Leading Order Radiative Corrections
to Deep Inelastic ep Scattering to $\mathcal{O}(\alpha^2)$
for
Different Kinematical Variables

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1. The Different Variables
2. The Corrections up to $\mathcal{O}(\alpha^2 L^2)$
3. Numerical Results
4. Conclusions
1. The Different Variables

**Goal:**

Measurement of a Born Cross section: $2 \rightarrow 2$ Reaction

$\leftrightarrow$ Integrating over the DOF of the radiated Photon(s).

$\leftrightarrow$ Different Correction Functions for Different Variables are obtained!

\[
\text{CALCULATE: } K\text{-FACTORS: } S_{NC,CC}(x_1,y_1) = \frac{\sigma^{(0)} + \sigma^{corr}}{\sigma^{(0)}}.
\]

- Double Angle Method $\Theta_e, \Theta_3$
- $\theta_e$ & $y_J$
- Jet Measurement: NC $Q^2_J, y_J$
- Jet Measurement: CC $-$ $-$
- Mixed Variables ($Q^2_e, y_J$) $\{+1\}$
- (Lepton Measurement) $Q^2_e, y_e, \Theta_e, \Theta_3$
Table 1: The shifted variables for different types of cross section measurement

\[ \hat{\nu} \text{ subsystem variable} \]
\[ \hat{\nu} \text{ 'tree level' variable} \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( \hat{s} )</th>
<th>( Q^2 )</th>
<th>( \hat{y} )</th>
<th>( J(x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>lepton measurement</td>
<td>( z )</td>
<td>( Q^2 z )</td>
<td>( (z + y - 1)/z )</td>
<td>( y/(z + y - 1) )</td>
</tr>
<tr>
<td>jet measurement</td>
<td>( z )</td>
<td>( Q^2(1 - y)/(1 - y/z) )</td>
<td>( y/z )</td>
<td>( (1 - y)/(z - y) )</td>
</tr>
<tr>
<td>mixed variables</td>
<td>( z )</td>
<td>( Q^2 )</td>
<td>( y/z )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>double angle method</td>
<td>( z )</td>
<td>( Q^2 z^2 )</td>
<td>( y )</td>
<td>( z )</td>
</tr>
<tr>
<td>( y_{JB} ) and ( \theta_e )</td>
<td>( z )</td>
<td>( Q^2 z(z - y)/(1 - y) )</td>
<td>( y/z )</td>
<td>( (z - y)/(1 - y) )</td>
</tr>
</tbody>
</table>

- \( \hat{\nu}_o \):

  - **Lepton Measurement**
  \[ \hat{x}(\hat{\nu}_o) = 1 \]

  - **Jet Measurement**

  - **Mixed Variables**
  \[ \hat{\nu}_o = y \]

  - **Double Angle**
  \[ \hat{\nu}_o = 0 \]

  \[ \frac{d}{d(x \to 0, \theta^2 \to 0)} \]

  \[ \text{for } z \to \hat{\nu}_o \]

**BUT:**

\[ 2E_e = E_e(1 - \cos \theta_e) + E_j(1 - \cos \theta_j) \geq 0 \]

**Fortunately:**

\[ \hat{\nu}_o = \frac{Q^2}{2E_e} \]

\[ \Rightarrow \text{ZEUS:} \text{this helps only in the case of the double angle method!} \]
2. The Corrections up to $O(\alpha^2)$

Contributions:

1. Bremsstrahlung: Diagrams a,b
2. Electron Pair Production: Diagram c
3. Fermion Pair Production: Diagram d, $f = e, \mu, \tau, u, d, s, c, b$

The **Radiator-Method** is applied.

Meaning of the bullet: Collinear Bremsstrahlung contribution including soft & virtual corrections.

An individual consideration of initial and final state bremsstrahlung is possible.

- **Apply the Renormalization Group Equation**
- Trivial Coefficient Functions in $O(\alpha^N\lambda^N)$
- Factorization for each charged ('massless') fermion line.

\[
\begin{align*}
  &\gamma & e & \bar{e} \\
  e \quad (a) & \quad \quad & \quad \quad & \quad \quad & \quad \quad & f & \bar{f} \\
  & NS & NS & S & \text{Running} & \alpha_{QED}
\end{align*}
\]
Figure 1: Diagrams contributing to the radiative corrections up to $O(\alpha^2L^2)$.

