu^{em}(\xi), J_\nu^{em}(\xi)\right] |P\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho \left(q \cdot PS^\sigma - q \cdot SP^\sigma\right)}{(q \cdot P)^2} g_2(x, Q^2)$$

- F_L, F_2, g_1 and g_2 contain contributions from both, charm and bottom quarks.

Introduction

Why are Heavy Flavor Contributions important?

- They form a significant contribution to all structure functions particularly at small x and high Q^2.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, m_c and perhaps m_b.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

NNLO: [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

$\alpha_s(M_Z^2) = 0.1147 \pm 0.0008$

$m_c(m_c) = 1.252 \pm 0.018(exp)\ {}^{+0.03}_{-0.02}\ (scale),\ {}^{+0.00}_{-0.07}\ (thy)\mathrm{GeV}$ ($\overline{\mathrm{MS}}$-scheme)

Yet approximate NNLO treatment for F_2 (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]
NS & PS corrections are exact [J. Ablinger et al. (Nucl. Phys. B886 & B890 (2014))]
EIC: many more high precision data ahead for varios detailed unpolarized and polarized precision measurements.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \; \delta(x - yz) f(y) g(z) \, .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \; x^{N-1} f(x) \, .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) \, .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j \rangle \, .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO (for unpolarized scattering).

[Moch, Vermaseren, Vogt (Nucl.Phys.B (2005))]

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Unpolarized

Leading Order: [Witten (1976); Babcock, Sivers, Wolfram (1978); Shifman, Vainshtein, Zakharov (1978); Leveille, Weiler (1979); Glück, Reya (1979); Glück, Hoffmann, Reya (1982)]

Next-to-Leading Order:

full m dependence (numeric) [Laenen, van Neerven, Riemersma, Smith (1993)]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven (1996)]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein (2007)]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein (2008, 2009)]

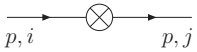
Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein (2009)] using `MATAD` [Steinhauser (2000)]
- Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments N = 2,4,6 [Blümlein, Wißbrock (2011)]

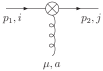
At 3-loop order for general values of N and extension to polarized scattering: Topic of this talk.

Calculation of the 3-loop operator matrix elements

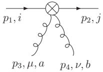
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



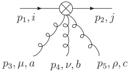
$$\delta^{ij} \not\Delta \gamma_{\pm} (\Delta \cdot p)^{N-1} \,, \quad N \geq 1$$



$$g t^a_{ji} \Delta^{\mu} \not\Delta \gamma_{\pm} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} \,, \quad N \geq 2$$



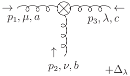
$$g^2 \Delta^{\mu} \Delta^{\nu} \not\Delta \gamma_{\pm} \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \Big[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \Big] \,, \quad N \geq 3$$



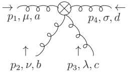
$$\begin{aligned}
& g^3 \Delta_{\mu} \Delta_{\nu} \Delta_{\rho} \not\Delta \gamma_{\pm} \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2} \\
& \Big[(t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\
& + (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\
& + (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1} \\
& + (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \\
& + (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1} \\
& + (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big] \,, \\
& N \geq 4
\end{aligned}$$

$$\gamma_+ = 1 \,, \quad \gamma_- = \gamma_5 \,.$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2} \Big[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_{\mu} p_{\nu} + \Delta_{\nu} p_{\mu}) \Delta \cdot p + p^2 \Delta_{\mu} \Delta_{\nu} \Big] \,, \quad N \geq 2$$



$$\begin{aligned}
& -ig \frac{1+(-1)^N}{2} f^{abc} \Bigg(\\
& \Big[(\Delta_{\nu} g_{\lambda\mu} - \Delta_{\lambda} g_{\mu\nu}) \Delta \cdot p_1 + \Delta_{\mu} (p_{1,\nu} \Delta_{\lambda} - p_{1,\lambda} \Delta_{\nu}) \Big] (\Delta \cdot p_1)^{N-2} \\
& + \Delta_{\lambda} \Big[\Delta \cdot p_1 p_{2,\mu} \Delta_{\nu} + \Delta \cdot p_2 p_{1,\nu} \Delta_{\mu} - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_{\mu} \Delta_{\nu} \Big] \\
& \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\
& + \left\{ {p_1 \to p_2 \to p_3 \to p_1 \atop \mu \to \nu \to \lambda \to \mu} \right\} + \left\{ {p_1 \to p_3 \to p_2 \to p_1 \atop \mu \to \lambda \to \nu \to \mu} \right\} \Bigg) \,, \quad N \geq 2
\end{aligned}$$



