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Harmonic Sums and Mellin Transforms up to Two-Loop Order

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DESY

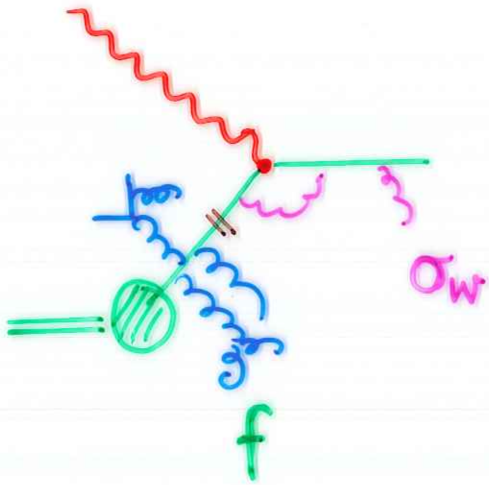
J. Blümlein, S. Kurth, Phys. Rev. **D60**, (1999) 014080
J. Blümlein, P. Will, in preparation.

1. INTRODUCTION

- STUDY OF MASSLESS FIELD THEORIES

QCD, QED $\quad m_i \to 0$

$\longrightarrow$ SIMPLE PHASE SPACE, NO MASS THRESHOLDS



$$\sigma = \sigma_W \otimes f$$

↑ MELLIN CONVOLUTION

σ_W WILSON COEFFICIENT

f PARTON DENSITY

X – SPACE FUNCTIONS :

MULTIPLE HARMONIC SUMS (N)

$\Updownarrow$

NIELSEN INTEGRALS : (x) $(\leq O(\alpha^2))$

N. NIELSEN 1909

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{dz}{z} \log^{n-1}(z) \log^p(1-zx)$$

$\tau = p + n.$

SPECIAL CASES :

$\mathrm{Li}_n(x) = S_{n-1,1}(x)$ $\quad \tau = n$

$\frac{d\,\mathrm{Li}_2(\pm x)}{d\log(x)} = -\log(1 \mp x)$ $\quad \tau = 1$

$\mathrm{Li}_0(x) = \frac{x}{1-x}$ $\quad \tau = 0$

$\frac{dx}{1 \pm x}, \; \frac{dx}{x}$ $\quad \tau = 1$

IT MAY BE USEFUL TO CONSIDER ALSO GENERALIZATIONS AS:

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!\,p!\,q!} \int_0^1 \frac{dz}{z} \log^{n-1}(z) \cdot \log^p(1-zx) \cdot \log^q(1+zx)$$

$\tau = n+p+q$

CF. ALSO: VERMASEREN, REMIDDI '99

JB 2000

→ THE NIELSEN INTEGRALS (AND THEIR GENERALIZATIONS) ARE HELPFUL TO BE CONSIDERED SINCE THE ARGUMENT-STRUCTURE IS KEPT TRACTABLE.

→ NON MINIMAL SET OF REPRESENTATION

MELLIN CONVOLUTION:

$$[A \otimes B](x) = \int_0^1 dx_1 \int_0^1 dx_2\, \delta(x - x_1 x_2)\, A(x_1)\, B(x_2)$$

MELLIN TRANSFORM:

$$\mathbb{M}[C(x)](N) := \int_0^1 dx\, x^{N-1}\, C(x) \quad , \quad N \in \mathbb{C}.$$

$$\boxed{\mathbb{M}[A \otimes B](N) = \mathbb{M}[A](N) \cdot \mathbb{M}[B](N)}$$

$\mathbb{M}$ TRANSFORMS NESTED INTEGRALS OF THE ABOVE TYPE INTO ORDINARY PRODUCTS.

N-SPACE OBJECTS ARE OFTEN SIMPLER THAN X-SPACE OBJECTS.

EXAMPLE: $O(\alpha_s)$: $\quad P_{qg} = x^2 + (1-x)^2$

WHAT IS $P_{qg}^{-1}(x)$?

$$P_{qg}^{-1}(N) = \frac{1}{P_{qg}(N)} = \frac{N(N+1)(N+2)}{N^2+N+2}. \quad (\mathbb{M}[\delta(1-x)] = 1)$$

$$P_{qg}^{-1}(x) = 4\,\delta(1-x) - 2x\,\delta'(1-x) - 4\sqrt{x}\left[\cos\left(\frac{\sqrt{7}}{2}\ln x\right) - \frac{3}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}\ln x\right)\right]$$

SO FAR: NON-ALTERNATING SUMS.

NOT SUFFICIENT!

∴ ALSO PROPAGATORS $\propto \frac{1}{1+x}$ EMERGE.

$$S_{-k}(N) = \sum_{\ell=1}^{N} \frac{(-1)^\ell}{\ell^k}$$

$$\lim_{N\to\infty} S_{-1}(N) = -\log 2$$; TRANSCD. 1

MULTIPLE HARMONIC SUMS:

$$S_{k_1,\dots,k_m}(N) = \sum_{n_1=1}^{N} \frac{[\mathrm{sign}(k_1)]^{n_1}}{n_1^{|k_1|}} \sum_{n_2=1}^{n_1} \frac{[\mathrm{sign}(k_2)]^{n_2}}{n_2^{|k_2|}} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{[\mathrm{sign}(k_m)]^{n_m}}{n_m^{|k_m|}}$$

$N \in \mathbb{N}$, $\forall \ell$, $k_\ell \neq 0$.

(LATER: $N \in \mathbb{C}$)

$$\begin{aligned}
\frac{d}{dx}\left[-H_{a_1,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) + H_{a_1,\ldots,a_n}(x) \sqcup\!\sqcup\, H_{b_1,\ldots,b_m}(x)\right] &= \\
-f(a_1;x)\left[H_{a_2,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) - H_{a_2,\ldots,a_n}(x) \sqcup\!\sqcup\, H_{b_1,\ldots,b_m}(x)\right] & \\
-f(b_1;x)\left[H_{a_1,\ldots,a_n}(x)H_{b_2,\ldots,b_m}(x) - H_{a_1,\ldots,a_n}(x) \sqcup\!\sqcup\, H_{b_2,\ldots,b_m}(x)\right] &= 0\,.
\end{aligned}$$

$$\begin{aligned}
\left[-H_{a_1,\ldots,a_n}(x)H_{b_1,\ldots,b_m}(x) + H_{a_1,\ldots,a_n}(x) \sqcup\!\sqcup\, H_{b_1,\ldots,b_m}(x)\right] &= \\
-\int_0^x dz f(a_1;z)\left[H_{a_2,\ldots,a_n}(z)H_{b_1,\ldots,b_m}(z) - H_{a_2,\ldots,a_n}(z) \sqcup\!\sqcup\, H_{b_1,\ldots,b_m}(z)\right] & \\
-\int_0^x dz f(b_1;z)\left[H_{a_1,\ldots,a_n}(z)H_{b_2,\ldots,b_m}(x) - H_{a_1,\ldots,a_n}(z) \sqcup\!\sqcup\, H_{b_2,\ldots,b_m}(z)\right] &= 0\,.
\end{aligned}$$

$$\mathbf{M}[H_{a_1,\ldots,a_n}(x)|_{reg}](N) = \int_0^1 dx\; x^{N-1}\;\; H_{a_1,\ldots,a_n}(x)|_{reg}$$

$$
\begin{aligned}
f(0;x) &= \frac{1}{x} \\
f(1;x) &= \frac{1}{1-x} \\
f(-1;x) &= \frac{1}{1+x}
\end{aligned}
$$

$$
\begin{aligned}
H_a(x) &= \int_0^x dz f(a;z) \\
H_{\pm1,b_1\ldots b_m}(x) &= \int_0^x dz f(\pm1;z) H_{b_1,\ldots,b_m}(z) \, , \\
H_{b_1+1,b_2,\ldots,b_m}(x) &= \int_0^x dz f(0;z) H_{b_1,\ldots,b_m}(z) \, ,
\end{aligned}
$$

$$
H_{a,b_1\ldots b_m}(x)\big|_{a=0,b_i=0} = \frac{1}{(m+1)!} \ln^{m+1}(x) \, .
$$

$$
H_{a_1,\ldots,a_n}(x) H_{b_1,\ldots,b_m}(x) = H_{a_1,\ldots,a_n}(x) ⧢ H_{b_1,\ldots,b_m}(x) \, .
$$

$$
\begin{aligned}
H_{a_1,\ldots,a_n}(x) &= H_{a_1}(x) H_{a_2,\ldots,a_n}(x) - H_{a_2,a_1}(x) H_{a_3,\ldots,a_n}(x) \\
&\quad + \ldots + (-1)^n H_{a_n,\ldots,a_1}(x)
\end{aligned}
$$

$$
H_{a_1}(x) H_{a_2,a_3}(x) - H_{a_3}(x) H_{a_2,a_1}(x) = H_{a_1}(x) ⧢ H_{a_2,a_3}(x) - H_{a_3}(x) ⧢ H_{a_2,a_1}(x)
$$

$$
H_{a_1,a_2,a_3}(x) = H_{a_1}(x) H_{a_2,a_3}(x) - H_{a_2,a_1}(x) H_{a_3}(x) + H_{a_3,a_2,a_1}(x)
$$

$$
\begin{aligned}
S_{a,a} &= \frac{1}{2}\left[S_a^2 + S_{2a}\right] \\
S_{a,a,a} &= \frac{1}{6}\left[S_a^3 + 3S_aS_{2a} + 2S_{3a}\right] \\
S_{a,a,a,a} &= \frac{1}{24}\left[S_a^4 + 6S_a^2 + S_{2a} + 3S_{2a}^2 + 8S_aS_{3a} + 6S_{4a}\right] \\
S_{a,a,a,a,a} &= \frac{1}{120}\left[S_a^5 + 10S_a^3S_{2a} + 20S_a^2S_{3a} + 30S_aS_{4a} + 15S_aS_{2a}^2 + 20S_{2a}S_{3a} + 24S_{5a}\right] \\
S_{a,a,a,a,a,a} &= \frac{1}{720}\left[S_a^6 + 15S_{2a}S_a^4 + 40S_{3a}S_a^3 + 90S_{4a}S_a^2 + 144S_aS_{5a} + 45S_a^2S_{2a}^2 \right. \\
&\quad \left. +120S_aS_{2a}S_{3a} + 15S_{2a}^3 + 90S_{2a}S_{4a} + 40S_{3a}^2 + 120S_{6a}\right] .
\end{aligned}
$$

3. LINEAR REPRESENTATIONS

$$S^{\tau}_{k_1 \dots k_m}(N) = M[F^{\tau}(x)](N) + P\left(S^{\tau'}_{k_1 \dots k_r},\ \sigma^{\tau''}_{k_1 \dots k_p}\right)$$

$\sigma^{\tau''}_{k_1 \dots k_p}$ ← transcendentals

$S^{\tau'}_{k_1 \dots k_r}$ ← lower harmonic sums

$\tau', \tau'' < \tau$

→ CALCULATE ALL ALTERNATING AND NON-ALTERNATING SUMS.

		Σ
$\tau = 1$	2	2
$\tau = 2$	6	8
$\tau = 3$	18	26
$\tau = 4$	54	80 $= 3^{\tau} - 1$
⋮		

$3^{\tau} - 1$: COVERS 2-LOOP LEVEL

2. HARMONIC SUMS AND MELLIN TRANSFORMS

THE SIMPLEST EXAMPLE:

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \frac{2}{(1-x)_+} + \dots$$

$$\int_0^1 dx\, x^{N-1} \frac{1}{(1-x)_+} = -\int_0^1 dx\, \frac{x^{N-1}-1}{x-1}$$

$$= -\sum_{k=0}^{N-2} \int_0^1 dx\, x^k = -\sum_{k=1}^{N-1} \frac{1}{k} = -S_1(N-1).$$

SINGLE HARMONIC SUMS:

$$S_k(N) = \sum_{l=1}^{N} \frac{1}{l^k}$$

'TRANSCENDENTALITY': $\equiv k$

$$\lim_{N\to\infty} S_k(N) = \zeta_k, \qquad k \geq 2$$

$$\lim_{N\to\infty} S_1(N) = \sigma_1 = \infty.$$

EXAMPLE:

$$S_{-1,3} = - S_{3,-1} + S_{-1} S_3 + S_{-4}$$

3-FOLD SUMS: SITA RAMACHANDRA RAO, SUBBARAO 1984

$$\sum_{perm} S_{\ell,m,n} = S_\ell S_m S_n + \sum_{perm} S_\ell S_{m \wedge n} + 2 S_{\ell \wedge m \wedge n}$$

EXAMPLE:

F₁ case ↓

$$S_{-1,-1,-2} = - S_{-2,-1,-1} - S_{-1,-2,-1} + \frac{1}{2}\left[S_{-1}^2 + S_2\right] S_{-2} + S_{-1} S_3 + S_{-4}$$

↑ F_1 - case

LIMIT $N \to \infty$: $\exists$ IF $n_1 \neq 1$.

EULER - ZAGIER MULTIPLE ζ - VALUES

(D. BROADHURST)

- NUMBER THEORY, TOPOLOGY, KNOT THEORY.

EXAMPLE :

$$\lim_{N \to \infty} S_{5,1,2,1}(N) = \frac{361}{144}\zeta_9 - \frac{5}{8}\zeta_7\zeta_2 + \frac{203}{48}\zeta_6\zeta_3 - \frac{35}{8}\zeta_5\zeta_4 - \zeta_3^3 .$$

(BROADHURST) TRANSCEDENTALITY 9.