Figure 2: $e^- \rightarrow e^+$ transition probability for different values of $Q^2$. Full line: $Q^2 = 10 \text{ GeV}^2$, dashed line: $Q^2 = 100 \text{ GeV}^2$, and dotted line: $Q^2 = 1000 \text{ GeV}^2$. 
\[ \frac{d^2 \sigma^{(2)}}{dx dy} = \frac{d^2 \sigma^{(0)}}{dx dy} + \frac{\alpha}{2\pi} \ln \left( \frac{Q^2}{m^2} \right) \int_0^1 P_{ee}^{(1)}(z) \left\{ \theta(z - z_0) J(x, y, z) \frac{d^2 \sigma^{(0)}}{dx dy} \bigg|_{x = \hat{x}, y = \hat{y}, z = \hat{z}} - \frac{d^2 \sigma^{(0)}}{dx dy} \right\} \]
\[ + \frac{1}{2} \left[ \frac{\alpha}{2\pi} \ln \left( \frac{Q^2}{m^2} \right) \right]^2 \int_0^1 P_{ee}^{(2,1)}(z) \left\{ \theta(z - z_0) J(x, y, z) \frac{d^2 \sigma^{(0)}}{dx dy} \bigg|_{x = \hat{x}, y = \hat{y}, z = \hat{z}} - \frac{d^2 \sigma^{(0)}}{dx dy} \right\} \bigg\} O(\alpha^2) \]
\[ + \left( \frac{\alpha}{2\pi} \right)^3 \int_0^1 \left\{ \ln^2 \left( \frac{Q^2}{m^2} \right) P_{ee}^{(2,2)}(z) + \sum_{j=1}^n \ln^2 \left( \frac{Q^2}{m_j^2} \right) P_{ee, j}^{(2,2)}(z) \right\} J(x, y, z) \frac{d^2 \sigma^{(0)}}{dx dy} \bigg|_{x = \hat{x}, y = \hat{y}, z = \hat{z}} \bigg\} O(\alpha^3) \]

\[ J(x, y, z) = \left| \begin{array}{ccc} \frac{\partial \hat{x}}{\partial x} & \frac{\partial \hat{y}}{\partial x} \\ \frac{\partial \hat{x}}{\partial y} & \frac{\partial \hat{y}}{\partial y} \end{array} \right| \]

\[ Z < 1 \]

\[ O(\alpha) \]

\[ P_{ee}^{(1)}(z) = \frac{1 + z^2}{1 - z} \]

\[ O(\alpha^2) \]

\[ P_{ee}^{(3,1)}(z) = \frac{1}{2} \left[ P_{ee}^{(1)} \otimes P_{ee}^{(1)} \right](z) \]

\[ = \frac{1 + z^2}{1 - z} \left[ 2 \ln(1 - z) - \ln z + \frac{3}{2} \right] + \frac{1}{2} (1 + z) \ln z - (1 - z) \]

\[ O(\alpha^3) \]

\[ P_{ee}^{(3,2)}(z) = \frac{1}{2} \left[ P_{ee}^{(1)} \otimes P_{ee}^{(1)} \right](z) \]

\[ \equiv (1 + z) \ln z + \frac{1}{2} (1 - z) + \frac{21}{3} (1 - z^2) \]

\[ P_{ee, j}^{(3,2)}(z) = N_c(f_j) \frac{1}{3} P_{ee}^{(1)}(z) \theta \left( 1 - z - \frac{2m_j}{E_e} \right) \]

\[ \times \left[ 1 - \exp(-A^2 \hat{Q}^2) \right] \text{ with } A^2 = 3.37 \text{ GeV}^{-2}. \]
REMARK:

FBR : $O(\alpha^2)$ FROM LEPTONS.

\[ E_{e'} = E_e + E_{\gamma_i} + E_{e^-e^+} + \sum_i E_{\gamma_i} \]

COLLECT ALL RADIATED ENERGY IN THE ANGULAR VICINITY OF $e'$!

