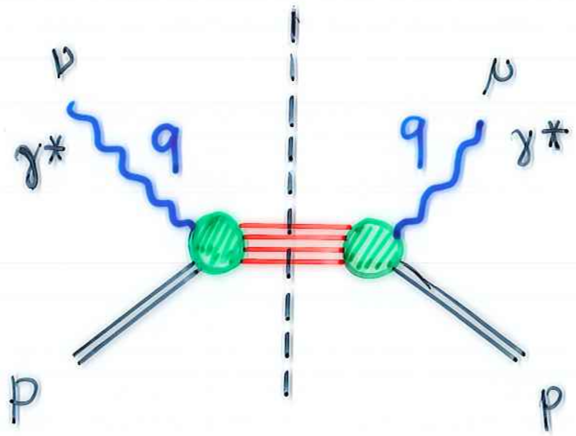
WÜRZBURG
2/98

ANOMALOUS DIMENSIONS AND EVOLUTION EQUATIONS IN NON-FORWARD SCATTERING

J. BLÜMLEIN, DESY

1. DIS : THE FORWARD CASE
2. NON- FORWARD SCATTERING
3. LC-OPERATOR EVOLUTION
4. 2-VARIABLE PARTITION FUNCTIONS
5. 1-VARIABLE PARTITION FUNCTIONS
   - B & L
   - A P
6. CONCLUSIONS

1. DEEP INELASTIC SCATTERING: THE FORWARD CASE



$$p+q \equiv p+q$$

$$M_\nu(p,q) \qquad M^*_\mu(p,q)$$

$$|M|^2_{\mu\nu} \propto W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(q\cdot p, q^2)$$
$$+ \frac{1}{M_p^2}\left[\left(p_\mu - \frac{qp}{q^2} q_\mu\right)\left(p_\nu - \frac{qp}{q^2} q_\nu\right)\right] W_2(q\cdot p, q^2)$$

- THE HADRONIC TENSOR (AND THE STRUCTURE FCTS.) CAN BE DECOMPOSED INTO: COEFFICIENT FUNCTIONS & OPERATOR MATRIX ELEMENTS (LCE, BJ-LIMIT)

BJORKEN LIMIT: $Q^2 \to \infty$ $\qquad \frac{Q^2}{2M\nu} = x = \text{fixed}.$

$p\cdot q \propto \nu \to \infty$

W-COEFF. OPM

$$W_i(x,Q^2) = \sum_{\substack{\ell=0 \\ n}}^{\infty} \underbrace{\left(\frac{\Delta^2}{Q^2}\right)^{\ell}}_{\text{TWIST}} \overbrace{C^n_{i\ell}}^{\text{W-COEFF.}}(x,Q^2) \underbrace{\bullet}_{\text{CONNECTOR}} \overbrace{\langle p|O^n|p\rangle}^{\text{OPM}}(x,Q^2)$$

CONNECTOR

TWIST

LEADING TWIST : SPIN - CAN. DIM OF O^n MINIMAL

$(\ell = 0)$

$$W_i(x,Q^2) = \sum_n C^n_i(x,Q^2) \otimes \underbrace{\langle p|O^n|p\rangle}_{\text{PARTONDENSITY}}(x,Q^2)$$

PARTONDENSITY.

WHAT IS UNIVERSAL IN THIS PICTURE ?

(PROCESS-INDEPENDENCE)

$\longrightarrow$ PARTONDENSITIES.

THEY STILL DEPEND ON THE STATE !

$|p\rangle$!

MORE GENERAL APPROACH : STUDY JUST THE RENORMALIZATION OF THE OPERATORS WITHOUT REFERENCE TO : $\langle H|_{in}$ & $|H\rangle_{OUT}$.

IF THE EVOLUTION EQUATIONS ARE KNOWN ONE CAN SANDWICH BETWEEN STATES.

IN GENERAL :	$\langle p_1 \vert$	$\rightarrow$	$\vert p_2 \rangle$
FORWARD :	$\langle p \vert$	$\rightarrow$	$\vert p \rangle$
VACUUM → MESON:	$\langle 0 \vert$	$\rightarrow$	$\vert p \rangle$

FORWARD EVOLUTION EQUS:

RGE FOR THE EXPECTATION VALUES OF LOCAL OPERATORS O^q, O^G (O^{NS})

UV-SINGULARITY

CALLEN-SYMANZIK EQ.