TRANSCENDENTALITY OF A HARMONIC SUM:

$$\tau = \sum_{l=1}^{m} |k_l| .$$

$$
\begin{aligned}
S_1(N) &= \int_0^1 dx \frac{x^N - 1}{x-1} \equiv \mathbf{M}\left[\left(\frac{1}{x-1}\right)_+\right](N) \\
S_{-1}(N) &= \int_0^1 dx \frac{(-x)^N - 1}{x+1} \equiv (-1)^N \mathbf{M}\left[\left(\frac{1}{x+1}\right)\right](N) - \log(2)\,.
\end{aligned}
$$

$$
\int_0^1 dx g(x)[f(x)]_+ = \int_0^1 dx\, [g(x) - g(1)]\, f(x).
$$

$$
\begin{aligned}
S_k(N) &= \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \cdots \int_0^{x_{k-1}} \frac{x_k^N - 1}{x_k - 1} \\
&= \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 dx \log^{k-1}(x) \frac{x^N - 1}{x-1}
\end{aligned}
$$

$$
\begin{aligned}
S_{-k}(N) &= \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \cdots \int_0^{x_{k-1}} \frac{(-x_k)^N - 1}{x_k + 1} \\
&= \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 dx \log^{k-1}(x) \frac{(-x)^N - 1}{x+1}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{x^k}{k^l} &= \frac{(-1)^{l-1}}{(l-1)!} \int_0^x dx \log^{l-1}(x) \frac{x^n - 1}{x-1} \\
\sum_{k=1}^{n} \frac{(-x)^k}{k^l} &= \frac{(-1)^{l-1}}{(l-1)!} \int_0^x dx \log^{l-1}(x) \frac{(-x)^n - 1}{x+1}\,.
\end{aligned}
$$

$$
\tilde{\sigma}_{k_l,\ldots,k_1} = \lim_{N\to\infty} S_{k_l,\ldots,k_1}(N)
$$

$$
\sigma_{+1} \stackrel{\text{Def}}{=} \int_0^1 dx \frac{1}{1-x}\,.
$$

$$
\begin{aligned}
S_{-1,-1,2}(N) &= \mathrm{M}\left\{\left[\frac{1}{x-1}\left(F_1(x)+\log(1+x)\mathrm{Li}_2(1-x)\right)\right]_+\right\}(N) \\
&\quad +\zeta(2)S_{-1,-1}(N)-\left[\zeta(3)-\frac{3}{2}\zeta(2)\log(2)\right]S_{-1}(N) \\
&\quad +\left[\frac{1}{8}\zeta(3)-\frac{1}{6}\log^3(2)\right]S_1(N) \\
&= -S_{2,-1,-1}(N)-S_{-1,2,-1}(N)+S_{-1}(N)S_{-3}(N)+S_4(N) \\
&\quad +\frac{1}{2}\left[S_2(N)S_{-1}(N)+S_2^2(N)\right] \qquad (126)
\end{aligned}
$$

$$
\begin{aligned}
S_{-1,1,-2}(N) &= -\mathrm{M}\left\{\left[\frac{1}{x-1}\left(\mathrm{Li}_2(-x)\log(1+x)+\mathrm{S}_{1,2}(x)+\frac{1}{2}\log(x)\log^2(1+x)\right)\right]_+\right\}(N) \\
&\quad -\frac{1}{2}\zeta(2)\left[S_{-1,1}(N)-S_{-1,-1}(N)\right] \\
&\quad -\left[\frac{1}{8}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right]\left[S_1(N)-S_{-1}(N)\right] \qquad (127)
\end{aligned}
$$

$$
\begin{aligned}
S_{-1,1,2}(N) &= (-1)^N\mathrm{M}\left[\frac{\mathrm{Li}_3(1-x)}{1+x}\right](N)+\zeta(2)S_{-1,1}(N)-\zeta(3)S_{-1}(N) \\
&\quad -\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{9}{20}\zeta^2(2)-\frac{7}{8}\zeta(3)\log(2)-\frac{1}{2}\zeta(2)\log^2(2)-\frac{1}{24}\log^4(2) \qquad (128)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,-1,-2}(N) &= -\mathrm{M}\left\{\left[\frac{1}{x-1}\left(F_1(x)+\log(1-x)\mathrm{Li}_2(-x)\right)\right]_+\right\}(N) \\
&\quad -\mathrm{M}\left\{\left[\frac{1}{x-1}\left(\frac{1}{2}\mathrm{S}_{1,2}(x^2)-\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(-x)\right)\right]_+\right\}(N) \\
&\quad +\frac{1}{2}\zeta(2)\left[S_{1,1}(N)-S_{1,-1}(N)\right]+\left[\frac{9}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)\right]S_1(N) \qquad (129)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,-1,2}(N) &= (-1)^N\mathrm{M}\left\{\frac{1}{1+x}\left[F_1(x)+\log(1+x)\mathrm{Li}_2(1-x)\right]\right\}(N)+\zeta(2)S_{1,-1}(N) \\
&\quad -\left[\zeta(3)-\frac{3}{2}\zeta(2)\log(2)\right]S_1(N)-\left[\frac{1}{8}\zeta(3)-\frac{1}{6}\log^3(2)\right]S_{-1}(N) \\
&\quad +\frac{7}{8}\zeta(2)^2-2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{11}{4}\zeta(3)\log(2)+\frac{5}{4}\zeta(2)\log^2(2)+\frac{1}{12}\log^4(2) \\
&= -S_{2,1,-1}(N)-S_{1,2,-1}(N)+S_2(N)S_{1,-1}(N)+S_{1,-3}(N)+S_{3,-1}(N) \qquad (130)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,1,-2}(N) &= (-1)^N\mathrm{M}\left[\frac{\mathrm{S}_{1,2}(-x)+\log^2(2)\log(x)/2+\log(2)\mathrm{Li}_2(-x)}{1+x}\right](N) \\
&\quad +\frac{1}{2}\zeta(2)\left[S_{1,-1}(N)-S_{1,1}(N)\right] \\
&\quad -\left[\frac{1}{8}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right]\left[S_1(N)-S_{-1}(N)\right] \\
&\quad +\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{2}{5}\zeta(2)^2+\zeta(3)\log(2)-\frac{1}{2}\zeta(2)\log^2(2)+\frac{1}{24}\log^4(2) \qquad (131)
\end{aligned}
$$

4. ALGEBRAIC RELATIONS

THE ALGEBRAIC RELATIONS FOLLOW FROM PERMUTATION SYMMETRY.

BASIC QUESTION: CAN THE NUMBER OF HARMONIC SUMS AS BEING OBTAINED IN THE LINEAR REPRESENTATION BE REDUCED ?

(COMPLETE OR PARTIAL).

→ LESS IMPORTANT

WITH:

$S_{k_1 \ldots k_n}$

- INTEGERS ($S_{k_1 \ldots k_r}$ SUMS) (POWERS) (RAMANUJAN)
- HARMONIC SUMS
- ROOTS OF ALGEBRAIC EQUATIONS (FAA DI BRUNO)

IMPORTANT FOR ANALYTIC CONTINUATIONS. (LESS FUNCTIONS).

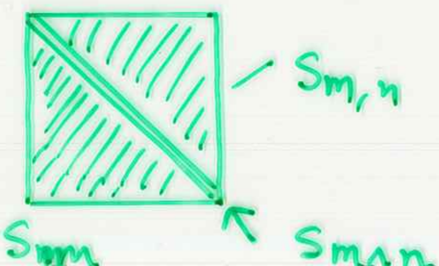
INVARIANT THEORY

- SINGLE HARMONIC SUMS (IRREDUCIBLE)
- DOUBLE SUMS: L. EULER 1775

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m \wedge n}$$

$m, n \in I$
$m, n \neq 0$

$$m \wedge n = [|m| + |n|]\, \mathrm{sign}(m)\, \mathrm{sign}(n)$$



Complete Permutations of the Index–Set

$$
\begin{aligned}
S_{a_1,a_2} + S_{a_2,a_1} &= S_{a_1} S_{a_2} + S_{a_1 \wedge a_2}, \\
\sum_{\mathrm{perm}\{a_1,a_2,a_3\}} S_{a_1,a_2,a_3} &= S_{a_1} S_{a_2} S_{a_3} + \sum_{\text{inv perm}} S_{a_1} S_{a_2 \wedge a_3} + 2 S_{a_1 \wedge a_2 \wedge a_3}, \\
\sum_{\mathrm{perm}\{a_1,a_2,a_3,a_4\}} S_{a_1,a_2,a_3,a_4} &= S_{a_1} S_{a_2} S_{a_3} S_{a_4} + \sum_{\text{inv perm}} S_{a_1} S_{a_2} S_{a_3 \wedge a_4} \\
&\quad + \sum_{\text{inv perm}} S_{a_1 \wedge a_2} S_{a_3 \wedge a_4} + 2 \sum_{\text{inv perm}} S_{a_1} S_{a_2 \wedge a_3 \wedge a_4} + 6 S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\mathrm{perm}} S_{a_1,a_2,a_3,a_4,a_5} &= S_{a_1} S_{a_2} S_{a_3} S_{a_4} S_{a_5} + \sum_{\text{inv perm}} S_{a_1 \wedge a_2} S_{a_3} S_{a_4} S_{a_5} \\
&\quad + \sum_{\text{inv perm}} S_{a_1 \wedge a_2} S_{a_3 \wedge a_4} S_{a_5} + 2 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3} S_{a_4} S_{a_5} \\
&\quad + 2 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3} S_{a_4 \wedge a_5} + 6 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4} S_{a_5} \\
&\quad + 24 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5} \\
\sum_{\mathrm{perm}} S_{a_1,a_2,a_3,a_4,a_5,a_6} &= S_{a_1} S_{a_2} S_{a_3} S_{a_4} S_{a_5} S_{a_6} + \sum_{\text{inv perm}} S_{a_1 \wedge a_2} S_{a_3} S_{a_4} S_{a_5} S_{a_6} \\
&\quad + \sum_{\text{inv perm}} S_{a_1 \wedge a_2} S_{a_3 \wedge a_4} S_{a_5} S_{a_6} + \sum_{\text{inv perm}} S_{a_1 \wedge a_2} S_{a_3 \wedge a_4} S_{a_5 \wedge a_6} \\
&\quad + 2 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3} S_{a_4} S_{a_5} S_{a_6} + 2 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3} S_{a_4 \wedge a_5} S_{a_6} \\
&\quad + 4 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3} S_{a_4 \wedge a_5 \wedge a_6} + 6 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4} S_{a_5} S_{a_6} \\
&\quad + 6 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4} S_{a_5 \wedge a_6} + 24 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5} S_{a_6} \\
&\quad + 120 \sum_{\text{inv perm}} S_{a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6}.
\end{aligned}
$$

3 FOLD SUMS :

$$\sum_{\text{perm}} S_{\ell,m,n} = S_\ell S_m S_n + \sum_{\text{perm}} S_\ell \, S_{m\wedge} S_n + 2 S_{\ell\wedge m\wedge n}$$

$N \to \infty$

- SITA RAMACHANDRA RAO, SUBBARAO 1984.

$$\begin{aligned}
S_{-1,-1,-1} &= \frac{1}{6}S_{-1}^3 + \frac{1}{2}S_{-1}S_2 + \frac{1}{3}S_{-3} \\
&= \frac{1}{3}\left[S_{-1,-1}S_{-1} + S_{-1}S_2 + S_{-3}\right] \\
S_{-1,-1,1} + S_{-1,1,-1} + S_{1,-1,-1} &= \frac{1}{2}\left[S_{-1}^2 + S_2\right]S_1 + S_{-1}S_{-2} + S_3 \\
&= S_{-1,-1}S_1 + S_{-2}S_{-1} + S_3 \\
S_{-1,1,1} + S_{1,-1,1} + S_{1,1,-1} &= \frac{1}{2}\left[S_1^2 + S_2\right]S_{-1} + S_1S_{-2} + S_{-3} \\
&= S_{1,1}S_{-1} + S_{-2}S_1 + S_{-3} \\
S_{1,1,1} &= \frac{1}{6}S_1^3 + \frac{1}{2}S_1S_2 + \frac{1}{3}S_3 \\
&= \frac{1}{3}\left[S_{+1,+1}S_{+1} + S_{+1}S_2 + S_{+3}\right] \\
S_{2,1,1} + S_{1,2,1} + S_{1,1,2} &= \frac{1}{2}\left[S_1^2 + S_2\right]S_2 + S_1S_3 + S_4 \\
&= S_{1,1}S_2 + S_1S_3 + S_4 \\
S_{2,-1,1} + S_{2,1,-1} + S_{1,2,-1} & \\
+S_{1,-1,2} + S_{-1,2,1} + S_{-1,1,2} &= S_1S_{-1}S_2 + S_2S_{-2} + S_1S_{-3} + S_{-1}S_3 + 2S_{-4} \\
S_{-2,-1,1} + S_{-2,1,-1} + S_{1,-2,-1} & \\
+S_{1,-1,-2} + S_{-1,-2,1} + S_{-1,1,-2} &= S_1S_{-1}S_{-2} + S_{-2}^2 + S_1S_3 + S_{-1}S_{-3} + 2S_4 \\
S_{2,-1,-1} + S_{-1,2,-1} + S_{-1,-1,2} &= \frac{1}{2}\left[S_{-1}^2 + S_2\right]S_2 + S_{-1}S_{-3} + S_4 \\
&= S_{-1,-1}S_2 + S_{-1}S_{-3} + S_4 \\
S_{-2,1,1} + S_{1,-2,1} + S_{1,1,-2} &= \frac{1}{2}\left[S_1^2 + S_2\right]S_{-2} + S_{-1}S_{-3} + S_{-4} \\
&= S_{1,1}S_{-2} + S_{-1}S_{-3} + S_{-4} \\
S_{-2,-1,-1} + S_{-1,-2,-1} + S_{-1,-1,-2} &= \frac{1}{2}\left[S_{-1}^2 + S_2\right]S_{-2} + S_{-1}S_3 + S_{-4} \\
&= S_{-1,-1}S_{-2} + S_{-1}S_3 + S_{-4}
\end{aligned}$$