RADIATION OF THE QUARK LINES:

ABSORB INTO SCALING VIOL. OF THE QUARK DISTR. (& RUNNING QED EFFECTS (S & N S) ARE TAKEN INTO ACC.).
SOFT EXPONENITIATION :

SOLVE : LO - GRIBOV LIPATOV eq. (NS) FOR \( z \to 1 \)

\[
D_{NS}(z, Q^2) = \zeta (1 - z)^{-1} \exp \left[ \frac{3}{2} - 2 \gamma_E \right] \frac{\zeta (z - 2 \gamma_E)}{\Gamma(1 + \zeta)}
\]

with

\[
\zeta = -3 \ln \left[ 1 - (\alpha/3\pi) \ln (Q^2/m_e^2) \right]
\]

(RUNNING \( \alpha_{QED} \) )

THESE TERMS WERE TAKEN INTO ACC. ALREADY

\[
P_{ee}^{>2, \text{soft}}(z, Q^2) = D_{NS}(z, Q^2) - \frac{\alpha}{2\pi} \ln \left( \frac{Q^2}{m_e^2} \right) \left\{ \frac{2}{1 - z} \left[ \frac{11}{6} + 2 \ln(1 - z) \right] \right\}
\]

and

\[
\frac{d^2\sigma^{(>2, \text{soft})}}{dz dy} = \int_0^1 dz P_{ee}^{(>2)}(z) \left\{ \delta(z - z_0) J(z, y, z) \frac{d^2\sigma^{(0)}}{dz dy} \bigg|_{z=x,y=v,x=v} - \frac{d^2\sigma^{(0)}}{dz dy} \right\}
\]

\( \rightarrow \) NOTE: NO 'UNIQUE' EXPONENTIATION EXISTS!
3. Numerical Results

- **UPDATE: \( O(\alpha) \) K-Factors**
  - \( \rightarrow \) **GOOD NUMERICAL AGREEMENT BETWEEN COMPLETE \( O(\alpha) \) \& \( O(\alpha L) \).**

- \( O(\alpha^2 L^2) \) IN ALL VARIABLES

- **CONSIDER ISR ONLY (LEPTONS)**
  - \( \rightarrow \) **FSR INTEGRATED BY THE CALORIMETRIC MEASUREMENT**
  - \( \rightarrow \) **QUARK LINE RAD \( \rightarrow O(\alpha_s) \) MOD.**
  - **OF SCAL. VIOL.**
  - **COMPTON PEAK: EXCL. TREATMENT!**

- **PDF’S: MRST D\(^-\), SHT. RES: MRSH CTEQ2.**
Lepton Measurement

\( Q^2, y_{\text{lep}} \)

\( x = 10^{-2} \) dashed line
\( x = 10^{-3} \) dotted line

\( x = 0.1 \)
\( x = 0.3 \)
\( x = 0.9 \)
$O(a^2 L^2)$

**Lepton Measurement**

- $x = 0.01$, dotted line
- $x = 0.001$, dashed line
- $x = 0.0001$, full line

*High y range: important for the measurement of $f_L(x, Q^2)$ and $g(x, Q^2)!$*
Figure 3: Leptonic initial state radiative corrections $\delta_{NC}(x, y) = (d\sigma^{(2+>2,soft)/dzdy})/(d\sigma^0/dzdy)$ in LLA for $e^- p$ deep inelastic scattering in the case of jet measurement for $\sqrt{s} = 314$ GeV, $A = 0$, and $Q^2 \geq 5$ GeV$^2$. Full lines: $O(\alpha^2)$ corrections; dotted lines: contributions due to $e^- \rightarrow e^+$ conversion eq. (13), $\delta_{NC}^{e^- \rightarrow e^+}(x, y) = (d\sigma^{(2,e^- \rightarrow e^+)/dzdy})/(d\sigma^0/dzdy)$ scaled by $\times 50$; upper line: $x = 0.01$, middle line: $x = 0.0001$, lower line $x = 0.9$. 
Figure 4: $\delta_{CC}(x,y) = (d\sigma^{(2\rightarrow2,soft)}_{CC}/dzdy)/(d\sigma^0_{CC}/dzdy)$ for deep inelastic $e^-p$ scattering in the case of jet measurement. Dotted lines: $\delta_{CC}^e e^+ (x,y)$. The other parameters are the same as in figure 3.
mixed variables