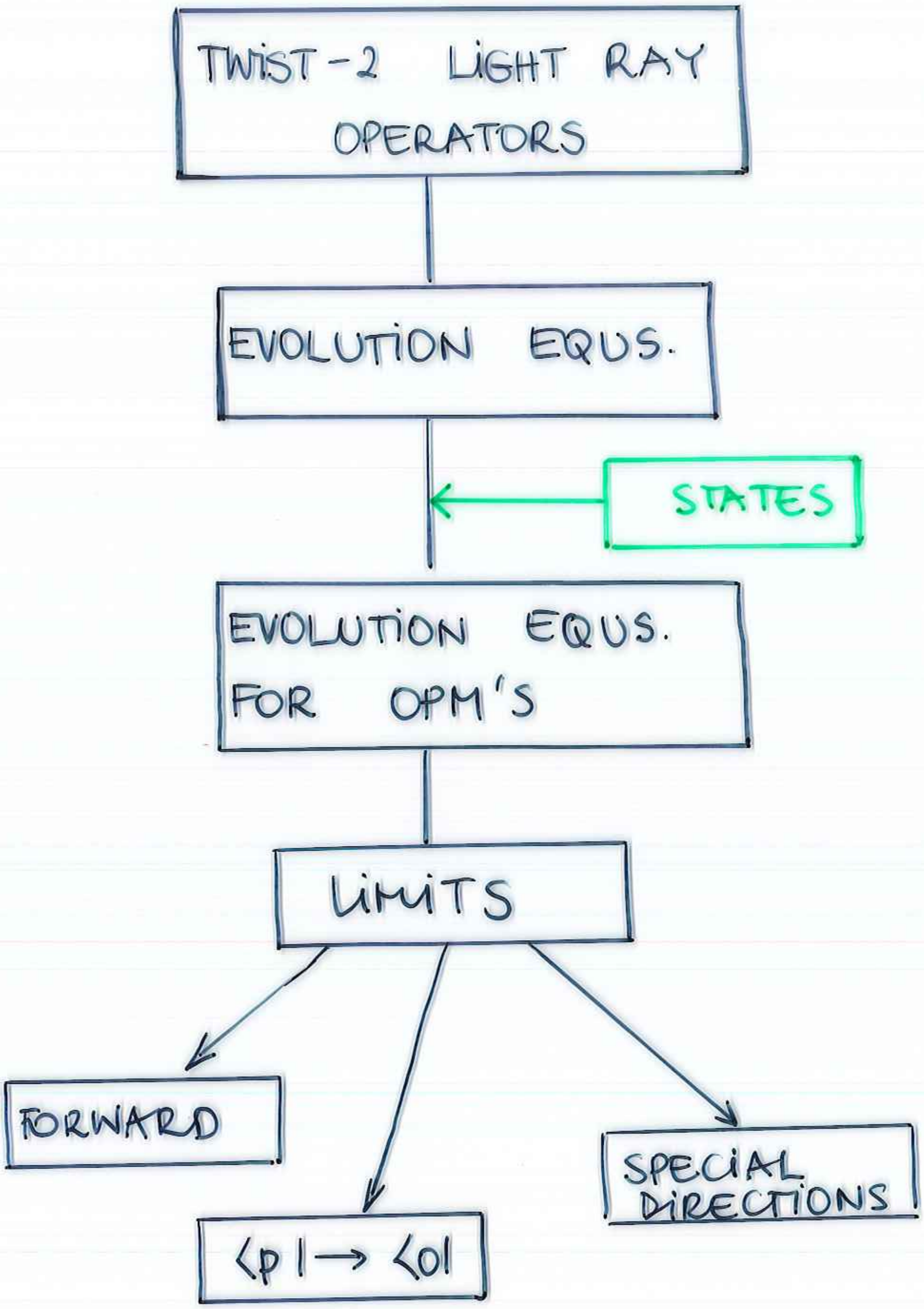
GEORGI, POLITZER, GROSS, WILCZEK

(GRIBOV, LIPATOV, ALTARELLI, PARISI, DOKSHITSER, KIM, SCHILCHER, SUSSKIND ...)

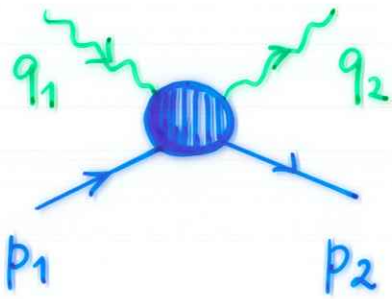
$$\frac{\partial F(x,Q^2)}{\partial \log Q^2} = \mathbb{P}(x,Q^2) \otimes F(x,Q^2)$$

$$\mathbb{P}(x,Q^2) = \sum_{l=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^l P_l(x)$$

, $\mathbb{P}$ IS A MATRIX (MORE OPERATORS) IN GENERAL.


TWIST-2 LIGHT RAY OPERATORS
EVOLUTION EQUS.
STATES
EVOLUTION EQUS. FOR OPM'S
LIMITS
FORWARD
⟨p| → ⟨0|
SPECIAL DIRECTIONS

2. NON - FORWARD SCATTERING



$$q_1 - q_2 = \Delta = p_2 - p_1 \quad ; \quad \Delta \neq 0$$

COMPTON SCATTERING $\quad q_i^2$ (ONE OF THEM AT LEAST & $q_i \cdot p_j$ LARGE).

FIG.

A MORE GENERAL LABORATORY FOR QCD TESTS !

OLDER AND RECENT DEVELOPMENTS:

- BRODSKY, LEPAGE: EVOLUTION MESON-WAVEFCT.
- EFREMOV, RADYUSCHKIN
- NLO NS: DITTES, RADYUSHKIN
- GEYER, ROBASCHIK, DITTES, HOREJSI, BRAUNSCHWEIG, D. MÜLLER
- BALITSKII, V. BRAUN

1996/97

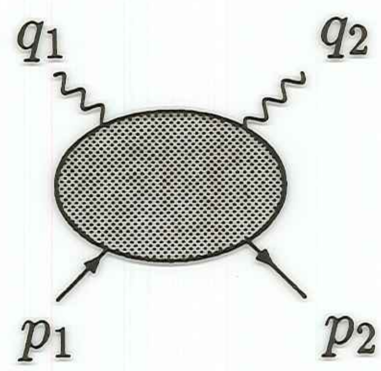
THOROUGH STUDIES:

- RADYUSHKIN
- JI
- JB, GEYER, ROBASCHIK 1 LOOP UNPOL + POL.
- COLLINS et al.
- BALITSKII, BRAUN.
- (CHEN, MANKIEWICZ et al, SCHÄFER et al. ...)

⋮

Compton Amplitude

$$T_{\mu\nu}(p_+, p_-, Q) \equiv i \int d^4x e^{iqx} \langle p_2 | T(J_\mu(x/2) J_\nu(-x/2)) | p_1 \rangle,$$



$p_1 + q_1 = p_2 + q_2$

$p_+ = p_1 + p_2$
$p_- = p_2 - p_1$
$Q = \frac{1}{2}(q_1 + q_2)$

OPE: (no states)

$$T(J_\mu(x/2)J_\nu(-x/2)) \approx \int_{-\infty}^{+\infty} d\kappa_- \int_{-\infty}^{+\infty} d\kappa_+ \left[C_a(x^2, \kappa_-, \kappa_+, \mu^2) S_{\mu\nu}{}^{\rho\sigma} \tilde{x}_\rho O^a_\sigma(\kappa_+\tilde{x}, \kappa_-\tilde{x}, \mu^2) \right.$$
$$\left. + C_{a,5}(x^2, \kappa_-, \kappa_+, \mu^2) \varepsilon_{\mu\nu}{}^{\rho\sigma} \tilde{x}_\rho O^a_{5,\sigma}(\kappa_+\tilde{x}, \kappa_-\tilde{x}, \mu^2)\right] \quad (2)$$