5.2 Partial Permutations and Triple Harmonic Sums

$$T_{a,b,c}(N) = \sum_{k=1}^{N} \frac{1}{k^a} \left[\sum_{l=1}^{k} \frac{1}{l^b}\right] \left[\sum_{m=1}^{k} \frac{1}{m^b}\right]$$

Example: Borwein & Girgensohn 1996.

$$\begin{aligned}
T &= S_{a,b,c} + S_{a,c,b} - S_{a\wedge b,c} - S_{a\wedge c,b} - S_{a,b\wedge c} + S_{a\wedge b\wedge c} \\
T &= S_c S_{a,b} - S_{c,a,b} + S_{c,a\wedge b} - S_c S_{a\wedge b} \\
T &= S_b S_{a,c} - S_{b,a,c} + S_{b,a\wedge c} - S_b S_{a\wedge c} \\
T &= S_{b,c,a} + S_{c,b,a} - S_{b\wedge c,a} - S_c S_{b,a} + S_b S_{a,c} - S_b S_{a\wedge c}
\end{aligned}$$

$$S_{-1,-1,1} = \frac{1}{2}\left(-S_{-1,1,-1} + S_{-1}S_{-1,1} + S_{-1,-2} + S_{2,1}\right) \tag{173}$$

$$S_{1,-1,-1} = \frac{1}{2}\left(-S_{-1,1,-1} + S_{-1}S_{1,-1} - S_{2,1} - S_{-1,-2} + S_1 S_2 + S_{-1}S_{-2} + 2S_3\right) \tag{174}$$

$$S_{-1,1,1} = S_{1,1,-1} - S_1 S_{1,-1} + S_{-2,1} + S_{-1,2} + \frac{1}{2}\left(S_1^2 S_{-1} - S_{-1}S_2\right) - S_{-3} \tag{175}$$

$$S_{1,-1,1} = -2S_{1,1,-1} + S_1 S_{1,-1} - S_{-2,1} - S_{-1,2} + S_1 S_{-2} + S_{-1}S_2 + 2S_{-3}\,. \tag{176}$$

The relations for the threefold level–4 sums are :

$$S_{1,2,1} = -2S_{2,1,1} + S_{3,1} + S_1 S_{2,1} + S_{2,2} \tag{177}$$

$$S_{1,1,2} = S_{2,1,1} + \frac{1}{2}\left[S_1\left(S_{1,2} - S_{2,1}\right) + S_{1,3} - S_{3,1}\right] \tag{178}$$

$$S_{1,-2,1} = -2S_{-2,1,1} + S_{-3,1} + S_1 S_{-2,1} + S_{-2,2} \tag{179}$$

$$S_{1,1,-2} = S_{-2,1,1} + S_{-2}S_2 - S_{-2,2} - S_{-2}S_{1,1} + S_1 S_{1,-2} + S_{1,-3} - S_1 S_{-3} \tag{180}$$

$$S_{-1,-1,2} = \frac{1}{2}\left(S_{-1}S_{-1,2} + S_{-1,-3} + S_{2,2} - S_{-1,2,-1}\right) \tag{181}$$

$$S_{2,-1,-1} = \frac{1}{2}\left(-S_{-1,2,-1} + S_{-3,-1} + S_{-1}S_{2,-1} + S_{2,2}\right) \tag{182}$$

$$S_{-1,-1,-2} = \frac{1}{2}\left(-S_{-1,-2,-1} + S_{2,-2} + S_{-1}S_{-1,-2} + S_{-1,3}\right) \tag{183}$$

F1 reg.

$$S_{-2,-1,-1} = \frac{1}{2}\left(-S_{-1,-2,-1} + S_{-2,2} + S_{3,-1} + S_{-1}S_{-2,-1}\right) \tag{184}$$

$$\begin{aligned}
S_{-2,-1,1} = &-S_{1,-2,-1} - S_{-2,1,-1} + S_{3,1} + S_{-3,-1} + S_4 \\
&+ S_1 S_{-2,-1} + S_{1,3} - S_1 S_3 + S_{-2,-2}
\end{aligned} \tag{185}$$

$$S_{1,-1,-2} = -S_{-1,-2,1} - S_{-1,1,-2} + S_{-1,-3} + S_{-2,-2} + S_1 S_{-1,-2} \tag{186}$$

$$S_{1,-2,-1} = S_{-1,-2,1} - S_{-1}S_{-2,1} - S_{-1,-3} + S_{-1}S_{-3} + S_1 S_{-2,-1} + S_{1,3} - S_1 S_3 \tag{187}$$

$$\begin{aligned}
S_{2,-1,1} = &-S_{1,2,-1} - S_{2,1,-1} + S_{3,-1} + S_2\left(S_{-1,1} + S_{1,-1}\right) \\
&+ S_{2,-2} - S_2 S_{-2} - S_1 S_{-1,2} + S_1 S_{-3}
\end{aligned} \tag{188}$$

$$S_{1,-1,2} = -S_{1,2,-1} - S_{2,1,-1} + S_{1,-3} + S_{3,-1} + S_2 S_{1,-1} \tag{189}$$

$$\begin{aligned}
S_{-1,1,2} = &\; S_{1,2,-1} + S_{2,1,-1} - S_{-1,2,1} + S_{-2,2} + S_{-3,1} \\
&- S_{-4} + S_{-1,3} - S_{3,-1} - S_2 S_{1,-1} + S_1 S_{-1,2} - S_1 S_{-3}
\end{aligned} \tag{190}$$

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4-FOLD SUMS:

$$\sum_{\text{perm}} S_{klmn} = S_k S_l S_m S_n + \sum S_k S_l S_{m\wedge n} + \sum S_{k\wedge l} S_{m\wedge n} + 2 \sum S_k S_{l\wedge m\wedge n} + 6\, S_{k\wedge l\wedge m\wedge n}$$

(NEW).

$$S_{-1,-1,-1,-1} = \frac{1}{4}S_4 + \frac{1}{8}S_2^2 + \frac{1}{3}S_{-3}S_{-1} + \frac{1}{4}S_2 S_{-1}^2 + \frac{1}{24}S_{-1}^4$$
$$= \frac{1}{4}\left[S_{-1,-1,-1}S_{-1} + S_{-1,-1}S_2 + S_{-1}S_{-3} + S_4\right]$$

$$S_{1,-1,-1,-1} + S_{-1,1,-1,-1} + S_{-1,-1,1,-1} + S_{-1,-1,-1,1} = \frac{1}{6}S_1^3 S_{-1} + \frac{1}{3}S_{-1}S_3 + S_1 S_{-3} + S_{-4} + \frac{1}{2}\left[S_1^2 S_{-2} + S_1 S_{-1} S_2 + S_2 S_{-2}\right]$$
$$= S_{1,1,1}S_{-1} + S_{1,1}S_{-2} + S_1 S_{-3} + S_{-4}$$

$$S_{-1,-1,1,1} + S_{-1,1,-1,1} + S_{-1,1,1,-1} + S_{1,1,-1,-1} + S_{1,-1,1,-1} + S_{1,-1,-1,1} = S_{-1}^2 S_1^2 + S_{-1}^2 S_2 + 4S_{-1}S_1S_{-2} + S_2S_1^2 + S_2^2 + 2S_{-2}^2 + 4S_{-1}S_{-3} + 4S_1S_3 + S_4$$
$$= 4\left[S_{1,1}S_{-1,-1} + S_{-2,-2} + S_{-1,-3} + S_{-3,-1} + S_{1,3} + S_{3,1} - S_4\right]$$

$$S_{-1,1,1,1} + S_{1,-1,1,1} + S_{1,1,-1,1} + S_{1,1,1,-1} = \frac{1}{6}S_{-1}^3 S_1 + \frac{1}{3}S_{-3}S_1 + S_{-1}S_3 + S_{-4} + \frac{1}{2}\left[S_{-1}S_1S_2 + S_{-1}^2 S_{-2} + S_{-2}S_2\right]$$
$$= S_{-1,-1,-1}S_1 + S_{-1,-1}S_{-2} + S_{-1}S_3 + S_{-4}$$

$$S_{1,1,1,1} = \frac{1}{4}S_4 + \frac{1}{8}S_2^2 + \frac{1}{3}S_3S_1 + \frac{1}{4}S_2S_1^2 + \frac{1}{24}S_1^4$$

PERMUTAT. SYMMETRY

MULTIPLE ⊗ CONVOLUTIONS ↔ PRODUCTS

$$\frac{(-1)^n}{\Gamma(n)}\left(\frac{\log^{n-1}(x)}{x-1}\right)_+ = \mathbf{M}^{-1}\left[S_n(N)\right](x)$$

$$\frac{(-1)^N}{x+1} - \log(2)\delta(1-x) = \mathbf{M}^{-1}\left[S_{-1}(N)\right](x),$$

$$\frac{(-1)^{N+n-1}}{\Gamma(n)}\frac{\log^{n-1}(x)}{x+1} - \left(1 - \frac{1}{2^{n-1}}\right)\zeta(n)\delta(1-x) = \mathbf{M}^{-1}\left[S_{-n}(N)\right](x),$$

THERE ARE MORE RELATIONS, WHICH WE WILL USE.

5. Analytic Continuation

$$S_k(N) = \frac{(-1)^{k+1}}{\Gamma(k)}\psi^{(k-1)}(N+1) + c_k, \quad c_1 = \gamma_E, \quad c_k = \zeta(k), \quad k > 1,$$

$$S_{-k}(N) = (-1)^N \frac{(-1)^{k+1}}{\Gamma(k)}\beta^{(k-1)}(N+1) + e_k, \quad e_1 = \log(2),$$

$$e_k = \left(1 - \frac{1}{2^{k-1}}\right)\zeta(k),$$

$$\beta(N) = \frac{1}{2}\left[\psi\left(\frac{1+N}{2}\right) - \psi\left(\frac{N}{2}\right)\right].$$

$$\mathbf{M}\left[\log^k(x) f(x)\right](N) = \frac{\partial^N}{\partial N^k}\mathbf{M}\left[f(x)\right](N).$$

$$\mathbf{M}\left[\frac{\log^k(1+x)}{1+x}\right](N) = \frac{1}{k+1}\log^k(2) - \frac{N}{k+1}\mathbf{M}\left[\log^{k+1}(1+x)\right](N-1)$$

$$\int_0^1 dx \frac{x^N}{x+1} f(x) = \log(2) f(1) - \int_0^1 dx \log(1+x) N x^{N-1} f(x) + \int_0^1 dx x^N \log(1+x) f'(x),$$

if $f(x)$ is differentiable in $[0,1]$. The function $\log(1+x)$ can be approximated by

$$\ln(1+x) \simeq \sum_{k=1}^{8} a_k x^k ,$$

which are as accurate as 10^{-8}. The coefficients a_k read

$$a_1 = 0.9999964239 \quad a_2 = -0.4998741238 \quad a_3 = 0.3317990258 \quad a_4 = -0.2407338084$$
$$a_5 = 0.1676540711 \quad a_6 = -0.0953293897 \quad a_7 = 0.0360884937 \quad a_8 = -0.0064535442 ,$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N) = \zeta(2)\log(2) - \sum_{k=1}^{8} a_k \left[\frac{N}{N+k}\zeta(2) + \frac{k}{(N+k)^2} S_1(N+k)\right.$$

$$\mathbf{M}\left[\frac{\log(x)\mathrm{Li}_2(x)}{1+x}\right](N) = \sum_{k=1}^{8} a_k \left\{\frac{k\zeta(2)}{(N+k)^2} - \frac{k}{(N+k)^2}\left[S_2(N+k) - \zeta(2)\right]. - \frac{2k}{(N+k)^3} S_1(N+k)\right\}$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_3(x)}{1+x}\right](N) = \log(2)\zeta(3) - \sum_{k=1}^{8} a_k \left[\frac{N}{N+k}\zeta(3) + \frac{k}{(N+k)^2}\zeta(2) - \frac{k}{(N+k)^3} S_1(N+k)\right]$$

$$\mathbf{M}\left[\frac{\mathrm{S}_{1,2}(x)}{1+x}\right](N) = \log(2)\zeta(3) - \sum_{k=1}^{8} a_k \left\{\frac{N}{N+k}\zeta(3) + \frac{k}{(N+k)^2}\frac{1}{2}\left[S_1^2(N+k) + S_2(N+k)\right]\right\}$$

THE OTHER CASES LOOK SIMILAR,
SOME OF THEM ARE LENGTHLY.