$$\begin{aligned}
& g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \\
& + f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) \,, \\
& O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_{\nu} \Delta_{\lambda} \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \\
& + [p_{4,\mu} \Delta_{\sigma} - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} \\
& - [p_{1,\sigma} \Delta_{\mu} - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \\
& + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_{\mu} \Delta_{\sigma} - \Delta \cdot p_4 p_{1,\sigma} \Delta_{\mu} - \Delta \cdot p_1 p_{4,\mu} \Delta_{\sigma}] \\
& \times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\} \\
& - \left\{ {p_1 \leftrightarrow p_2 \atop \mu \leftrightarrow \nu} \right\} - \left\{ {p_3 \leftrightarrow p_4 \atop \lambda \leftrightarrow \sigma} \right\} + \left\{ {p_1 \leftrightarrow p_2, \; p_3 \leftrightarrow p_4 \atop \mu \leftrightarrow \nu, \; \lambda \leftrightarrow \sigma} \right\} \,, \quad N \geq 2
\end{aligned}$$

Method of calculation

- Resummation of operator insertion into propagator structure ($\Delta \cdot \Delta = 0$):

$$\sum_{N=0}^{\infty} t^N (\Delta \cdot k)^N = \frac{1}{1 - t\,\Delta \cdot k}$$

- Reduction to master integrals using IBP relations implemented in `Reduze2`[Manteuffel, Studerus (2012)] .
- Solution of master integrals obtained by different methods:
  - direct integration using Mellin-Barnes and ${}_pF_q$-techniques
  - differential equations in resummation variable t
- method of arbitrary high moments, i.e. reconstructing all-N solution from a large number of fixed moments

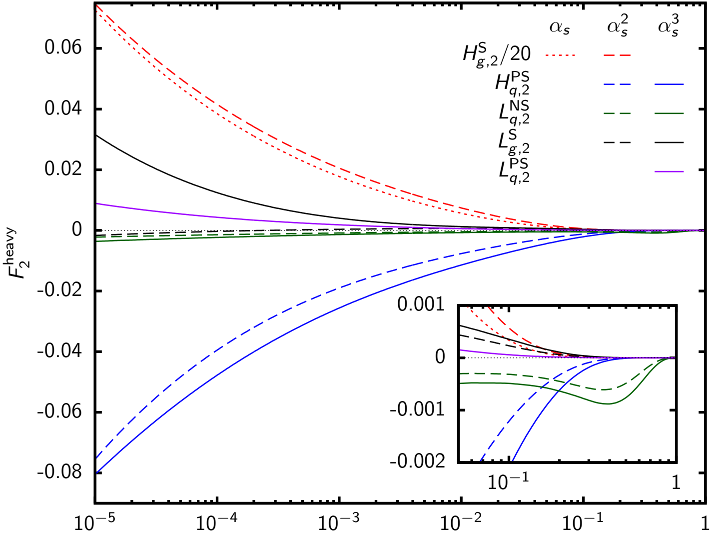
$\Rightarrow$ All these methods use the packages `Sigma`, `EvaluateMultiSums` [Schneider (2007-)] and `HarmonicSums` [Ablinger et al. (2010-)] which have been developed with these calculations.

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\rm NS}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right] \\
L_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right] \\
L_{g,(2,L)}^{\rm S}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right. \\
&\quad \left. + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right] \\
H_{q,(2,L)}^{\rm PS}(N_F+1) &= a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right] \\
&\quad + a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right] \\
H_{g,(2,L)}^{\rm S}(N_F+1) &= a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right] \\
&\quad + a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right] \\
&\quad + a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right. \\
&\quad \left. + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]
\end{aligned}
$$

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{Qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.

Heavy Flavor contribution to F_2



The NC PS contributions to $F_2(x, Q^2)$ and $F_L(x, Q^2)$

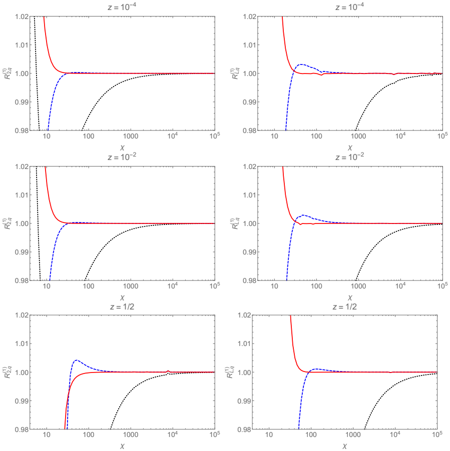


Figure 1: The ratios $R^{(1)}_{2,q}$ (left) and $R^{(1)}_{L,q}$ (right) as a function of $\chi = Q^2/m^2$ for different values of z gradually improved with κ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines : $O((m^2/Q^2)^2)$ improved.