Anikin, Zavialov.

with $S_{\mu\nu\rho\sigma} \equiv g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}$ and $\varepsilon_{\mu\nu\rho\sigma}$ denoting the Levi–Civita symbol. The light–like vector

$$\tilde{x} \equiv x + r(x.r/r.r)\left[\sqrt{1 - x.xr.r/(x.r)^2} - 1\right] \quad (3)$$

THE RELATION BETWEEN LOCAL AND 'NON'- LOCAL OPERATORS:

LOCAL:

$$\psi(x)\, \partial_{\nu_1} \dots \partial_{\nu_N}\, \gamma^{\mu}\, \psi(0)\, \Delta^{\nu_1} \dots \Delta^{\nu_N} \Delta_{\mu}$$

NON - LOCAL:

$$\psi(\tilde{x}\kappa_1)\, \tilde{x}^{\mu}\gamma_{\mu}\, \psi(\tilde{x}\kappa_2) \equiv O(\kappa_1 - \kappa_2) = O(\kappa_-)$$

$$O(\kappa_-) = \sum_{\ell=0}^{\infty} \frac{\kappa_-^{\ell}}{\ell!}\, \psi(x)\, \partial_{\nu_1} \dots \partial_{\nu_\ell}\, \gamma^{\mu}\, \psi(0) \cdot \tilde{x}^{\nu_1} \dots \tilde{x}^{\nu_\ell}\, \tilde{x}_{\mu}$$

$\tilde{x}$ light cone vector : $\tilde{x}\cdot\tilde{x} = 0$, $\Delta\cdot\Delta = 0$.

'NON'- LOCAL OPERATORS ARE SUMMED - UP LOCAL OPERATORS.

$\longrightarrow$ NOTION OF SPIN CAN BE SUMMED!

IMPORTANT FOR HIGHER TWIST DIAGONALITY!

(LEADING TWIST NO ADVANTAGE).

Twist 2 Light-Ray Operators

UNPOLARIZED

$$O^{\rm NS}(\kappa_1,\kappa_2) = \frac{i}{2}\left[\overline{\psi_a}(\kappa_1\tilde{x})\lambda_f\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_2\tilde{x}) - \overline{\psi_a}(\kappa_2\tilde{x})\lambda_f\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_1\tilde{x})\right] \quad (4)$$

$$O_5^{\rm NS}(\kappa_1,\kappa_2) = \frac{i}{2}\left[\overline{\psi_a}(\kappa_1\tilde{x})\gamma_5\lambda_f\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_2\tilde{x}) + \overline{\psi_a}(\kappa_2\tilde{x})\gamma_5\lambda_f\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_1\tilde{x})\right] \quad (5)$$

$$O^{q}(\kappa_1,\kappa_2) = \frac{i}{2}\left[\overline{\psi_a}(\kappa_1\tilde{x})\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_2\tilde{x}) - \overline{\psi_a}(\kappa_2\tilde{x})\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_1\tilde{x})\right] \quad (6)$$

$$O_5^{q}(\kappa_1,\kappa_2) = \frac{i}{2}\left[\overline{\psi_a}(\kappa_1\tilde{x})\gamma_5\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_2\tilde{x}) + \overline{\psi_a}(\kappa_2\tilde{x})\gamma_5\gamma_\mu\tilde{x}^\mu\psi_a(\kappa_1\tilde{x})\right] \quad (7)$$

$$O^{G}(\kappa_1,\kappa_2) = \tilde{x}^\mu F_{a\mu}{}^{\nu}(\kappa_1\tilde{x})\tilde{x}^{\mu'}F^{a}{}_{\mu'\nu}(\kappa_2\tilde{x}) \quad (8)$$

$$O_5^{G}(\kappa_1,\kappa_2) = \frac{1}{2}\left[\tilde{x}^\mu F_{a\mu}{}^{\nu}(\kappa_1\tilde{x})\tilde{x}^{\mu'}\widetilde{F}^{a}{}_{\mu'\nu}(\kappa_2\tilde{x}) - \tilde{x}^\mu F^{a}{}_{\mu\nu}(\kappa_2\tilde{x})\tilde{x}^{\mu'}\widetilde{F}^{a}{}_{\mu'}{}^{\nu}(\kappa_1\tilde{x})\right], \quad (9)$$

POLARIZED

RGE : FOR OPERATORS

$$\mu^2\frac{d}{d\mu^2}O^{\rm NS}_{(5)}(\kappa_1,\kappa_2) = \frac{\alpha_s(\mu^2)}{2\pi}\int_0^1 d\alpha_1\int_0^1 d\alpha_2\theta(1-\alpha_1-\alpha_2)K^{\rm NS}(\alpha_1,\alpha_2)O^{\rm NS}_{(5)}(\kappa_1',\kappa_2'),$$

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix}O^q(\kappa_1,\kappa_2)\\ O^G(\kappa_1,\kappa_2)\end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi}\int_0^1 d\alpha_1\int_0^1 d\alpha_2\theta(1-\alpha_1-\alpha_2)\boldsymbol{K}(\alpha_1,\alpha_2)\begin{pmatrix}O^q(\kappa_1',\kappa_2')\\ O^G(\kappa_1',\kappa_2')\end{pmatrix}$$

(ALPHA - REPRESENTATION)

KERNELS :