4. Linear representations

80 SUMS
UP TO LEVEL 4

First Order Sums

$$S_{-1}(N) = (-1)^N \mathbf{M}\left[\frac{x^N}{1+x}\right](N) - \log(2) = (-1)^N \beta(N+1) - \log(2) \quad (69)$$

$$S_1(N) = \mathbf{M}\left[\left(\frac{1}{x-1}\right)_+\right](N) = \psi(N+1) + \gamma_E \quad (70)$$

Second Order Sums

$$S_{-2}(N) = (-1)^{N+1}\mathbf{M}\left[\frac{\log x}{1+x}\right](N) - \frac{1}{2}\zeta(2) = (-1)^{N+1}\beta'(N+1) - \frac{1}{2}\zeta(2) \quad (71)$$

$$S_2(N) = -\mathbf{M}\left[\frac{\log x}{x-1}\right](N) + \zeta(2) = -\psi'(N+1) + \zeta(2) \quad (72)$$

$$S_{-1,-1}(N) = -\mathbf{M}\left[\left(\frac{\log(1+x)}{x-1}\right)_+\right](N) + \log(2)\left[S_1(N) - S_{-1}(N)\right] = \frac{1}{2}\left\{\left[\beta(N+1) - (-1)^N \log(2)\right]^2 + \zeta(2) - \psi'(N+1)\right\} \quad (73)$$

$$S_{-1,1}(N) = (-1)^{N+1}\mathbf{M}\left[\frac{\log(1-x)}{1+x}\right](N) - \frac{1}{2}\left[\zeta(2) - \log^2(2)\right] = (-1)^N \mathbf{M}\left[\frac{\log(1+x)}{1+x}\right](N) + S_1(N)S_{-1}(N) + S_{-2}(N) + \left[S_1(N) - S_{-1}(N)\right]\log(2) - \frac{1}{2}\log^2(2) \quad (74)$$

$$S_{1,-1}(N) = (-1)^{N+1}\mathbf{M}\left[\frac{\log(1+x)}{1+x}\right](N) - \left[S_1(N) - S_{-1}(N)\right]\log(2) + \frac{1}{2}\log^2(2) \quad (75)$$

$$S_{1,1}(N) = -\mathbf{M}\left[\left(\frac{\log(1-x)}{x-1}\right)_+\right](N) = \frac{1}{2}\left\{\left[\psi(N+1) + \gamma_E\right]^2 + \zeta(2) - \psi'(N+1)\right\} \quad (76)$$

Third Order Sums

$$S_{-3}(N) = (-1)^N \frac{1}{2} \mathrm{M}\left[\frac{\log^2 x}{1+x}\right](N) - \frac{3}{4}\zeta(3)$$
$$= (-1)^N \frac{1}{2}\beta''(N+1) - \frac{3}{4}\zeta(3) \quad (77)$$

$$S_3(N) = \frac{1}{2}\mathrm{M}\left[\frac{\log^2 x}{x-1}\right](N) + \zeta(3)$$
$$= \frac{1}{2}\psi''(N+1) + \zeta(3) \quad (78)$$

$$S_{-2,-1}(N) = -\mathrm{M}\left[\left(\frac{\mathrm{Li}_2(-x)}{x-1}\right)_+\right](N) + \log(2)\left[S_2(N) - S_{-2}(N)\right] - \frac{\zeta(2)}{2}S_1(N) \quad (79)$$

$$S_{-2,1}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N) + \zeta(2)S_{-1}(N) - \frac{5}{8}\zeta(3) + \zeta(2)\log(2) \quad (80)$$

$$S_{2,-1}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{\mathrm{Li}_2(-x)}{1+x}\right](N) - \log(2)\left[S_2(N) - S_{-2}(N)\right] - \frac{1}{2}\zeta(2)S_{-1}(N)$$
$$+\frac{1}{4}\zeta(3) - \frac{1}{2}\zeta(2)\log(2) \quad (81)$$

$$S_{2,1}(N) = \mathrm{M}\left[\left(\frac{\mathrm{Li}_2(x)}{x-1}\right)_+\right](N) + \zeta(2)S_1(N) \quad (82)$$

$$S_{-1,-2}(N) = \mathrm{M}\left\{\left[\frac{\log(1+x)\log(x) + \mathrm{Li}_2(-x)}{x-1}\right]_+\right\}(N) + \frac{1}{2}\zeta(2)\left[S_1(N) - S_{-1}(N)\right] \quad (83)$$

$$S_{-1,2}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{\mathrm{Li}_2(1-x)}{1+x}\right](N) + \zeta(2)S_{-1}(N) - \zeta(3) + \frac{3}{2}\zeta(2)\log(2) \quad (84)$$

$$S_{1,-2}(N) = (-1)^N\mathrm{M}\left[\frac{\log(x)\log(1+x) + \mathrm{Li}_2(-x)}{1+x}\right](N) - \frac{1}{2}\zeta(2)\left[S_1(N) - S_{-1}(N)\right]$$
$$-\frac{1}{8}\zeta(3) + \frac{1}{2}\zeta(2)\log(2) \quad (85)$$

$$S_{1,2}(N) = -\mathrm{M}\left[\frac{\mathrm{Li}_2(1-x)}{x-1}\right](N) + \zeta(2)S_1(N) - \zeta(3) \quad (86)$$

$$S_{-1,-1,-1}(N) = (-1)^{N+1}\mathrm{M}\left[\frac{\mathrm{Li}_2[(1-x)/2] - \log(2)\log(1-x)}{1+x}\right](N)$$
$$+\log(2)\left[S_{-1,1}(N) - S_{-1,-1}(N)\right]$$
$$+\frac{1}{2}\left[\zeta(2) - \log^2(2)\right]S_{-1}(N) - \frac{1}{4}\zeta(3) + \zeta(2)\log(2) - \frac{2}{3}\log^3(2)$$
$$= \frac{1}{6}\left[(-1)^N\beta(N+1) - \log(2)\right]^3$$
$$-\frac{1}{2}\left[(-1)^N\beta(N+1) - \log(2)\right]\left[\psi'(N+1) - \zeta(2)\right]$$
$$+\frac{1}{3}\left[(-1)^N\frac{1}{2}\beta''(N+1) - \frac{3}{4}\zeta(3)\right] \quad (87)$$

$$S_{-1,-1,1}(N) = \mathbf{M}\left\{\left[\frac{1}{x-1}\left(\log(1-x)\log\left(\frac{1+x}{2}\right)+\mathrm{Li}_2\left(\frac{1-x}{2}\right)\right)\right]_+\right\}(N)$$
$$-\frac{1}{2}\left[\zeta(2)-\log^2(2)\right]S_{-1}(N)$$
$$= \frac{1}{2}\left[-S_{-1,1,-1}(N)+S_{-1}(N)S_{-1,1}(N)+S_{-1,-2}(N)+S_{2,1}(N)\right] \quad (88)$$

$$S_{-1,1,-1}(N) = \frac{1}{2}\mathbf{M}\left\{\left[\frac{\log^2(1+x)}{x-1}\right]_+\right\}(N)+\log(2)\left[S_{-1,-1}(N)-S_{-1,1}(N)\right]$$
$$-\frac{1}{2}\log^2(2)\left[S_1(N)-S_{-1}(N)\right] \quad (89)$$

$$S_{-1,1,1}(N) = \frac{1}{2}(-1)^N\mathbf{M}\left[\frac{\log^2(1-x)}{1+x}\right](N)-\frac{7}{8}\zeta(3)+\frac{1}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)$$
$$= S_{1,1,-1}(N)-S_1(N)S_{1,-1}(N)+S_{-2,1}(N)+S_{-1,2}(N)$$
$$+\frac{1}{2}S_1^2(N)S_{-1}(N)-\frac{1}{2}S_{-1}(N)S_2(N)-S_{-3}(N) \quad (90)$$

$$S_{1,-1,-1}(N) = \mathbf{M}\left\{\left[\frac{1}{x-1}\left(\log\left(\frac{1-x}{2}\right)\log(1+x)+\mathrm{Li}_2\left(\frac{1+x}{2}\right)\right)\right]_+\right\}(N)$$
$$+\log(2)\left[S_{1,1}(N)-S_{1,-1}(N)\right]-\frac{1}{2}\left[\zeta(2)-\log^2(2)\right]S_1(N)$$
$$= \frac{1}{2}\Big[-S_{-1,1,-1}(N)+S_{-1}(N)S_{1,-1}(N)-S_{2,1}(N)-S_{-1,-2}(N)$$
$$+S_1(N)S_2(N)+S_{-1}(N)S_{-2}(N)+2S_3(N)\Big] \quad (91)$$

$$S_{1,-1,1}(N) = (-1)^N\mathbf{M}\left\{\frac{1}{1+x}\left[\log\left(\frac{1+x}{2}\right)\log(1-x)+\mathrm{Li}_2\left(\frac{1-x}{2}\right)\right]\right\}(N)$$
$$-\frac{1}{2}\left[\zeta(2)-\log^2(2)\right]S_1(N)+\frac{1}{8}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)+\frac{1}{3}\log^3(2)$$

$$= -2S_{1,1,-1}(N)+S_1(N)S_{1,-1}(N)-S_{-2,1}(N)-S_{-1,2}(N)$$
$$+S_1(N)S_{-2}(N)+S_{-1}(N)S_2(N)+2S_{-3}(N) \quad (92)$$

$$S_{1,1,-1}(N) = (-1)^N\frac{1}{2}\mathbf{M}\left[\frac{\log^2(1+x)}{1+x}\right](N)+\log(2)\left[S_{1,-1}(N)-S_{1,1}(N)\right]$$
$$+\frac{1}{2}\log^2(2)\left[S_1(N)-S_{-1}(N)\right]-\frac{1}{6}\log^3(2) \quad (93)$$

$$S_{1,1,1}(N) = \frac{1}{2}\mathbf{M}\left[\left(\frac{\log^2(1-x)}{x-1}\right)_+\right](N)$$
$$= \frac{1}{6}\left[\psi(N+1)+\gamma_E\right]^3-\frac{1}{2}\left[\psi(N+1)+\gamma_E\right]\left[\psi'(N+1)-\zeta(2)\right]$$
$$= +\frac{1}{3}\left[\frac{1}{2}\psi''(N+1)+\zeta(3)\right] \quad (94)$$

Fourth Order Sums

$$\begin{aligned} S_{-4}(N) &= (-1)^{N+1}\frac{1}{6}\mathbf{M}\left[\frac{\log^3 x}{1+x}\right](N) - \frac{7}{20}\zeta^2(2) \\ &= (-1)^{N+1}\frac{1}{6}\beta^{(3)}(N+1) - \frac{7}{20}\zeta^2(2) \end{aligned} \tag{95}$$

$$\begin{aligned} S_4(N) &= -\frac{1}{6}\mathbf{M}\left[\frac{\log^3 x}{x-1}\right](N) + \frac{2}{5}\zeta^2(2) \\ &= -\frac{1}{6}\psi^{(3)}(N+1) + \frac{2}{5}\zeta^2(2) \end{aligned} \tag{96}$$

$$\begin{aligned} S_{-2,-2}(N) &= \mathbf{M}\left\{\left[\frac{1}{x-1}\left(\log(x)\mathrm{Li}_2(-x) - 2\mathrm{Li}_3(-x)\right)\right]_+\right\}(N) \\ &\quad +\frac{1}{2}\zeta(2)\left[S_2(N) - S_{-2}(N)\right] - \frac{3}{2}S_1(N) \\ &= \frac{1}{2}\left\{\left[\beta'(N+1) + (-1)^N\frac{1}{2}\zeta(2)\right]^2 - \frac{1}{6}\psi^{(3)}(N+1) + \frac{2}{5}\zeta(2)^2\right\} \end{aligned} \tag{97}$$

$$\begin{aligned} S_{-2,2}(N) &= (-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[2\mathrm{Li}_3(x) - \log(x)\left(\mathrm{Li}_2(x) + \zeta(2)\right)\right]\right\}(N) \\ &\quad +\zeta(2)S_{-2}(N) + 2\zeta(3)S_{-1}(N) \\ &\quad +\frac{71}{40}\zeta(2)^2 - 4\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{3}{2}\zeta(3)\log(2) + \zeta(2)\log^2(2) - \frac{1}{6}\log^4(2) \end{aligned} \tag{98}$$

$$\begin{aligned} S_{2,-2}(N) &= (-1)^N\mathbf{M}\left\{\frac{1}{1+x}\left[\mathrm{Li}_2(-x)\log(x) - 2\mathrm{Li}_3(-x)\right]\right\}(N) - \frac{3}{2}\zeta(3)S_{-1}(N) \\ &\quad +\frac{1}{2}\zeta(2)\left[S_{-2}(N) - S_2(N)\right] - \frac{11}{8}\zeta(2)^2 + 4\mathrm{Li}_4\left(\frac{1}{2}\right) \\ &\quad +2\zeta(3)\log(2) - \zeta(2)\log^2(2) + \frac{1}{6}\log^4(2) \end{aligned} \tag{99}$$

6. A Dictionary of Mellin Transforms

ABOUT 80 entries.