Different convergence range for F_2 and F_L w.r.t Q^2 at $O(\alpha_s^2)$.

The variable flavor number scheme

- Matching conditions for parton distribution functions:

$$
\begin{aligned}
f_k(N_F+2) + f_{\bar{k}}(N_F+2) &= A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \left[f_k(N_F) + f_{\bar{k}}(N_F)\right] + \frac{1}{N_F} A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) \\
&+ \frac{1}{N_F} A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,, \\
f_Q(N_F+2) + f_{\bar{Q}}(N_F+2) &= A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\right) \cdot \Sigma(N_F) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,, \\
\Sigma(N_F+2) &= \left[A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot \Sigma(N_F) \\
&+ \left[A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot G(N_F)\,, \\
G(N_F+2) &= A_{gq,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) + A_{gg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,.
\end{aligned}
$$

Polarized OMEs

- Polarized OMEs for heavy flavor production at 2-loop order have been calculated before.
  [Buza et al (1996), Klein (2009), Hasselhuhn (2013)]
- Calculation of OMEs relied on tensor-decomposition in order to arrive at Larin scheme, i.e.

$$\gamma_5 \to \frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

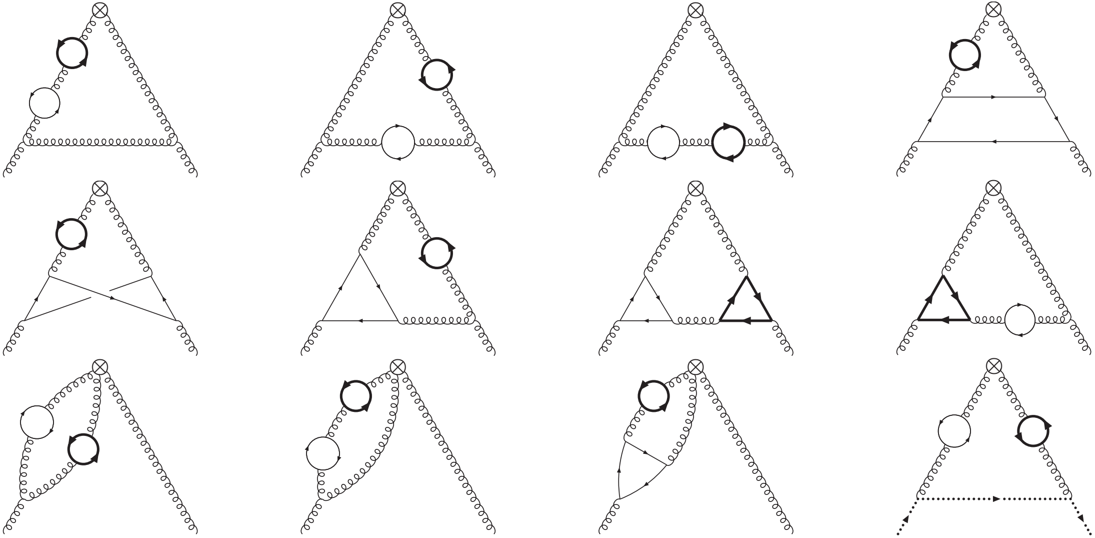
- We found out that a change of projector can accomplish the same:

$$\epsilon_{\mu\nu\rho\sigma}\text{tr}\left[\not{p}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma G_q\right] \to \epsilon_{\mu\nu\rho\sigma}p^\rho\Delta^\sigma\text{tr}\left[\not{p}\gamma^\mu\gamma^\nu G_q\right]$$

$\Rightarrow$ this allows to apply all technologies for the unpolarized OMEs directly to the polarized ones, i.e.