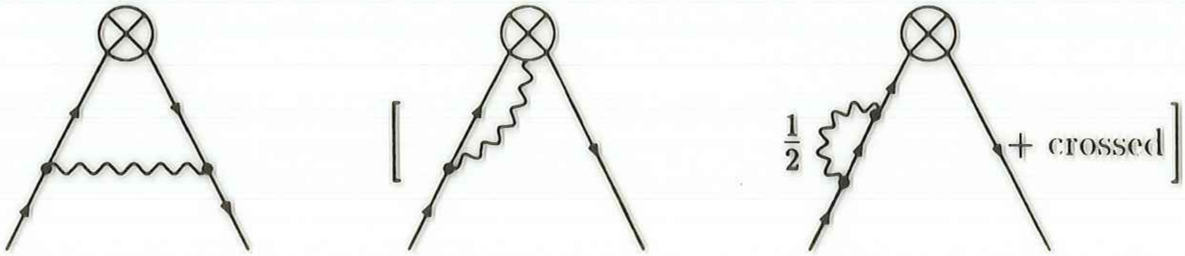
$$\boldsymbol{K} = \begin{pmatrix}K^{qq} & K^{qG}\\ K^{Gq} & K^{GG}\end{pmatrix} \quad \text{and} \quad \Delta\boldsymbol{K} = \begin{pmatrix}\Delta K^{qq} & \Delta K^{qG}\\ \Delta K^{Gq} & \Delta K^{GG}\end{pmatrix},$$

UNPOLARIZED POLARIZED

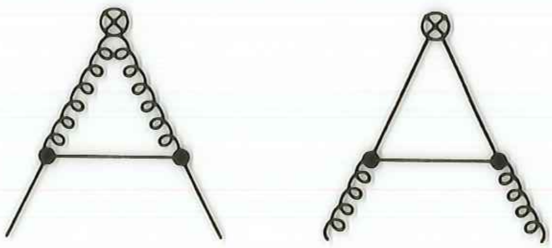
DIAGRAMS IN $O(\alpha_s)$: (AXIAL GAUGE)

$\hat{O}$

qq (NS)

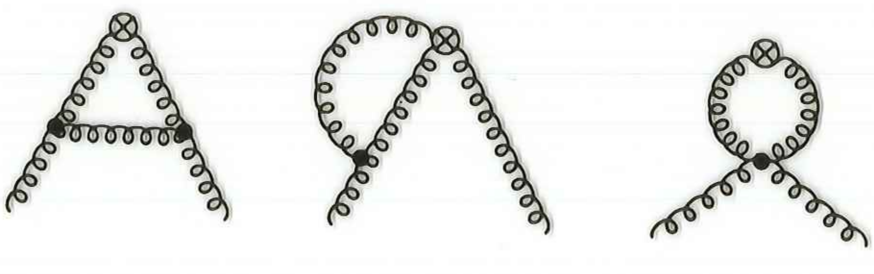


Gq



qG

GG



Z - FACTORS.

Two-Variable Partition Functions

$$\frac{\langle p_1|O^q|p_2\rangle}{(i\tilde{x}p_+)} \equiv e^{-i\kappa_+\tilde{x}p_-}\int_{-\infty}^{+\infty} dz_+ \int_{-\infty}^{+\infty} dz_- e^{-i\kappa_-(\tilde{x}p_+z_+ + \tilde{x}p_-z_-)} F_q(z_-, z_+)$$

$$\frac{\langle p_1|O^G|p_2\rangle}{(i\tilde{x}p_+)^2} \equiv e^{-i\kappa_+\tilde{x}p_-}\int_{-\infty}^{+\infty} dz_+ \int_{-\infty}^{+\infty} dz_- e^{-i\kappa_-(\tilde{x}p_+z_+ + \tilde{x}p_-z_-)} F_G(z_-, z_+)\,.$$

(FOURIER-REPRESENT.)

The evolution equations for the partition functions read

$(z_+, z_-; z'_+, z'_-)$

$$\mu^2\frac{d}{d\mu^2}F^{\rm NS}(z_+, z_-) = \frac{\alpha_s(\mu^2)}{2\pi}\int_{-\infty}^{+\infty}\frac{dz'_+}{|z'_+|}\int_{-\infty}^{+\infty} dz'_- \widetilde{K}^{\rm NS}(\cancel{\alpha_1, \alpha_2})F^{\rm NS}(z'_+, z'_-)$$

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix} F^q(z_+, z_-) \\ F^G(z_+, z_-)\end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi}\int_{-\infty}^{+\infty}\frac{dz'_+}{|z'_+|}\int_{-\infty}^{+\infty} dz'_- \widetilde{\boldsymbol{K}}(z_+, z_-; z'_+, z'_-)\begin{pmatrix} F^q(z'_+, z'_-) \\ F^G(z'_+, z'_-)\end{pmatrix},$$

where $F^{\rm NS}(z_+, z_-) = F^{q_i}(z_+, z_-) - F^{\bar{q}_j}(z_+, z_-)$, and

$$\widetilde{K}^{ij}(\alpha_1, \alpha_2) \equiv \frac{1}{2}\int_0^1 dz''_+ \underline{\widetilde{O}^{ij}(z_+, z''_+)}\theta(1-\alpha'_+)\theta(\alpha'_+)\theta(\alpha'_+ + \alpha'_-)\theta(\alpha'_+ - \alpha'_-)K^{ij}(\alpha'_1, \alpha'_2)\,,$$

with $\widetilde{K}^{\rm NS} \equiv \widetilde{K}^{qq}$ and $\alpha'_\rho \equiv \alpha_\rho(z_+ \to z''_+)$,

$\alpha_{1,2} \equiv \alpha_{1,2}(z_+, z_-; z'_+, z'_-)$

$$\widetilde{O}^{ij}(z_+, z''_+) = \begin{pmatrix} \delta(z_+ - z''_+) & \partial_{z_+}\delta(z_+ - z''_+) \\ -\theta(z_+ - z''_+) & \delta(z_+ - z''_+)\end{pmatrix}$$

$$\alpha_1 \equiv \frac{\alpha_+ + \alpha_-}{2} \qquad \alpha_2 \equiv \frac{\alpha_+ - \alpha_-}{2},$$

$$\alpha_+ \equiv 1 - \frac{z_+}{z'_+} \qquad -\alpha_- \equiv \frac{z_+z'_- - z_-z'_+}{z'_+},$$