No.	$f(z)$	$M[f](N)$
1	$\delta(1-z)$	1
2	z^r	$\frac{1}{N+r}$
3	$\left(\frac{1}{1-z}\right)_+$	$-S_1(N-1)$
4	$\frac{1}{1+z}$	$(-1)^{N-1}[\log(2)-S_1(N-1)]$ $+\frac{1+(-1)^{N-1}}{2}S_1\left(\frac{N-1}{2}\right)-\frac{1-(-1)^{N-1}}{2}S_1\left(\frac{N-2}{2}\right)$
5	$z^r\log^n(z)$	$\frac{(-1)^n}{(N+r)^{n+1}}\Gamma(n+1)$
6	$z^r\log(1-z)$	$-\frac{S_1(N+r)}{N+r}$
7	$z^r\log^2(1-z)$	$\frac{S_1^2(N+r)+S_2(N+r)}{N+r}$
8	$z^r\log^3(1-z)$	$-\frac{S_1^3(N+r)+3S_1(N+r)S_2(N+r)+2S_3(N+r)}{N+r}$
9	$\left[\frac{\log(1-z)}{1-z}\right]_+$	$\frac{1}{2}S_1^2(N-1)+\frac{1}{2}S_2(N-1)$
10	$\left[\frac{\log^2(1-z)}{1-z}\right]_+$	$-\left[\frac{1}{3}S_1^3(N-1)+S_1(N-1)S_2(N-1)+\frac{2}{3}S_3(N-1)\right]$
11	$\left[\frac{\log^3(1-z)}{1-z}\right]_+$	$\frac{1}{4}S_1^4(N-1)+\frac{3}{2}S_1^2(N-1)S_2(N-1)$ $+\frac{3}{4}S_2^2(N-1)+2S_1(N-1)S_3(N-1)$ $+\frac{3}{2}S_4(N-1)$
12	$\frac{\log^n(z)}{1-z}$	$(-1)^{n+1}\Gamma(n+1)\left[S_{n+1}(N-1)-\zeta(n+1)\right]$

No.	$f(z)$	$M[f](N)$
13	$z^r \log(1+z)$	$\frac{(-1)^{N+r-1}}{N+r}\Big[-S_1(N+r)+\frac{1+(-1)^{N+r-1}}{2}$ $\times S_1\left(\frac{N+r-1}{2}\right)+\frac{1-(-1)^{N+r-1}}{2}S_1\left(\frac{N+r}{2}\right)\Big]$ $+\left[1+(-1)^{N+r+1}\right]\frac{\log(2)}{N+r}$
14	$z^r \log^2(1+z)$	$\frac{(-1)^{N+r}}{N+r}\Big[-2S_{-1,1}(N+r)+2S_1(N+r)S_{-1}(N+r)$ $+2S_1(N+r)\log 2-2S_{-1}(N+r)\log 2+2S_{-2}(N+r)-\log^2 2\Big]$ $+\frac{1}{N+r}\log^2 2$
15	$\frac{\log^n(z)}{1+z}$	$(-1)^n\Gamma(n+1)\Big\{(-1)^N\left[S_{n+1}(N-1)-\zeta(n+1)\right]$ $+\frac{1+(-1)^{N-1}}{2^{n+1}}\left[S_{n+1}\left(\frac{N-1}{2}\right)-\zeta(n+1)\right]$ $-\frac{1-(-1)^{N-1}}{2^{n+1}}\left[S_{n+1}\left(\frac{N-2}{2}\right)-\zeta(n+1)\right]\Big\}$
16	$\frac{\log(1-z)}{1+z}$	$(-1)^N\left[S_{-1,1}(N-1)+\frac{\zeta(2)}{2}-\frac{\log^2 2}{2}\right]$

No.	$f(z)$	$M[f](N)$
20	$\frac{1}{1-z}\log(z)\log(1-z)$	$\zeta(3)+\zeta(2)S_1(N-1)-S_1(N-1)S_2(N-1)-S_3(N-1)$
21	$z^r\log(z)\log(1+z)$	$(-1)^{N+r}\frac{\zeta(2)}{2(N+r)}-\frac{1+(-1)^{N+r-1}}{(N+r)^2}\log(2)+\frac{(-1)^{N+r}}{N+r}\times$ $\times\Big[-S_2(N+r)+\frac{1+(-1)^{N+r-1}}{4}S_2\Big(\frac{N+r-1}{2}\Big)+$ $+\frac{1-(-1)^{N+r-1}}{4}S_2\Big(\frac{N+r}{2}\Big)\Big]+$ $+\frac{(-1)^{N+r}}{(N+r)^2}\Big[-S_1(N+r)+\frac{1+(-1)^{N+r-1}}{2}S_1\Big(\frac{N+r-1}{2}\Big)+$ $+\frac{1-(-1)^{N+r-1}}{2}S_1\Big(\frac{N+r}{2}\Big)\Big]$
22	$\frac{1}{1+z}\log(z)\log(1+z)$	$(-1)^{N-1}\Big[\frac{\zeta(2)}{2}S_1(N-1)+2S_2(N-1)\log 2$ $-2S_1(N-1)S_2(N-1)-S_{-2,1}(N-1)-S_{-1,2}(N-1)$ $-2S_3(N-1)-\frac{\zeta(3)}{8}\Big]$ $+\frac{1+(-1)^{N-1}}{4}\Big[S_1(N-1)S_2\Big(\frac{N-1}{2}\Big)$

No.	$f(z)$	$M[f](N)$
24	$\frac{1}{1-z}\log(z)\log^2(1-z)$	$S_2^2(N-1)-\zeta(2)S_2(N-1)+2S_4(N-1)$ $+S_1^2(N-1)[S_2(N-1)-\zeta(2)]$ $+2S_1(N-1)[S_3(N-1)-\zeta(3)]-2\zeta(4)$
25	$\frac{1}{1-z}\log^2(z)\log(1-z)$	$-2\zeta(3)S_1(N-1)+2S_1(N-1)S_3(N-1)-2\zeta(2)S_2(N-1)$ $+S_2^2(N-1)+3S_4(N-1)-\frac{\zeta(4)}{2}$
26	$\frac{1}{1+z}\log^2(z)\log(1-z)$	$2(-1)^N\Big[S_{-3,1}(N-1)+S_{-1,3}(N-1)+S_{-2,2}(N-1)$ $-\zeta(2)S_{-2}(N-1)-\zeta(3)S_{-1}(N-1)$ $+2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{1}{5}\zeta^2(2)-\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{12}\log^4 2\Big]$
27	$\frac{1}{1+z}\log(z)\log^2(1-z)$	$2(-1)^N\Big[S_{-2,1,1}(N-1)+S_{-1,1,2}(N-1)+S_{-1,2,1}(N-1)$ $-\zeta(2)S_{-1,1}(N-1)-\zeta(3)S_{-1}(N-1)$ $+3\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{11}{20}\zeta^2(2)+\frac{1}{8}\log^4 2\Big]$

No.	$f(z)$	$M[f](N)$
30	$\frac{1}{1+z}\log z\log^2(1+z)$	$2(-1)^N\Big[S_{1,1,-2}(N-1)+S_{1,2,-1}(N-1)+S_{2,1,-1}(N-1)$ $-S_{1,-2}(N-1)\log 2-S_{2,-1}(N-1)\log 2+S_1(N-1)S_2(N-1)\log 2$ $+\frac{1}{4}\zeta(2)S_1^2(N-1)+S_3(N-1)\log 2+\frac{1}{4}\zeta(2)S_2(N-1)$ $-\frac{1}{2}S_2(N-1)\log^2 2+\frac{1}{2}S_{-2}(N-1)\log^2 2-\frac{1}{8}\zeta(3)S_1(N-1)$ $-\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{2}{5}\zeta^2(2)-\frac{7}{8}\zeta(3)\log 2$ $+\frac{1}{4}\zeta(2)\log^2 2-\frac{1}{24}\log^4 2\Big]$
31	$z^r\log z\log^2(1+z)$	$-\frac{2}{N+r}\Big\{(-1)^{N+r}\Big[\frac{\zeta(2)}{2}S_1(N+r)+2S_2(N+r)\log 2$ $-2S_1(N+r)S_2(N+r)-S_{-2,1}(N+r)-S_{-1,2}(N+r)$ $-2S_3(N+r)-\frac{\zeta(3)}{8}\Big]$ $+\frac{1+(-1)^{N+r}}{4}\Big[S_1(N+r)S_2\left(\frac{N+r}{2}\right)+2S_1\left(\frac{N+r}{2}\right)S_2(N+r)$ [illegible]

No.	$f(z)$	$M[f](N)$
32	$z^r \log z \log(1+z)\log(1-z)$	$-\frac{(-1)^N}{N}\Big[S_{-2,1}(N)+S_{-1,2}(N)-\zeta(2)S_{-1}(N)$ $+\frac{13}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\Big]$ $-\frac{1}{N}\Big[S_{-1}(N)S_{-2}(N)+S_3(N)-S_2(N)\log 2$ $+S_{-2}(N)\log 2+\frac{1}{2}\zeta(2)S_{-1}(N)-\zeta(3)+\frac{3}{2}\zeta(2)\log 2\Big]$ $-\frac{(-1)^N}{N^2}\Big[S_{-1,1}(N)+\frac{1}{2}\zeta(2)-\frac{1}{2}\log^2 2\Big]$ $-\frac{1}{N^2}\Big[\frac{1}{2}S_2(N)+\frac{1}{2}S_{-1}^2(N)+S_{-1}(N)\log 2$ $-S_1(N)\log 2-\frac{1}{2}\zeta(2)+\frac{1}{2}\log^2 2\Big]$
33	$\frac{\log z}{1+z}\log(1+z)\log(1-z)$	$(-1)^N\Big\{S_{1,-2,1}(N-1)+S_{1,-1,2}(N-1)$ $+S_{-1,-1,-2}(N-1)+S_{-1,-2,-1}(N-1)$ $+S_{2,-1,1}(N-1)+S_{-2,-1,-1}(N-1)$ $-\zeta(2)S_{1,-1}(N-1)-S_{-1,2}(N-1)\log 2$ $-S_{-2,1}(N-1)\log 2+S_{-1}(N-1)S_{-1}(N-1)\log 2$

No.	$f(z)$	$M[f](N)$
36	$\frac{\mathrm{Li}_2(z)}{1+z}$	$(-1)^{N-1}\Big[\zeta(2)S_{-1}(N-1) - S_{-2,1}(N-1)$ $-\frac{5}{8}\zeta(3) + \zeta(2)\log 2\Big]$
37	$\frac{\mathrm{Li}_2(z)}{1-z}\log z$	$2\zeta(2)S_2(N-1) - \frac{1}{2}S_2^2(N-1) - \frac{1}{2}S_4(N-1)$ $-2S_{3,1}(N-1) - \frac{3}{10}\zeta^2(2)$
38	$z^r\mathrm{Li}_2(z)\log z$	$\frac{1}{(N+r)^2}\Big[-2\zeta(2) + \frac{2S_1(N+r)}{N+r} + S_2(N+r)\Big]$
39	$z^r\mathrm{Li}_2(z)\log(1-z)$	$\frac{1}{N+r}\Big\{-2\zeta(3) - \zeta(2)S_1(N+r)+$ $+\frac{1}{N+r}\Big[S_1^2(N+r) + S_2(N+r)\Big] + S_{2,1}(N+r)\Big\}$
40	$\left[\frac{\log(1-z)}{1-z}\right]_+\mathrm{Li}_2(z)$	$-2S_{2,1,1}(N-1) - S_{1,2,1}(N-1) + \frac{1}{2}\zeta(2)S_1^2(N-1)$ $+\frac{1}{2}\zeta(2)S_2(N-1) + 2\zeta(3)S_1(N-1) + \frac{6}{5}\zeta^2(2)$
41	$z^r\mathrm{Li}_2(-z)$	$\frac{1}{N+r}\Big\{-\frac{\zeta(2)}{2} + \frac{1+(-1)^{N+r-1}}{N+r}\log(2) + \frac{(-1)^{N+r-1}}{N+r}$

No.	$f(z)$	$M[f](N)$
43	$z^r \mathrm{Li}_2(-z)\log z$	$\frac{(-1)^{N+r-1}}{(N+r)^2}\left[\frac{2S_1(N+r)}{N+r}+S_2(N+r)\right]$ $-\frac{1+(-1)^{N+r-1}}{2(N+r)^2}\left[\frac{2}{N+r}S_1\left(\frac{N+r-1}{2}\right)+\frac{1}{2}S_2\left(\frac{N+r-1}{2}\right)\right.$ $\left.+\frac{4\log 2}{N+r}\right]$ $+\frac{1-(-1)^{N+r-1}}{2(N+r)^2}\left[\frac{2}{N+r}S_1\left(\frac{N+r}{2}\right)+\frac{1}{2}S_2\left(\frac{N+r}{2}\right)+\zeta(2)\right]$
44	$\frac{\mathrm{Li}_2(-z)}{1-z}\log z$	$-2S_{-3,-1}(N-1)-\frac{1}{2}S_{-2}^2(N-1)-\frac{1}{2}S_4(N-1)$ $+2S_3(N-1)\log 2-2S_{-3}(N-1)\log 2-\frac{1}{2}\zeta(2)S_2(N-1)$ $-\frac{1}{2}\zeta(2)S_{-2}(N-1)-4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{71}{40}\zeta^2(2)$ $-\frac{7}{2}\zeta(3)\log 2+\zeta(2)\log^2 2-\frac{1}{6}\log^4 2$
45	$\frac{\mathrm{Li}_2(-z)}{1+z}\log z$	$(-1)^{N-1}\left[2S_{3,-1}(N-1)+S_{2,-2}(N-1)-2S_{-3}(N-1)\log 2\right.$ $+2S_3(N-1)\log 2+\frac{1}{2}\zeta(2)S_2(N-1)+\frac{1}{2}\zeta(2)S_{-2}(N-1)$ $-4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{13}{8}\zeta^2(2)-\frac{7}{2}\zeta(3)\log 2$ $\left.+\zeta(2)\log^2 2-\frac{1}{6}\log^4 2\right]$