$\delta_{NC}(x,y)$

$Q^2_{\text{lept.}}$

$y_{JB}$

$x = 0.5$

$x = 0.01$

$x = 0.0001$
Figure 5: $\delta_{NC}(x, y)$ for the case of mixed variables. Dotted lines: $\delta_{NC}^{-}(x, y)$; upper line: $z = 0.5$, lower line $z = 0.01$. The other parameters are the same as in figure 3.
Figure 6: $\delta_{NC}(x,y)$ for the case of the double angle method for $A = 35\,\text{GeV}$. Full lines: $\delta^{(1+2+2,soft)}_{NC}(x,y)$, dashed lines: $\delta^{(1)}_{NC}(x,y)$. Dotted lines: $\delta^{\psi^\pm,\gamma^\pm}_{NC}(x,y)$ scaled by $\times 100$; upper line: $x = 0.5$, middle line: $x = 0.01$, lower line: $x = 0.0001$. The other parameters are the same as in figure 3.
A DANGEROUS CASE: θ_e & y_f

RESCALING: ISR

\[ Q^2 = Q^2 \frac{z-y}{1-y} \]
\[ x = x \frac{z(2-z-y)}{1-y} \]
\[ z_0 = y \]

ZEUS:

\[ z_0 = \max \left\{ \frac{35 \text{ GeV}}{2E_e}, y \right\} \]

\[ \delta_{NC}(x,y) \text{ JUMPS! AT } y \geq \frac{\phi}{2E_e}, \phi = 35 \text{ GeV.} \]

\[ \frac{\sigma(\phi^2, x \rightarrow 0)}{\sigma(Q^2, x)} ! \]

NO CONTROL ON INPUT AT ALL !

\[ \rightarrow \text{ UNFORTUNATE CHOICE OF VARIABLES.} \]
Figure 7: $\delta_{NC}(x,y)$ for the measurement based on $\theta_e$ and $y_{BB}$ for $A = 35$ GeV. Full lines: $\delta_{NC}^{(1+2+2^{+2,\omegaH})}(x,y)$, dotted lines: $\delta_{NC}^{(4)}(x,y)$. Dashed lines: $\delta_{NC}^{+-+}(x,y)$; upper line: $z = 0.5$, middle line: $z = 10^{-2}$, lower line: $z = 10^{-4}$. The other parameters are the same as in figure 3.
Comparison with a Full $\mathcal{O}(\alpha)$ Calculation

TERAD, D.Y. BARDIN ET AL.

\begin{align*}
\delta_{lep}[\%] \\
\text{LEPT. VAR. } x = 10^{-3} \\
\hline
10^{-2} & 10^{-1} & 0.5 & 0.9 \\
\hline
\end{align*}

HERA ep \quad \underline{- -} \text{complete} \quad \underline{-} \text{leading log}

\begin{align*}
\delta_{jb}[\%] \\
x = 10^{-3} & 10^{-2} & 10^{-1} & 0.5 & 0.9 \\
\hline
\end{align*}

HERA ep \quad \underline{- -} \text{complete} \quad \underline{-} \text{leading log}
$\delta_{\text{mix}}[\%]$

**MIXED VAR.**

$x = 0.5$

$10^{-4}$

$10^{-3}$

$10^{-2}$

$10^{-1}$

HERA ep

---

complete

leading log
POLYDAMAS ADVISES HECTOR TO MAKE THE ASSAULT ONTO THE CAMP OF THE GREEKS ON FOOT.

engraving by J. FLAXMAN 1780ies.
4. Conclusions

1. The $O(\alpha_L)$ and $O(\alpha^2 L^2)$ radiative corrections have been calculated for:
   - Leptonic variables
   - Jet measurement: NC & CC
   - Mixed variables
   - Double angle method
   - Variables based on $\theta_e / y_{3b}$.

2. The dominance & stability of RC's in $O(\alpha')$ is established, except of the high $y$ range for lept. variables & the $(\theta_e, y_{3b})$ case.

3. The double angle method is the ideal way to measure $d^2\sigma / d\theta_c dy_{3b}$ with respect to RC's, due to their flat behaviour & smallness.

4. The method based on $\theta_e$ & $y_{3b}$ is problematic due to a jump at the cut threshold $y_{cut}$. The reason for this is the mapping $d^2\sigma \rightarrow d^2\sigma (x=0, y^2=0)$.

5. The inclusion of the $O(\alpha' L^2)$ is required to reach accuracy at the $\%$ level.

6. The use of RGE methods considerably simplifies the calculation of dominant terms & provides a faster way to recognize instable mappings under RC if compared to full calculations.