Anomalous Dimensions

UNPOLARIZED:

$$
\begin{aligned}
K^{qq}(\alpha_1,\alpha_2) &= C_F\left\{1-\delta(\alpha_1)-\delta(\alpha_2)+\delta(\alpha_1)\left[\frac{1}{\alpha_2}\right]_+ +\delta(\alpha_2)\left[\frac{1}{\alpha_1}\right]_+ +\frac{3}{2}\delta(\alpha_1)\delta(\alpha_2)\right\}\\
K^{qG}(\alpha_1,\alpha_2) &= -2N_fT_R\underline{\kappa_-}\{1-\alpha_1-\alpha_2+4\alpha_1\alpha_2\}\\
K^{Gq}(\alpha_1,\alpha_2) &= -C_F\frac{1}{\underline{\kappa_-}}\{\delta(\alpha_1)\delta(\alpha_2)+2\}\\
K^{GG}(\alpha_1,\alpha_2) &= C_A\{4(1-\alpha_1-\alpha_2)+12\alpha_1\alpha_2\\
&\quad +\delta(\alpha_1)\left(\left[\frac{1}{\alpha_2}\right]_+-2+\alpha_2\right)+\delta(\alpha_2)\left(\left[\frac{1}{\alpha_1}\right]_+-2+\alpha_1\right)\Big\}\\
&\quad +\frac{\beta_0}{2}\delta(\alpha_1)\delta(\alpha_2),
\end{aligned}
$$

· Braunschweig, Geyer, Ro 87
· BALITSKY, BRAUN
· RADYUSHKIN
· JB, GEYER, ROBASCHIK

POLARIZED CASE:

$$
\begin{aligned}
\Delta K^{qq}(\alpha_1,\alpha_2) &= K^{qq}(\alpha_1,\alpha_2) \equiv K^{NS}(\alpha_1,\alpha_2)\\
\Delta K^{qG}(\alpha_1,\alpha_2) &= -2N_fT_R\underline{\kappa_-}\{1-\alpha_1-\alpha_2\}\\
\Delta K^{Gq}(\alpha_1,\alpha_2) &= -C_F\frac{1}{\underline{\kappa_-}}\{\delta(\alpha_1)\delta(\alpha_2)-2\}\\
\Delta K^{GG}(\alpha_1,\alpha_2) &= K^{GG}(\alpha_1,\alpha_2)-12C_A\alpha_1\alpha_2\,.
\end{aligned}
$$

(COEFFICIENT FCTS. BECOME NONTRIVIAL ONLY FOR NLO NF-PDF's)

· JB, GEYER, ROBASCHIK
· BALITSKY, RADYUSHKIN

$T_R=\frac{1}{2}$, $C_A=N_c\equiv 3$

$C_F=\frac{N_c^2-1}{2N_c}=\frac{4}{3}$.

$\beta_0=\frac{11}{3}C_A-\frac{4}{3}T_RN_f$

$$\int_0^1 dx\,[f(x,y)]_+\,\varphi(x)=\int_0^1 dx\,f(x,y)\,[\varphi(x)-\varphi(y)]\;,$$

One-Variable Partition Functions

$$\left.\frac{\langle p_1|O^q(-\kappa_-\tilde{x},\kappa_-\tilde{x})|p_2\rangle}{(i\tilde{x}p_+)}\right|_{\tilde{x}p_-=\tau\tilde{x}p_+} = \int_{-\infty}^{+\infty} dt e^{-i\kappa_-\tilde{x}p_+t} F_q(t)$$

$$\left.\frac{\langle p_1|O^G(-\kappa_-\tilde{x},\kappa_-\tilde{x})|p_2\rangle}{(i\tilde{x}p_+)^2}\right|_{\tilde{x}p_-=\tau\tilde{x}p_+} = \int_{-\infty}^{+\infty} dt e^{-i\kappa_-\tilde{x}p_+t}\, tF_G(t)\,.$$

$\tau = \frac{\tilde{x}p_-}{\tilde{x}p_+}$

$$\mu^2\frac{d}{d\mu^2}F^{\rm NS}(t) = \frac{\alpha_s(\mu^2)}{2\pi}\int_{-\infty}^{+\infty} dt' V_{ext}^{\rm NS}(t,t',\tau)F^{\rm NS}(t')$$

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix} F^q(t)\\ F^G(t)\end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi}\int_{-\infty}^{+\infty} dt' \mathbf{V}_{ext}(t,t',\tau)\begin{pmatrix} F^q(t')\\ F^G(t')\end{pmatrix}$$

2-dim Kernels.

↓

$$V_{ext}^{ij}(t,t',\tau) = \int_0^1 d\alpha_1\int_0^{1-\alpha_1} d\alpha_2 K^{ij}(\alpha_1,\alpha_2)\frac{1}{2\pi}\int_{-\infty}^{+\infty} d(p_+\tilde{x}\kappa_-)(\kappa_-)^{-a_{ij}}$$
$$\times\left[ip_+\tilde{x}(\kappa_-)\right]^{a_{ij}}\frac{t'^{a_j}}{t^{a_i}}\exp\left\{ip_+\kappa_-\tilde{x}\left[t-(1-\alpha_1-\alpha_2)t'+\tau(\alpha_1-\alpha_2)\right]\right\},$$

The Evolution Kernels

$$
\begin{aligned}
V^{qq}_{ext}(t,t',\tau) &= \frac{1}{2}\left\{V^{qq}(x,y)\rho(x,y) + V^{qq}(\overline{x},\overline{y})\rho(\overline{x},\overline{y}) + \frac{3}{2}C_F\delta(x-y)\right\}\frac{1}{\tau} \equiv V^{NS}_{ext}(t,t',\tau) \\
V^{qG}_{ext}(t,t',\tau) &= \frac{1}{2}\left(\frac{2y-1}{2}\right)\left\{V^{qG}(x,y)\rho(x,y) - V^{qG}(\overline{x},\overline{y})\rho(\overline{x},\overline{y})\right\}\frac{1}{\tau} \\
V^{Gq}_{ext}(t,t',\tau) &= \frac{1}{2}\left(\frac{2}{2x-1}\right)\left\{V^{Gq}(x,y)\rho(x,y) - \overline{V}^{Gq}(\overline{x},\overline{y})\rho(\overline{x},\overline{y})\right\}\frac{1}{\tau}
\end{aligned}
$$