No.	$f(z)$	$M[f](N)$
46	$z^r \mathrm{Li}_2(-z)\log(1+z)$	$\frac{(-1)^{N+r}}{N+r}\Big\{\Big[S_2(N+r)+\frac{\zeta(2)}{2}\Big]\Big[\frac{1+(-1)^{N+r}}{2}S_1\left(\frac{N+r}{2}\right)$ $+\frac{1-(-1)^{N+r}}{2}S_1\left(\frac{N+r-1}{2}\right)-S_1(N+r)\Big]$ $-\Big[\frac{1+(-1)^{N+r}}{4}S_2\left(\frac{N+r}{2}\right)+\frac{1-(-1)^{N+r}}{4}S_2\left(\frac{N+r-1}{2}\right)$ $-2S_2(N+r)\Big]\log 2$ $+\frac{1+(-1)^{N+r}}{8}S_3\left(\frac{N+r}{2}\right)+\frac{1-(-1)^{N+r}}{8}S_3\left(\frac{N+r-1}{2}\right)$ $-S_3(N+r)-S_{-1,2}(N+r)+\frac{1-(-1)^{N+r}}{2}\zeta(2)\log 2-\frac{\zeta(3)}{4}\Big\}$ $+\frac{2(-1)^{N+r}}{(N+r)^2}\Big\{-S_{-1,1}(N+r)-S_1^2(N+r)-S_2(N+r)$ $+2S_1(N+r)\log 2$ $+\frac{1+(-1)^{N+r}}{2}\Big[S_1(N+r)S_1\left(\frac{N+r}{2}\right)-S_1\left(\frac{N+r}{2}\right)\log 2$ $+\frac{1}{2}S_2\left(\frac{N+r}{2}\right)\Big]$ $+\frac{1-(-1)^{N+r}}{2}\Big[S_1(N+r)S_1\left(\frac{N+r-1}{2}\right)$ $-S_1\left(\frac{N+r-1}{2}\right)\log 2$ $+\frac{1}{2}S_2\left(\frac{N+r-1}{2}\right)-\log^2 2\Big]\Big\}$
48	$\frac{\mathrm{Li}_2(-z)}{1+z}\log(1-z)$	$(-1)^{N-1}\Big\{S_{2,-1,1}(N-1)+S_{-2,-1,-1}(N-1)$ $+S_{-1,-2,-1}(N-1)$ $-S_{-2,1}(N-1)\log 2-S_{-1,2}(N-1)\log 2+\frac{1}{2}\zeta(2)S_{-1,1}(N-1)$ $+S_{-1}(N-1)S_{-2}(N-1)\log 2+S_3(N-1)\log 2$ $+\frac{1}{2}\Big[\zeta(2)-\log^2(2)\Big]S_2(N-1)-\frac{1}{2}\Big[\zeta(2)-\log^2(2)\Big]S_{-2}(N-1)$ $+\frac{5}{8}\zeta(3)S_{-1}(N-1)-4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{3}{2}\zeta^2(2)$ $-\frac{21}{8}\zeta(3)\log 2+\frac{3}{4}\zeta(2)\log^2 2-\frac{1}{6}\log^4 2\Big\}$

No.	$f(z)$	$M[f](N)$
49	$\frac{\mathrm{Li}_2(-z)}{1+z}\log(1+z)$	$(-1)^{N-1}\Big\{S_{1,2,-1}(N-1)+2S_{2,1,-1}(N-1)$ $+\Big[S_{2,1}(N-1)-S_{1,-2}(N-1)-2S_{2,-1}(N-1)$ $+S_1(N-1)S_2(N-1)+S_3(N-1)-\frac{1}{2}\zeta(2)S_{-1}(N-1)\Big]\log 2$ $+\frac{1}{2}\zeta(2)S_{1,-1}(N-1)-\Big[S_2(N-1)-S_{-2}(N-1)\Big]\log^2 2$ $-\Big[\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log 2\Big]S_1(N-1)$ $-3\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{6}{5}\zeta^2(2)-\frac{21}{8}\zeta(3)\log 2$ $+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{8}\log^4 2\Big\}$
50	$\mathrm{Li}_2(1-z)$	$-\frac{1}{N}\left[S_2(N)-\zeta(2)\right]$
51	$\mathrm{Li}_2(1-z)\log z$	$\frac{1}{N^2}[S_2(N)-\zeta(2)]+\frac{2}{N}[S_3(N)-\zeta(3)]$
52	$\mathrm{Li}_2(1-z)\log(1-z)$	$\frac{2}{N}\left[S_1(N)S_2(N)-\zeta(2)S_1(N)+S_3(N)-\frac{1}{2}S_{2,1}(N)\right]$
53	$\frac{\mathrm{Li}_2(1-z)}{1-z}$	$S_1(N-1)S_2(N-1)-\zeta(2)S_1(N-1)+S_3(N-1)$ $-S_{2,1}(N-1)+\zeta(3)$
54	$\mathrm{Li}_2(1-z)\frac{\log(1-z)}{1-z}$	$-2S_{1,1,2}(N-1)-S_{1,2,1}(N-1)+\zeta(2)S_1^2(N-1)$ $+\zeta(2)S_2(N-1)-\frac{2}{5}\zeta^2(2)$
55	$\mathrm{Li}_2(1-z)\frac{\log z}{1-z}$	$-2S_1(N-1)S_3(N-1)+2\zeta(3)S_1(N-1)-\frac{1}{2}S_2^2(N-1)$ $+\zeta(2)S_2(N-1)-\frac{5}{2}S_4(N-1)+2S_{3,1}(N-1)-\frac{5}{4}\zeta(4)$
56	$\frac{\mathrm{Li}_2(1-z)\log z}{1+z}$	$(-1)^{N-1}\Big[S_{-2,2}(N-1)+2S_{-1,3}(N-1)-\zeta(2)S_{-2}(N-1)$ $-2\zeta(3)S_{-1}(N-1)+4\mathrm{Li}_4\left(\frac{1}{2}\right)$ $-\frac{33}{40}\zeta^2(2)-\zeta(2)\log^2 2+\frac{1}{6}\log^4 2\Big]$

No.	$f(z)$	$M[f](N)$
57	$\tilde{\Phi}(z)$	$\frac{1}{N^3}+2\frac{(-1)^N}{N}\left[S_2(N)-\zeta(2)\right]$ $-\frac{1+(-1)^N}{2N}\left[S_2\left(\frac{N}{2}\right)-\zeta(2)\right]$ $+\frac{1-(-1)^N}{2N}\left[S_2\left(\frac{N-1}{2}\right)-\zeta(2)\right]$
58	$\frac{\tilde{\Phi}(z)}{1+z}$	$(-1)^{N-1}\Big\{-\zeta(2)S_1(N-1)$ $-2S_1(N-1)\Big[\frac{1+(-1)^{N-1}}{4}S_2\left(\frac{N-1}{2}\right)$ $+\frac{1-(-1)^{N-1}}{4}S_2\left(\frac{N-2}{2}\right)-S_2(N-1)\Big]$ $-\frac{1+(-1)^{N-1}}{8}S_3\left(\frac{N-1}{2}\right)-\frac{1-(-1)^{N-1}}{8}S_3\left(\frac{N-2}{2}\right)$ $+S_3(N-1)+2S_{-2,1}(N-1)+\frac{\zeta(3)}{2}\Big\}$
59	$\tilde{\Phi}(z)\frac{\log(1-z)}{1+z}$	$(-1)^{N-1}\Big\{2S_{1,-2,1}(N-1)+2S_{1,-1,2}(N-1)$ $+2S_{-1,-1,-2}(N-1)-S_{-2,2}(N-1)-S_{-1,3}(N-1)$ $-S_{-3,1}(N-1)-2\zeta(2)S_{1,-1}(N-1)$ $+\frac{1}{2}\zeta(2)S_{-1}^2(N-1)+\frac{1}{2}\zeta(2)S_2(N-1)+\zeta(2)S_{-2}(N-1)$ $+\left[\frac{13}{4}\zeta(3)-3\zeta(2)\log 2\right]S_1(N-1)$ $-\left[\frac{9}{4}\zeta(3)-3\zeta(2)\log 2\right]S_{-1}(N-1)$ $+2\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{7}{10}\zeta^2(2)+\zeta(2)\log^2 2+\frac{1}{12}\log^4 2\Big\}$
60	$z^r\,\mathrm{Li}_3(z)$	$\frac{1}{N+r}\left[\zeta(3)-\frac{\zeta(2)}{N+r}+\frac{S_1(N+r)}{(N+r)^2}\right]$
61	$\left(\frac{1}{1-z}\right)_+\mathrm{Li}_3(z)$	$-\zeta(3)S_1(N-1)+\zeta(2)S_2(N-1)-S_{3,1}(N-1)-\frac{5}{4}\zeta(4)$
62	$z^r\,\mathrm{Li}_3(-z)$	$(-1)^{N+r-1}\frac{S_1(N+r)}{(N+r)^3}$ $-\frac{1+(-1)^{N+r-1}}{2(N+r)^3}\left[S_1\left(\frac{N+r-1}{2}\right)+2\log 2\right]$ $+\frac{1-(-1)^{N+r-1}}{2(N+r)^3}S_1\left(\frac{N+r}{2}\right)+\frac{\zeta(2)}{2(N+r)^2}-\frac{3\zeta(3)}{4(N+r)}$
63	$\left(\frac{1}{1-z}\right)_+\mathrm{Li}_3(-z)$	$-S_{-3,-1}(N-1)+\left[S_3(N-1)-S_{-3}(N-1)\right]\log 2$ $-\frac{1}{2}\zeta(2)S_2(N-1)+\frac{3}{4}\zeta(3)S_1(N-1)-2\mathrm{Li}_4\left(\frac{1}{2}\right)$ $+\frac{11}{10}\zeta^2(2)-\frac{7}{4}\zeta(3)\log 2+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2$

No.	$f(z)$	$M[f](N)$
70	$z^r \mathrm{S}_{1,2}(z)$	$\frac{\zeta(3)}{N+r} - \frac{S_1^2(N+r)+S_2(N+r)}{2(N+r)^2}$
71	$\left(\frac{1}{1-z}\right)_+ \mathrm{S}_{1,2}(z)$	$S_{2,1,1}(N-1) - \zeta(3)S_1(N-1) - \frac{6}{5}\zeta^2(2)$
72	$\frac{\mathrm{S}_{1,2}(z)}{1+z}$	$(-1)^N \Big[S_{-2,1,1}(N-1) - \zeta(3)S_{-1}(N-1) + \mathrm{Li}_4\left(\frac{1}{2}\right)$ $-\frac{1}{8}\zeta^2(2) - \frac{1}{8}\zeta(3)\log 2 - \frac{1}{4}\zeta(2)\log^2 2 + \frac{1}{24}\log^4 2\Big]$
73	$z^r \mathrm{S}_{1,2}(-z)$	$\frac{(-1)^N}{N^2}\Big[S_{-1,1}(N) + S_1^2(N) + S_2(N) - 2S_1(N)\log 2\Big]$ $-\frac{1+(-1)^N}{2N^2}\left[S_1(N)S_1\left(\frac{N}{2}\right) - S_1\left(\frac{N}{2}\right)\log 2 + \frac{1}{2}S_2\left(\frac{N}{2}\right)\right]$ $+\frac{1-(-1)^N}{2N^2}\Big[S_1(N)S_1\left(\frac{N-1}{2}\right) - S_1\left(\frac{N-1}{2}\right)\log 2$ $+\frac{1}{2}S_2\left(\frac{N-1}{2}\right) - \log^2 2\Big] + \frac{\zeta(3)}{8N}$
74	$\frac{\mathrm{S}_{1,2}(-z)}{1+z}$	$(-1)^N\Big\{S_{2,1,-1}(N-1) + [S_{2,1}(N-1) - S_{2,-1}(N-1)]\log 2$ $-\frac{1}{2}[S_2(N-1) - S_{-2}(N-1)]\log^2 2$ $-\frac{1}{8}\zeta(3)S_{-1}(N-1) - 3\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{6}{5}\zeta^2(2)$ $-\frac{11}{4}\zeta(3)\log 2 + \frac{3}{4}\zeta(2)\log^2 2 - \frac{1}{8}\log^4 2\Big\}$
75	$\frac{\mathrm{S}_{1,2}(1-z)}{1-z}$	$S_1(N-1)S_3(N-1) - \zeta(3)S_1(N-1) + S_4(N-1)$ $-S_{3,1}(N-1) + \frac{\zeta(4)}{4}$
76	$\mathrm{S}_{1,2}(1-z)$	$-\frac{1}{N}\Big[S_3(N) - \zeta(3)\Big]$
77	$\frac{\mathrm{S}_{1,2}(1-z)}{1+z}$	$(-1)^N\Big[S_{-1,3}(N-1) - \zeta(3)S_{-1}(N-1)$ $+\frac{19}{40}\zeta^2(2) - \frac{7}{4}\zeta(3)\log 2\Big]$

7. Applications

A. $F_L(x,Q^2)$ NLO

The longitudinal structure function $F_L(x,Q^2)$ has the representation

$$F_L(x,Q^2)=x\Big\{C_{\rm NS}(x,Q^2)\otimes f_{\rm NS}(x,Q^2)+\delta_f\Big[C_{\rm S}(x,Q^2)\otimes\Sigma(x,Q^2)+C_{\rm g}(x,Q^2)\otimes G(x,Q^2)\Big]\Big\} \quad (172)$$

in the case of pure photon exchange. The symbol $\otimes$ denotes the Mellin convolution

$$A(x,Q^2)\otimes B(x,Q^2)=\int_0^1 dx_1\int_0^1 dx_2\delta(x-x_1x_2)A(x_1,Q^2)B(x_2,Q^2)\,. \quad (173)$$

The combinations of parton densities are

$$f_{\rm NS}(x,Q^2) = \sum_{i=1}^{N_f} e_i^2\Big[q_i(x,Q^2)+\overline{q}_i(x,Q^2)\Big]\,, \quad (174)$$

$$\Sigma(x,Q^2) = \sum_{i=1}^{N_f}\Big[q_i(x,Q^2)+\overline{q}_i(x,Q^2)\Big]\,. \quad (175)$$

$G(x,Q^2)$ denotes the gluon density, e_i the electric charge, and $\delta_f=(\sum_{i=1}^{N_f}e_i^2)/N_f$, with N_f the number of active flavors.