1. the terms $\sim T_F$ of the polarized 3-loop anomalous dimensions from a massive calculation
   [Behring et al. (Nucl. Phys. B948 (2020))]
2. $A_{qq,Q}^{(3),\mathrm{PS}}$, $A_{Qq}^{(3),\mathrm{PS}}$, $A_{gq,Q}^{(3)}$ (single and 2-mass contributions)
   [Ablinger et al. (Nucl. Phys. B952,953,955 (2020)), Behring et al. (Nucl. Phys. B964 (2021))]

2-mass contributions



2-mass contributions

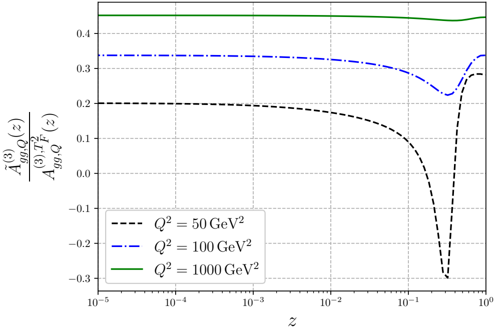
$A_{qq,Q}^{(3),\mathrm{NS}},\ A_{gq,Q}^{(3)}$	Harmonic Sums [Vermaseren '98; Blümlein, Kurth '98] $\sum_{i=1}^{N}\frac{1}{i^3}\sum_{j=1}^{i}\frac{1}{j}$	HPLs [Remiddi, Vermaseren '99] $\int_0^x \frac{d\tau_1}{1+\tau_1}\int_0^{\tau_1}\frac{d\tau_2}{1-\tau_2}$
$A_{gg,Q}^{(3)}$	Generalized harmonic and binomial sums [Ablinger, Blülein, Schneider '13] [Ablinger, Blülein, Raab, Schneider '14] $\sum_{i=1}^{N}\frac{4^i(1-\eta)^{-i}}{i\binom{2i}{i}}\sum_{j=1}^{i}\frac{(1-\eta)^j}{j^2}$	Iterated integrals over root and η valued letters [Ablinger, Blümlein, Raab, Schneider '14] $\int_0^x d\tau_1\frac{\sqrt{\tau_1(1-\tau_1)}}{1-\tau_1(1-\eta)}\int_0^{\tau_1}\frac{d\tau_2}{\tau_2}$
$A_{Qq}^{(3),\mathrm{PS}}$	–	Iterated integrals over root valued letters with restricted support $\theta(x-\eta_+)\int_0^{x(1-x)/\eta} d\tau\,\frac{\sqrt{1-4\tau}}{\tau}$

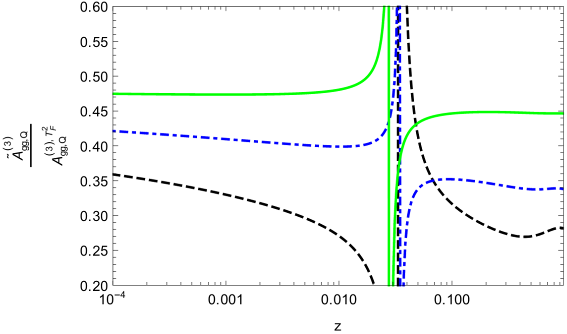
Results: $A_{gg,Q}^{(3)}$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3)}(N) = & \frac{1}{2}\left(1+(-1)^N\right)\Bigg\{T_F^3\left\{\frac{32}{3}\left(L_1^3+L_2^3\right)+\frac{64}{3}L_1L_2\left(L_1+L_2\right)+32\zeta_2\left(L_1+L_2\right)+\frac{128}{9}\zeta_3\right\} \\
& + C_F T_F^2\Bigg\{\cdots+32\left(\mathrm{H}_0^2(\eta)-\frac{1}{3}S_2\right)S_1+\frac{128}{3}S_{2,1}-\frac{64}{3}S_{1,1,1}\left(\frac{1}{1-\eta},1-\eta,1,N\right) \\
& -\frac{4P_{41}}{3(N-1)N^3(N+1)^2(N+2)(2N-3)(2N-1)}\left(\frac{\eta}{1-\eta}\right)^N\Bigg[\mathrm{H}_0^2(\eta) \\
& -2\mathrm{H}_0(\eta)S_1\left(\frac{\eta-1}{\eta},N\right)-2S_2\left(\frac{\eta-1}{\eta},N\right)+2S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)\Bigg]+\ldots\Bigg\} \\
& + C_A T_F^2\Bigg\{\cdots+\Bigg[\frac{8P_{65}}{3645\eta(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} \\
& +\frac{8P_{37}\mathrm{H}_0(\eta)}{45\eta(N-1)N^2(N+1)^2(N+2)}+\frac{2P_{23}\mathrm{H}_0^2(\eta)}{9\eta(N-1)N(N+1)^2}+\frac{32}{27}\mathrm{H}_0^3(\eta)+\frac{128}{9}\mathrm{H}_{0,0,1}(\eta) \\
& +\frac{64}{9}\mathrm{H}_0^2(\eta)\mathrm{H}_1(\eta)-\frac{128}{9}\mathrm{H}_0(\eta)\mathrm{H}_{0,1}(\eta)\Bigg]S_1 \\
& +\frac{2^{-1-2N}P_{47}}{45\eta^2(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}\Bigg\{\frac{1}{2}\mathrm{H}_0^2(\eta) \\
& S_{1,1}\left(\frac{\eta-1}{\eta},1,i\right)\Bigg\}+\ldots\Bigg\}
\end{aligned}
$$