UNPOLARIZED

$$
\begin{aligned}
V^{GG}_{ext}(t,t',\tau) &= \frac{1}{2}\left(\frac{2y-1}{2x-1}\right)\left\{V^{GG}(x,y)\rho(x,y) + V^{GG}(\overline{x},\overline{y})\rho(\overline{x},\overline{y})\right\}\frac{1}{\tau} \\
&\quad + \frac{1}{2}\frac{1}{2}\beta_0\delta(x-y)\frac{1}{\tau} \\
\Delta V^{qq}_{ext}(t,t',\tau) &= V^{qq}_{ext}(t,t',\tau) \\
\Delta V^{qG}_{ext}(t,t',\tau) &= \frac{1}{2}\left(\frac{2y-1}{2}\right)\left\{\Delta V^{qG}(x,y)\rho(x,y) - \Delta V^{qG}(\overline{x},\overline{y})\rho(\overline{x},\overline{y})\right\}\frac{1}{\tau} \\
\Delta V^{Gq}_{ext}(t,t',\tau) &= \frac{1}{2}\left(\frac{2}{2x-1}\right)\left\{\Delta V^{Gq}(x,y)\rho(x,y) - \Delta\overline{V}^{Gq}(\overline{x},\overline{y})\rho(\overline{x},\overline{y})\right\}\frac{1}{\tau} \\
\Delta V^{GG}_{ext}(t,t',\tau) &= \frac{1}{2}\left(\frac{2y-1}{2x-1}\right)\left\{\Delta V^{GG}(x,y)\rho(x,y) + \Delta V^{GG}(\overline{x},\overline{y})\rho(\overline{x},\overline{y})\right\}\frac{1}{\tau} \\
&\quad + \frac{1}{2}\frac{1}{2}\beta_0\delta(x-y)\frac{1}{\tau}
\end{aligned}
$$

POLARIZED

$$
V^{ij}_{ext}(t,t',\tau) = \frac{1}{\tau}V^{ij}_{ext}\left(\frac{t}{\tau},\frac{t'}{\tau},1\right) \qquad x = \frac{1}{2}\left(1+\frac{t}{\tau}\right), \quad y = \frac{1}{2}\left(1+\frac{t'}{\tau}\right) \qquad \rho(x,y) = \theta\left(1-\frac{x}{y}\right)\theta\left(\frac{x}{y}\right)\ \mathrm{sign}(y),
$$

UNPOLARIZED

$$V^{qq}(x,y) = C_F\left[\frac{x}{y} - \frac{1}{y} + \frac{1}{(y-x)_+}\right]$$

$$V^{qG}(x,y) = -2N_f T_R \frac{x}{y}\left[4(1-x) + \frac{1-2x}{y}\right]$$

$$V^{Gq}(x,y) = C_F\left[1 - \frac{x^2}{y}\right]$$

$$V^{GG}(x,y) = C_A\left[2\frac{x^2}{y}\left(3 - 2x + \frac{1-x}{y}\right) + \frac{1}{(y-x)_+} - \frac{y+x}{y^2}\right]$$

POLARIZED

$$\Delta V^{qq}(x,y) = V^{qq}(x,y)$$

$$\Delta V^{qG}(x,y) = -2N_f T_R \frac{x}{y^2}$$

$$\Delta V^{Gq}(x,y) = C_F\left[\frac{x^2}{y}\right]$$

$$\Delta V^{GG}(x,y) = C_A\left[2\frac{x^2}{y^2} + \frac{1}{(y-x)_+} - \frac{y+x}{y^2}\right]$$

Special Case, $x_i \ni$ ($t' = 1$)

$$
\begin{aligned}
K^{qq}(t,t',\xi) &= C_F \frac{t^2 + t'^2 - \xi^2/2}{(t'^2 - \xi^2/4)(t'-t)_+} + \frac{3}{2}\delta(t'-t) \\
K^{qG}(t,t',\xi) &= T_R N_f \frac{t^2 + (t'-t)^2 - \xi^2/4}{(t'^2 - \xi^2/4)^2} t' \\
K^{Gq}(t,t',\xi) &= C_F \frac{t'^2 + (t'-t)^2 - \xi^2/4}{t(t'^2 - \xi^2/4)} \\
K^{GG}(t,t',\xi) &= 2C_A \left(\frac{t'}{t}\right) \frac{1}{(t'^2 - \xi^2/4)^2} \left[\frac{(t'^2 - \xi^2/4)^2}{(t'-t)_+} + t'(t'^2 + \xi^2/4) \right. \\
&\qquad \left. - t(3t'^2 - \xi^2/4) - (t'+t)(t'-t)^2 \right] + \frac{\beta_0}{2}\delta(t'-t), \\
\Delta K^{qq}(t,t',\xi) &= K^{qq}(t,t',\xi) \\
\Delta K^{qG}(t,t',\xi) &= T_R N_f \frac{t^2 - (t'-t)^2 - \xi^2/4}{(t'^2 - \xi^2/4)^2} t' \\
\Delta K^{Gq}(x,\xi) &= C_F \frac{t' - (t'-t)^2 - \xi^2/4}{t(t'^2 - \xi^2/4)} \\
\Delta K^{GG}(x,\xi) &= 2C_A \left(\frac{t'}{t}\right) \frac{1}{(t'^2 - \xi^2/4)^2} \left[\frac{(t'^2 - \xi^2/4)^2}{(t'-t)_+} + t'(t'^2 + \xi^2/4) \right. \\
&\qquad \left. -t(3t'^2 - \xi^2/4) - 2t'(t'-t)^2 \right] + \frac{\beta_0}{2}\delta(t'-t) \; .
\end{aligned}
$$

The Brodsky–Lepage Limit

$\tau \to \pm 1$: $\langle p_2| \to \langle p|$, $\langle p_1| \to \langle 0|$

BRODSKY-LEPAGE-EFREMOV-RADYUSHKIN Kernels.

$$V^{qq}(x,y) = C_F\{\Theta(y-x)[\frac{x}{y} - \frac{1}{y} + \frac{1}{(y-x)_+}] + \Theta(x-y)[\frac{1-x}{1-y} - \frac{1}{1-y} + \frac{1}{(x-y)_+}\}.$$

etc.