The coefficient functions $C_i(x,Q^2)$ are given by

$$\begin{aligned} C_{\rm NS}(z,Q^2) &= a_s c_{L,q}^{(1)}(z)+a_s^2 c_{L,q}^{(2),NS}(z)\\ C_{\rm S}(z,Q^2) &= a_s^2 c_{L,q}^{(2),PS}(z)\\ C_{\rm g}(z,Q^2) &= a_s c_{L,g}^{(1)}(z)+a_s^2 c_{L,g}^{(2)}(z)\,, \end{aligned} \quad (176)$$

where $a_s=\alpha_s(Q^2)/(4\pi)$. For convenience we list as well the coefficient functions in x-space, since we will give the Mellin transforms of the individual contributing functions separately below. The leading order coefficient functions are given by [42]

$$c_{L,q}^{(1)}(z) = 4C_F z \quad (177)$$

$$c_{L,g}^{(1)}(z) = 8N_f z(1-z)\,. \quad (178)$$

In the $\overline{\rm MS}$ scheme the NLO coefficient functions read [4–6]

$$\begin{aligned} c_{L,q}^{(2),NS}(z) = {}& 4C_F(C_A-2C_F)z\Big\{4\frac{6-3z+47z^2-9z^3}{15z^2}\ln z\\ &-4{\rm Li}_2(-z)[\ln z-2\ln(1+z)]-8\zeta(3)-2\ln^2 z[\ln(1+z)+\ln(1-z)]\\ &+4\ln z\ln^2(1+z)-4\ln z{\rm Li}_2(z)+\frac{2}{5}(5-3z^2)\ln^2 z\\ &-4\frac{2+10z^2+5z^3-3z^5}{5z^3}[{\rm Li}_2(-z)+\ln z\ln(1+z)]\\ &+4\zeta(2)\left[\ln(1+z)+\ln(1-z)-\frac{5-3z^2}{5}\right]+8{\rm S}_{1,2}(-z)+4{\rm Li}_3(z)\\ &+4{\rm Li}_3(-z)-\frac{23}{3}\ln(1-z)-\frac{144+294z-1729z^2+216z^3}{90z^2}\Big\} \end{aligned}$$

$$
\begin{aligned}
& +8C_F^2 z\Big\{\mathrm{Li}_2(z) + \ln^2 z - 2\ln z \ln(1-z) + \ln^2(1-z) - 3\zeta(2) \\
& -\frac{3-22z}{3z}\ln z + \frac{6-25z}{6z}\ln(1-z) - \frac{78-355z}{36z}\Big\} \\
& -\frac{8}{3}C_F N_f z\left\{2\ln z - \ln(1-z) - \frac{6-25z}{6z}\right\}, \qquad (179)
\end{aligned}
$$

$$
\begin{aligned}
c_{L,q}^{(2),PS}(z) &= \frac{16}{9z}C_F N_f \Big\{3(1-2z-2z^2)(1-z)\ln(1-z) + 9z^2[\mathrm{Li}_2(z) + \ln^2 z - \zeta(2)] \\
&\quad + 9z(1-z-2z^2)\ln z - 9z^2(1-z) - (1-z)^3\Big\}, \qquad (180)
\end{aligned}
$$

$$
\begin{aligned}
c_{L,g}^{(2)}(z) &= C_F N_f \Big\{16z[\mathrm{Li}_2(1-z) + \ln z \ln(1-z)] \\
&+ \left(-\frac{32}{3}z + \frac{64}{5}z^3 + \frac{32}{15z^2}\right)[\mathrm{Li}_2(-z) + \ln z \ln(1+z)] + (8+24z-32z^2)\ln(1-z) \\
&- \left(\frac{32}{3}z + \frac{32}{5}z^3\right)\ln^2 z + \frac{1}{15}\left(-104 - 624z + 288z^2 - \frac{32}{z}\right)\ln z \\
&+ \left(-\frac{32}{3}z + \frac{64}{5}z^3\right)\zeta(2) - \frac{128}{15} - \frac{304}{5}z + \frac{336}{5}z^2 + \frac{32}{15z}\Big\} \\
&+ C_A N_f \Big\{-64\mathrm{Li}_2(1-z) + (32z + 32z^2)[\mathrm{Li}_2(-z) + \ln z \ln(1+z)] \\
&+ (16z - 16z^2)\ln^2(1-z) + (-96z + 32z^2)\ln z \ln(1-z) \\
&+ \left(-16 - 144z + \frac{464}{3}z^2 + \frac{16}{3z}\right)\ln(1-z) + 48z\ln^2 z + (16 + 128z - 208z^2)\ln z \\
&+ 32z^2\zeta(2) + \frac{16}{3} + \frac{272}{3}z - \frac{848}{9}z^2 - \frac{16}{9z}\Big\}, \qquad (181)
\end{aligned}
$$

with $C_A = N_c = 3, C_F = (N_c^2-1)/(2N_c) = 4/3$. The corresponding expressions in the DIS scheme are given in [39]. The class of basic functions is the same for both schemes.

B: $c_2^{\rm NS}(x,Q^2)$

$$
\begin{aligned}
c_{2,+}^{(2)}(x) \;=\; & C_F^2\Bigg\{\frac{1+x^2}{1-x}\Bigg[4\ln^3(1-x)-(14\ln(x)+9)\ln^2(1-x)-\frac{4}{3}\ln^3(x)-\frac{3}{2}\ln^2(x)\\
&-\Bigg[4{\rm Li}_2(1-x)-12\ln^2(x)-12\ln(x)+16\zeta(2)+\frac{27}{2}\Bigg]\ln(1-x)+48{\rm Li}_3(-x)\\
&+\Bigg[-24{\rm Li}_2(-x)+24\zeta(2)+\frac{61}{2}\Bigg]\ln(x)+12{\rm Li}_3(1-x)-12{\rm S}_{1,2}(1-x)\\
&+48{\rm Li}_3(-x)-6{\rm Li}_2(1-x)+32\zeta(3)+18\zeta(2)+\frac{51}{4}\Bigg]\\
&+(1+x)\Bigg[2\ln(x)\ln^2(1-x)+4\Big[{\rm Li}_2(1-x)-\ln^2(x)\Big]\ln(1-x)+\frac{5}{3}\ln^3(x)\\
&-4{\rm Li}_3(1-x)-4\left[{\rm Li}_2(1-x)+\zeta(2)\right]\ln(x)\Bigg]+\left(40+8x-48x^2-\frac{72}{5}x^3+\frac{8}{5x^2}\right)\\
&\times\left[{\rm Li}_2(-x)+\ln(x)\ln(1+x)\right]+(5+9x)\ln^2(1-x)+\frac{1}{2}(-91+141x)\ln(1-x)\\
&+(-8+40x)\left[\ln(x){\rm Li}_2(-x)+{\rm S}_{1,2}(1-x)-2{\rm Li}_3(-x)-\zeta(2)\ln(1-x)\right]\\
&-(28+44x)\ln(x)\ln(1-x)-(14+30x){\rm Li}_2(1-x)+\left(\frac{29}{2}+\frac{25}{2}x+24x^2+\frac{36}{3}x^3\right)\\
&\times\ln^2(x)+\frac{1}{10}\left(13-407x+144x^2-\frac{16}{x}\right)\ln(x)+\left(-10+6x-48x^2-\frac{72}{5}x^3\right)\zeta(2)\\
&+\frac{407}{20}-\frac{1917}{20}x+\frac{72}{5}x^2+\frac{8}{5x}+\left[6\zeta^2(2)-78\zeta(3)+69\zeta(2)+\frac{331}{8}\right]\delta(1-x)\Bigg\}\\
&+C_AC_F\Bigg\{\frac{1+x^2}{1-x}\Bigg[-\frac{11}{3}\ln^2(1-x)+\Bigg[4{\rm Li}_2(1-x)+2\ln^2(x)+\frac{44}{3}\ln(x)-4\zeta(2)\\
&+\frac{367}{18}\Bigg]\ln(1-x)-\ln^3(x)-\frac{55}{6}\ln^2(x)+\Bigg[4{\rm Li}_2(1-x)+12{\rm Li}_2(-x)\\
&-\frac{239}{6}\Bigg]\ln(x)-12{\rm Li}_3(1-x)+12{\rm S}_{1,2}(1-x)-24{\rm Li}_3(-x)+\frac{22}{3}{\rm Li}_2(1-x)+2\zeta(3)\\
&+\frac{22}{3}\zeta(2)-\frac{3155}{108}\Bigg]+4(1+x)\left[{\rm Li}_2(1-x)+\ln(x)\ln(1-x)\right]\\
&+\left(-20-4x+24x^2+\frac{36}{5}x^3-\frac{4}{5x^2}\right)\left[{\rm Li}_2(-x)+\ln(x)\ln(1+x)\right]\\
&+(4-20x)\left[\ln(x){\rm Li}_2(-x)+{\rm S}_{1,2}(1-x)-2{\rm Li}_3(-x)-\zeta(2)\ln(1-x)\right]\\
&+\left(\frac{133}{6}-\frac{1113}{18}x\right)\ln(1-x)+\left(-2+2x-12x^2-\frac{18}{5}x^3\right)\ln^2(x)\\
&+\frac{1}{30}\left(13+1753x-216x^2+\frac{24}{x}\right)\ln(x)+\left(-2-10x+24x^2+\frac{36}{5}x^3\right)\zeta(2)\\
&-\frac{9687}{540}+\frac{59157}{540}-\frac{36}{5}x^2-\frac{4}{5x}\\
&+\left[\frac{71}{5}\zeta^2(2)+\frac{140}{3}\zeta(3)-\frac{251}{3}\zeta(2)-\frac{5465}{72}\right]\delta(1-x)\Bigg\}\\
&+C_FN_F\Bigg\{\frac{1+x^2}{1-x}\Bigg[\frac{2}{3}\ln^2(1-x)-\left(\frac{8}{3}\ln(x)+\frac{29}{9}\right)\ln(1-x)-\frac{4}{3}{\rm Li}_2(1-x)+\frac{5}{3}\ln^2(x)\\
&+\frac{19}{3}\ln(x)-\frac{4}{3}\zeta(2)+\frac{247}{54}\Bigg]+\frac{1}{3}(1+13x)\ln(1-x)-\frac{1}{3}(7+19x)\ln(x)-\frac{23}{18}-\frac{27}{2}x\\
&+\left[\frac{4}{3}\zeta(3)+\frac{38}{3}\zeta(2)+\frac{457}{36}\right]\delta(1-x)\Bigg\}\,. \qquad (208)
\end{aligned}
$$

$$
\begin{aligned}
c_{2,-}^{(2)}(x) \;=\;& C_F\left(C_F-\frac{1}{2}C_A\right)\times \\
&\left\{\frac{1+x^2}{1+x}\left[\Big[4\ln^2(x)-16\ln(x)\ln(1+x)-16\mathrm{Li}_2(-x)-8\zeta(2)\Big]\ln(1-x)\right.\right. \\
&+\Big[-2\ln^2(x)+20\ln(x)\ln(1+x)-8\ln^2(1+x)+8\mathrm{Li}_2(1-x)+16\mathrm{Li}_2(-x)-8\Big]\ln(x) \\
&-16\ln(1+x)\mathrm{Li}_2(-x)-8\zeta(2)\ln(1+x)-16\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) \\
&\left.+16\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)-16\mathrm{Li}_3(1-x)+8\mathrm{S}_{1,2}(1-x)+8\mathrm{Li}_3(-x)-16\mathrm{S}_{1,2}(-x)+8\zeta(3)\right] \\
&+(4+20x)\Big[\ln^2(x)\ln(1+x)-2\ln(x)\ln^2(1+x)-2\zeta(2)\ln(1+x)-4\ln(1+x)\mathrm{Li}_2(-x) \\
&+2\mathrm{Li}_3(-x)-4\mathrm{S}_{1,2}(-x)+2\zeta(3)\Big]+\left(32+32x+48x^2-\frac{72}{5}x^3+\frac{8}{5x^2}\right) \\
&\times\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right]+8(1+x)\left[\mathrm{Li}(1-x)+\ln(x)\ln(1-x)\right]+16(1-x)\ln(1-x) \\
&+\left(-4-16x-24x^2+\frac{36}{5}x^3\right)\ln^2(x)+\frac{1}{5}\left(-26-106x+72x^2-\frac{8}{x}\right)\ln(x) \\
&\left.+\left(-4+20x+48x^2-\frac{72}{5}x^3\right)\zeta(2)+\frac{1}{5}\left(-162+82x+72x^2+\frac{8}{x}\right)\right\}. \qquad (210)
\end{aligned}
$$

$$c_2^{(2),\mathrm{NS}}(N) = C_F^2 B_{2,F}(N) + C_F C_A B_{2,A}(N) + C_F N_F B_{2,N_F}(N) \ .$$

N	$B_{2,F}(N)$	$B_{2,A}(N)$	$B_{2,N_F}(N)$
2	17.9078876005	−3.53561968276	−3.99999999998
4	21.4337645358	31.8422327506	−12.7409351851
6	49.6093136699	58.4783326748	−21.0097878018
8	76.6379137872	92.3938671695	−28.4436170993
10	101.906494546	130.266385188	−35.1459888118
12	125.462647526	170.288348629	−41.2458493431
14	147.482822156	211.433606893	−46.8491181970
16	168.152110570	253.091881557	−52.0380522803
18	187.635850771	294.886826041	−56.8764477704
20	206.075234490	336.580388240	−61.4143760545

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An expression for $c_2^{(2),\mathrm{NS}}(N)$ was given in ref. [55] in terms of a linear combination of harmonic sums $S_i|_{i=1}^4$. This representation equivalent to the one given above, in which we refer also to products of harmonic sums.