Results: $A_{gg,Q}^{(3)}$

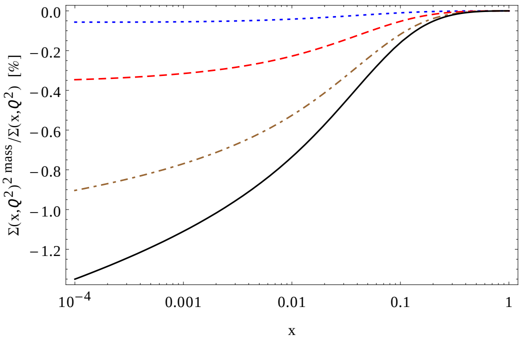
The two mass contributions over the whole T_F^2- contributions to the OME $A_{gg,Q}^{(3)}$:



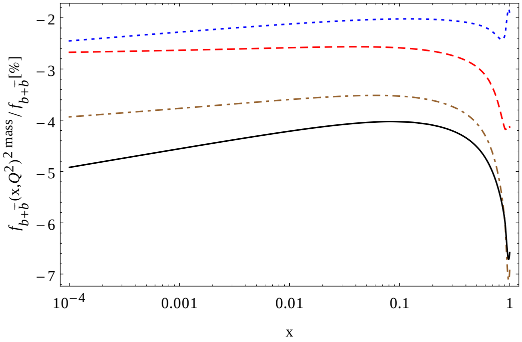


The variable flavor number scheme at NLO

$$\Sigma(x,Q^2)^{\text{2-mass}}/\Sigma(x,Q^2)$$



$$f_{b+\bar{b}}(x,Q^2)^{\text{2-mass}}/f_{b+\bar{b}}(x,Q^2)$$



- The ratio of the 2-mass contributions to the singlet parton distribution $\Sigma(x,Q^2)$ (left) and the heavy flavor parton distribution $f_{b+\bar{b}}(x,Q^2)$ (right) over their full form in percent for $m_c = 1.59$ GeV, $m_b = 4.78$ GeV in the on-shell scheme. Dash-dotted line: $Q^2 = 30$ GeV^2; Dotted line: $Q^2 = 30$ GeV^2; Dashed line: $Q^2 = 100$ GeV^2; Dash-dotted line: $Q^2 = 1000$ GeV^2; Full line: $Q^2 = 10000$ GeV^2.
- For the PDFs the `NNLO` variant of ABMP16 with $N_f = 3$ flavors was used.
  Alekhin et al., Phys. Rev. D 96 (2017) 1

Conclusions and Outlook

Conclusions

- The 2-loop contributions for the pure-singlet Wilson coefficients have been calculated for general kinematics.
- All operator matrix elements except of $A_{Qg}^{(3)}$ have been calculated for general values of N. $A_{Qg}^{(3)}$ is in progress.
- Calculation of 2-mass contributions to all operator matrix elements except of $A_{Qg}^{(3)}$ for general values of N. $A_{Qg}^{(3)}$ is in progress.
- Extension to the polarized case without the need of tensor-decomposition.
- The technologies have also been applied to QED ISR calculations for e^+e^- annihlation at higher order of massive leptons, incluing the electron.
- There were several new mathematica developments concerning massive Feynman integral calculations since RADCOR19.

Outlook

- Study of the VFNS in the polarized case.
- High accuracy reconstruction of x-space expression of $A_{Qg}^{(3)}$ from the large number of moments already constructed. Analytic solution requires more mathematical effort.
- 2-mass contributions to $A_{Qg}^{(3)}$ in an expansion in the mass ratio m_c/m_b.