The Altarelli–Parisi Limit

$\tau \to 0.$

(ONE POSSIBILITY).

In the case $t > \tau, t' > \tau$ we obtain another representation. First note :sign$\overline{y}$ = −sign$y$, $\Theta(1-\frac{\overline{x}}{\overline{y}}) = \Theta(y-x)$. Using these changes we obtain (For simplicity, we dropped here the +-prescriptions.)

$$\begin{aligned} V^{qq}(x,y) &= C_F\Theta(y-x)\{\frac{x}{y}[1+\frac{1}{y-x}] - \frac{1-x}{1-y}[1+\frac{1}{x-y}]\} \\ &= C_F\Theta(y-x)\frac{1}{y-x}[1+\frac{x\overline{x}}{y\overline{y}}], \quad \boxed{x = \frac{1}{2\tau}(\tau+t), \ y = \frac{1}{2\tau}(\tau+t').} \end{aligned}$$

$x = x(\tau)$
$y = y(\tau)$

Note, that the Altarelli-Parisi Limit is obtained by

$$\underline{lim_{\tau\to 0}}\, V^{qq}(x,y) = \frac{1}{|t'|}P^{qq}(\frac{t}{t'}) = \underline{\underline{\frac{1}{|t'|}C_F\frac{z^2+1}{1-z}}}.$$

SIMILARLY FOR THE OTHER KERNELS.

SECOND POSSIBILITY: $\langle p_2| \to \langle p_1| \equiv \langle p|$ RIGHT FROM THE BEGINNING.

$$f^q(z,\mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d(2p\tilde{x}\kappa_-)\langle p|O^q|p\rangle(\kappa_-,\mu)\frac{e^{2ip\tilde{x}\kappa_-}}{2ip\tilde{x}}$$

$$zf^G(z,\mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d(2p\tilde{x}\kappa_-)\langle p|O^G|p\rangle(\kappa_-,\mu)\frac{e^{2ip\tilde{x}\kappa_-}}{(2ip\tilde{x})^2}\,.$$

$$\mu^2\frac{d}{d\mu^2}f^{\rm NS}(z,\mu) = \frac{\alpha_s(\mu^2)}{2\pi}\int_{-\infty}^{+\infty}\frac{dz'}{|z'|}\widehat{P}^{\rm NS}\left(\frac{z}{z'}\right)f^{\rm NS}(z,\mu),$$

$$\mu^2\frac{d}{d\mu^2}\begin{pmatrix} f^q(z,\mu) \\ f^G(z,\mu)\end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi}\int_{-\infty}^{+\infty}\frac{dz'}{|z'|}\widehat{\boldsymbol{P}}\left(\frac{z}{z'}\right)\begin{pmatrix} f^q(z,\mu) \\ f^G(z,\mu)\end{pmatrix}.$$

$$\widehat{P}^{ij}(z) = P^{ij}(z)\theta(z)\theta(1-z)$$

$$P^{ij}(z) = \int_{-\infty}^{+\infty} du\,\underline{\widehat{O}^{ij}(u,z)}\int_0^1 d\xi(1-u)\widehat{K}^{ij}(\alpha_1,\alpha_2)\theta(1-u)\theta(u),$$

$$\widehat{\boldsymbol{K}} = \begin{pmatrix} K^{qq} & (1/\kappa_-)K^{qG} \\ (\kappa_- - i\epsilon)K^{Gq} & K^{GG}\end{pmatrix},$$

$$\widehat{O}^{ij}(u,z) = \begin{pmatrix} \delta(z-u) & \partial_z\delta(z-u) \\ -\theta(z-u)/z & \delta(z-u)/z\end{pmatrix}$$

$$
\begin{aligned}
P^{qq}(z) &= C_F\left(\frac{1+z^2}{1-z}\right)_+ \\
P^{qG}(z) &= 2N_fT_R\left[z^2+(1-z)^2\right] \\
P^{Gq}(z) &= C_F\frac{1+(1-z)^2}{z} \\
P^{GG}(z) &= 2C_A\left[\frac{1}{z}+\frac{1}{(1-z)_+}-2+z(1-z)\right]+\frac{\beta_0}{2}\delta(1-z), \\
\Delta P^{qq}(z) &= P^{qq}(z) \\
\Delta P^{qG}(z) &= 2N_fT_R\left[z^2-(1-z)^2\right] \\
\Delta P^{Gq}(z) &= C_F\frac{1-(1-z)^2}{z} \\
\Delta P^{GG}(z) &= 2C_A\left[1-2z+\frac{1}{(1-z)_+}\right]+\frac{\beta_0}{2}\delta(1-z).
\end{aligned}
$$

In deriving eq. (70) it is useful to apply the relations

$$
\theta(x)=\lim_{\varepsilon\to0^+}\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\frac{e^{ix\xi}}{i\xi+\varepsilon},\qquad \delta^{(k)}(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\,(i\xi)^k e^{ix\xi},
$$

which are valid for tempered distributions [16].

$$H(t,t') \equiv \int D\xi\, \Theta(\frac{t'-t}{t'+\tau(2\xi-1)})\Theta(\frac{t+\tau(2\xi-1)}{t'+\tau(2\xi-1)})\,[\frac{1}{|t'-t|} - \int_0^1 d\lambda \frac{1}{\lambda}\delta(t-t')]$$

$$\begin{aligned}\int dt' f(t')H(t,t') \equiv & \int dt' \int D\xi\Theta(\frac{t'-t}{t'+\tau(2\xi-1)})\Theta(\frac{t+\tau(2\xi-1)}{t'+\tau(2\xi-1)}) \\ & \{\frac{1}{|t'-t|} - \delta(t-t')\int dt"\,[\frac{1}{|t"-t|} - \frac{1}{|t"+\tau(2\xi-1)|}] \\ & \Theta(\frac{t"-t}{t"+\tau(2\xi-1)})\Theta(\frac{t+\tau(2\xi-1)}{t"+\tau(2\xi-1)})\}f(t')\end{aligned}$$

$$\begin{aligned}\int dt f(t)H(t,t') \equiv & \int dt \int D\xi\Theta(\frac{t'-t}{t'+\tau(2\xi-1)})\Theta(\frac{t+\tau(2\xi-1)}{t'+\tau(2\xi-1)}) \\ & [\frac{1}{|t'-t|} - \delta(t-t')\int dt"\Theta(\frac{t'-t"}{t'+\tau(2\xi-1)})\Theta(\frac{t"+\tau(2\xi-1)}{t'+\tau(2\xi-1)})\frac{1}{|t'-t"|}]f(t)\end{aligned}$$

For $V_{ext}^{GG}(t,t')$, $\Delta V_{ext}^{GG}(t,t')$ the factors $\frac{t'}{t}$ have to be included in the functions f.

13 Appendix: The +-prescription

Let us consider the typical terms

$$H(t,t') = \int D\alpha(\delta(\alpha_1) + \delta(\alpha_2)[\frac{1}{\alpha_1+\alpha_2}]_+ \delta(t - t'(1-\alpha_1-\alpha_2) + \tau(\alpha_1-\alpha_2))$$

$$\alpha_1 = \lambda\xi, \ \alpha_2 = \lambda(1-\xi),$$

$$H(t,t') = \int D\xi \, d\lambda \, [\frac{1}{\lambda}]_+ \, \delta(t - t'(1-\lambda) + \tau\lambda(2\xi - 1))$$

where λ is an arbitrary integration constant. In terms of the variables x, y we have to solve

$$\begin{aligned} tV^{Gq}_{ext}(t,t',\tau) &= \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 K^{Gq}(\alpha_1, \alpha_2'\kappa_-)\kappa_- \frac{1}{2}(\Theta(x-(1-\alpha_1-\alpha_2)y-\alpha_2) \\ &\quad -\Theta(-[x-(1-\alpha_1-\alpha_2)y-\alpha_2])+\lambda) \end{aligned}$$

Now we have to solve the integral

$$\begin{aligned} \int D\alpha(\Theta(x-(1-\alpha_1-\alpha_2)y-\alpha_2) &= (x-\frac{y}{2})+\Theta(y-x)\frac{(y-x)^2}{2y} \\ &\quad -\Theta(x-y)\frac{(\overline{y}-\overline{x})^2}{2\overline{y}} \end{aligned}$$

$$\int D\alpha(\Theta(x-(1-\alpha_1-\alpha_2)y-\alpha_2)\delta(\alpha_1)\delta(\alpha_2) = \Theta(x-y)$$

Furthermore we use

$$\int D\alpha(\Theta(-[x-(1-\alpha_1-\alpha_2)y-\alpha_2]) = \int D\alpha(\Theta(\overline{x}-(1-\alpha_1-\alpha_2)\overline{y}-\alpha_2)$$

Using these equations and taking into account the general structure [3] we obtain finally

$$\begin{aligned} (2x-1)V^{Gq}_{ext}(x,y) &= C_F\{-\lambda+\Theta(\frac{x}{y})\Theta(1-\frac{x}{y})\mathrm{sign}(y)(1-\frac{x^2}{y}) \\ &\quad -\Theta(\frac{\overline{x}}{\overline{y}})\Theta(1-\frac{\overline{x}}{\overline{y}})\mathrm{sign}(\overline{y})(1-\frac{\overline{x}^2}{\overline{y}})\}\frac{1}{\tau} \end{aligned}$$

12 Appendix: V_{ext}^{Gq}

We have to solve the equation (72)

$$V_{ext}^{Gq}(t,t',\tau) = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 K^{Gq}(\alpha_1, \alpha_2'\kappa_-)\kappa_- \frac{1}{2\pi}\int_{-\infty}^{+\infty} d(p_+\tilde{x}\kappa_-)$$
$$\times \left[ip_+\tilde{x}(\kappa_-)\right]^{-1} \frac{1}{t} \exp\left\{ip_+\kappa_-\tilde{x}\left[t-(1-\alpha_1-\alpha_2)t' + \tau(\alpha_1-\alpha_2)\right]\right\},$$

with

$$K^{Gq}(\alpha_1,\alpha_2) = -C_F\frac{1}{\kappa_-}\{\delta(\alpha_1)\delta(\alpha_2)+2\}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d(p_+\tilde{x}\kappa_-)}{ip_+\tilde{x}\kappa_-}\exp\{ip_+\kappa_-\tilde{x}X\} = \frac{1}{2}(\Theta(X)-\Theta(-X)+\lambda)$$

$$\lim_{\tau\to 0} V_{ext}^{Gq}(t,t') = \frac{1}{t'}C_F\left(\frac{1+(1-z)^2}{z} - \frac{\lambda}{z}\right), \qquad z = \frac{t}{t'}.$$

6. CONCLUSIONS

1) THE EVOLUTION OF LC-OPERATORS DEPENDS ON TWO SCALAR VARIABLES.

2) LO KERNELS ARE KNOWN NOW BOTH FOR UNPOLARIZED AND POLARIZED OPERATORS

3) TWO-VARIABLE PARTITION FUNCTIONS CAN BE STUDIED BY EXPERIMENT $\longleftrightarrow$ SCALING VIOL.

- NS, SINGLET
- POLARIZED, UNPOLARIZED

4) CONSIDER SPECIFIC DIRECTIONS $\frac{\tilde{x}p_-}{\tilde{x}p_+} = \tau = \text{CONST.}$

$\longrightarrow$ HOST OF TESTABLE ONE VARIABLE EVOLUTIONS.

5) IMPORTANT BREAKTHROUGH FOR HIGHER TWIST: SPIN CAN BE RESUMMED ALSO THERE.

$\longrightarrow$ HT EVOLUTION eqs. (JB, DR, WVN).