$$\begin{aligned}
c_2^{(2),\mathrm{NS}}(N) \;=\; & \theta(N-3)\left[S_{1,-2}(N-3)-S_{1,-2}(N-2)\right]\left(\frac{8}{5}C_FC_A-\frac{16}{5}C_F^2\right)\\
&+\delta_{N,2}\zeta(3)\left(\frac{12}{5}C_FC_A-\frac{24}{5}C_F^2\right)\\
&+\theta(N-2)\left\{\left[S_1(N-2)+S_2(N-2)\right]\left(\frac{8}{5}C_FC_A-\frac{16}{5}C_F^2\right)\right.\\
&+\Big[S_{-4}(N-1)(12C_FC_A-24C_F^2)-S_{-3,1}(N-1)(8C_FC_A-16C_F^2)\\
&+S_{-2}(N-1)(8C_FC_A-16C_F^2)-S_{-2,-2}(N-1)(24C_FC_A-48C_F^2)\\
&+S_1(N-1)\left(\frac{1585}{54}C_FC_A-\frac{89}{27}C_FN_F+\frac{5}{2}C_F^2\right)\\
&+S_1(N-1)\zeta(3)(-36C_FC_A+48C_F^2)-S_{1,-3}(N-1)(24C_FC_A-48C_F^2)\\
&+S_{1,-2}(N-1)(36C_FC_A-72C_F^2)+S_{1,-2,1}(N-1)(8C_FC_A-16C_F^2)\\
&+S_{1,1}(N-1)\left(\frac{311}{9}C_FC_A-\frac{26}{9}C_FN_F-43C_F^2\right)\\
&+S_{1,1,1}(N-1)\left(\frac{22}{3}C_FC_A-\frac{4}{3}C_FN_F+8C_F^2\right)\\
&+S_{1,1,-2}(N-1)(24C_FC_A-48C_F^2)+S_{1,1,1,1}(N-1)(24C_F^2)\\
&+S_{1,2}(N-1)\left(-\frac{22}{3}C_FC_A+\frac{4}{3}C_FN_F-4C_F^2\right)\\
&+S_{1,1,2}(N-1)(4C_FC_A-32C_F^2)+S_{1,2,1}(N-1)(-4C_FC_A-24C_F^2)\\
&+S_2(N-1)\left(-\frac{212}{5}C_FC_A+4C_FN_F+\frac{189}{5}C_F^2\right)\\
&+S_2(N-1)\left(-\frac{212}{5}C_FC_A+4C_FN_F+\frac{189}{5}C_F^2\right)\\
&+S_{1,3}(N-1)(12C_FC_A+4C_F^2)+S_{2,-2}(N-1)(-8C_FC_A+16C_F^2)\\
&+S_{2,1}(N-1)\left(-\frac{44}{3}C_FC_A+\frac{8}{3}C_FN_F+8C_F^2\right)\\
&-S_{2,1,1}(N-1)(24C_F^2)+S_{2,2}(N-1)(20C_F^2)\\
&+S_3(N-1)\left(\frac{55}{3}C_FC_A-\frac{10}{3}C_FN_F-18C_F^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+S_{3,1}(N-1)(8C_FC_A+8C_F^2)+S_4(N-1)(-12C_FC_A+14C_F^2)\Big]\\
&+\Big[S_1(N)\left(-\frac{4639}{45}C_FC_A+\frac{110}{9}C_FN_F+\frac{337}{5}C_F^2\right)\\
&+S_1(N)\zeta(3)(72C_FC_A-144C_F^2)+S_{1,-3}(N)(24C_FC_A-48C_F^2)\\
&+S_{1,-2}(N)(-56C_FC_A+112C_F^2)+S_{1,1,-2}(N)(-48C_FC_A+96C_F^2)\\
&+S_{1,1}(N)(-68C_FC_A+4C_FN_F+84C_F^2)+S_{1,1,1}(N)(-8C_F^2)\\
&+S_{1,3}(N)(-24C_FC_A+48C_F^2)+S_{2,-2}(N)(-16C_FC_A+32C_F^2)\\
&+S_2(N)(74C_FC_A-4C_FN_F-74C_F^2)+S_{2,1}(N)(16C_F^2)\\
&+S_3(N)(-20C_FC_A+28C_F^2)+S_{3,1}(N)(-8C_FC_A+16C_F^2)\\
&+S_4(N)(12C_FC_A-24C_F^2)\Big]\\
&+\Big[S_1(N+1)\left(\frac{3914}{27}C_FC_A-\frac{488}{27}C_FN_F-121C_F^2\right)\\
&+S_1(N+1)\zeta(3)(-84C_FC_A+144C_F^2)+S_{1,-3}(N+1)(-40C_FC_A+80C_F^2)\\
&+S_{1,-2}(N+1)(20C_FC_A-40C_F^2)+S_{1,-2,1}(N+1)(8C_FC_A-16C_F^2)\\
&+S_{1,1}(N+1)\left(\frac{668}{9}C_FC_A-\frac{68}{9}C_FN_F-68C_F^2\right)\\
&+S_{1,1,1}(N+1)\left(\frac{22}{3}C_FC_A-\frac{4}{3}C_FN_F+36C_F^2\right)\\
&+S_{1,1,-2}(N+1)(56C_FC_A-112C_F^2)+S_{1,1,1,1}(N+1)(24C_F^2)\\
&+S_{1,2}(N+1)\left(-\frac{22}{3}C_FC_A+\frac{4}{3}C_FN_F-32C_F^2\right)\\
&+S_{1,1,2}(N+1)(4C_FC_A-32C_F^2)+S_{1,2,1}(N+1)(-4C_FC_A-24C_F^2)\\
&+S_{1,3}(N+1)(28C_FC_A-28C_F^2)+S_2(N+1)\left(-\frac{1909}{15}C_FC_A+\frac{38}{3}C_FN_F+\frac{646}{5}C_F^2\right)\\
&+S_{2,1}(N+1)\left(-\frac{44}{3}C_FC_A+\frac{8}{3}C_FN_F-48C_F^2\right)\\
&+S_{2,1,1}(N+1)(-32C_F^2)+S_{2,2}(N+1)(28C_F^2)\\
&+S_3(N+1)\left(\frac{115}{3}C_FC_A-\frac{10}{3}C_FN_F-4C_F^2\right)\\
&+S_{3,1}(N+1)(8C_FC_A+24C_F^2)+S_4(N+1)(-12C_FC_A-6C_F^2)\Big]\\
&+\Big[-(S_1(N+2)-S_{1,-2}(N+2))\left(\frac{72}{5}C_FC_A-\frac{144}{5}C_F^2\right)\\
&+(S_2(N+2)+S_3(N+2))\left(\frac{72}{5}C_FC_A-\frac{144}{5}C_F^2\right))\Big]\\
&-(S_{1,-2}(N+3)+)+S_3(N+3)\left(\frac{72}{5}C_FC_A-\frac{144}{5}C_F^2\right)\\
&-\frac{5465}{72}C_FC_A+\frac{457}{36}C_FN_F+\frac{331}{8}C_F^2+\zeta(3)(54C_FC_A-72C_F^2)\Big\}
\end{aligned}
\tag{211}
$$

$$
\begin{aligned}
& - \ln x \ln(1-x) \ln(1+x) - \mathrm{Li}_3(1-x) - \mathrm{Li}_3\Big(-\frac{1-x}{1+x}\Big) - \mathrm{Li}_3\Big(\frac{1}{1+x}\Big) \\
& + \mathrm{Li}_3\Big(\frac{1-x}{1+x}\Big) \,, \\
H_{1,0,0}(x) &= -\frac{1}{2}\ln^2 x \ln(1-x) - \ln x \mathrm{Li}_2(x) + \mathrm{Li}_3(x) && (191) \\
&= \zeta_3 - \mathrm{S}_{n,p}(1-x) \,, \\
H_{1,0,1}(x) &= -2\zeta_3 - 2\ln(1-x)\zeta_2 + \ln(1-x)\mathrm{Li}_2(x) + \ln x \ln^2(1-x) + 2\mathrm{Li}_3(1-x) \,, && (192) \\
H_{1,1,-1}(x) &= \frac{1}{2}\zeta_2 \ln 2 - \frac{1}{8}\zeta_3 - \frac{1}{6}\ln^3 2 + \frac{1}{2}\ln(1-x)\zeta_2 + \frac{1}{2}\ln^2(1-x)\ln 2 && (193) \\
& - \ln(1-x)\ln(1+x)\ln 2 + \frac{1}{2}\ln(1-x)\ln^2(1+x) - \ln(1+x)\zeta_2 + \frac{1}{2}\ln(1+x)\ln^2 2 \\
& - \frac{1}{6}\ln^3(1+x) + \mathrm{Li}_3\Big(\frac{1+x}{2}\Big) + \mathrm{Li}_3\Big(-\frac{1-x}{1+x}\Big) \,, \\
H_{1,1,0}(x) &= \zeta_3 + \ln(1-x)\zeta_2 - \mathrm{Li}_3(1-x) \,, && (194) \\
H_{1,1,1}(x) &= -\frac{1}{6}\ln^3(1-x) \,. && (195)
\end{aligned}
$$

The function $\mathrm{S}_{n,p}$ in eqs.(178) and (191) denote the Nielsen functions [48], defined as

$$
\mathrm{S}_{n,p}(x) = \frac{(-1)^{p+n-1}}{p!\,(n-1)!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-xz) \,. \qquad (196)
$$

Appendix B

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Here we present the formulae for the Mellin moments of the 2-loop coefficient functions. We obtain,

$$
\begin{aligned}
c_{2,\mathrm{q}}^{(2),+\mathrm{ns}}(N) &= \delta(N-2)\Big\{\Big(\frac{11581}{360} - \frac{86}{9}\zeta_2 - \frac{32}{5}\zeta_2^2 - \frac{9}{5}\zeta_3\Big)C_F C_A - 4C_F n_f && (197) \\
& - \Big(\frac{24359}{1620} - \frac{172}{9}\zeta_2 - \frac{64}{5}\zeta_2^2 + \frac{142}{5}\zeta_3\Big)C_F^2\Big\} + \theta(N-3)\,(-1)^N \times \\
\Big[C_F C_A\Big\{ & (-1)^N\Big(-\frac{5465}{72} - 4\zeta_2 - \frac{32}{5}\zeta_2^2 + 54\zeta_3 - 12S_{-2}(N-1)\zeta_2 - 24S_{-2,-2}(N-1) \\
& + \Big(\frac{3155}{54} - 32\zeta_3\Big)S_1(N-1) + \Big(\frac{367}{9} - 16\zeta_2\Big)S_{1,1}(N-1) + \frac{44}{3}S_{1,1,1}(N-1) \\
& + 8S_{1,1,2}(N-1) - \frac{44}{3}S_{1,2}(N-1) - 8S_{1,2,1}(N-1) + 16S_{1,3}(N-1) \\
& - \Big(\frac{239}{3} - 12\zeta_2\Big)S_2(N-1) - \frac{88}{3}S_{2,1}(N-1) + \frac{110}{3}S_3(N-1) + 8S_{3,1}(N-1) \\
& - 12S_4(N-1)\Big) + 6S_{-4}(N-1) + 6S_{-4}(N+1) - 12S_{-4}(N) - \frac{43}{3}S_{-3}(N-1) \\
& + \frac{5}{3}S_{-3}(N+1) - \frac{156}{5}S_{-3}(N+2) + \frac{36}{5}S_{-3}(N+3) + \frac{110}{3}S_{-3}(N) - 4S_{-3,1}(N-1) \\
& - 4S_{-3,1}(N+1) + 8S_{-3,1}(N) - \frac{4}{5}S_{-2}(N-2) + \Big(\frac{616}{15} - 4\zeta_2\Big)S_{-2}(N-1) \\
& + \Big(\frac{1366}{15} - 16\zeta_2\Big)S_{-2}(N+1) + \frac{36}{5}S_{-2}(N+2) - \Big(\frac{2078}{15} - 20\zeta_2\Big)S_{-2}(N) \\
& + \frac{32}{3}S_{-2,1}(N-1) + \frac{32}{3}S_{-2,1}(N+1) - \frac{64}{3}S_{-2,1}(N) - \frac{2}{5}S_{-1}(N-3)\zeta_2 \\
& - \Big(\frac{4}{5} - \frac{2}{5}\zeta_2\Big)S_{-1}(N-2) - \Big(\frac{1414}{135} + 8\zeta_2 - 6\zeta_3\Big)S_{-1}(N-1)
\end{aligned}
$$

$$
\begin{aligned}
&-28S_{-2,2}(N+1)+56S_{-2,2}(N)+\frac{4}{5}S_{-1}(N-3)\zeta_2+\Big(\frac{8}{5}-\frac{4}{5}\zeta_2\Big)S_{-1}(N-2)\\
&-\Big(\frac{46}{5}-16\zeta_2-12\zeta_3\Big)S_{-1}(N-1)+\Big(\frac{471}{5}+8\zeta_2-108\zeta_3\Big)S_{-1}(N+1)\\
&+\Big(\frac{72}{5}-\frac{156}{5}\zeta_2\Big)S_{-1}(N+2)+\frac{36}{5}S_{-1}(N+3)\zeta_2-\Big(101-96\zeta_3\Big)S_{-1}(N)\\
&-\Big(32+8\zeta_2\Big)S_{-1,1}(N-1)+\Big(84-56\zeta_2\Big)S_{-1,1}(N+1)-\Big(52-64\zeta_2\Big)S_{-1,1}(N)\\
&-28S_{-1,1,1}(N-1)-36S_{-1,1,1}(N+1)+64S_{-1,1,1}(N)-24S_{-1,1,1,1}(N-1)\\
&-24S_{-1,1,1,1}(N+1)+48S_{-1,1,1,1}(N)+32S_{-1,1,2}(N-1)+32S_{-1,1,2}(N+1)\\
&-64S_{-1,1,2}(N)+32S_{-1,2}(N-1)+32S_{-1,2}(N+1)-64S_{-1,2}(N)+24S_{-1,2,1}(N-1)\\
&+24S_{-1,2,1}(N+1)-48S_{-1,2,1}(N)-20S_{-1,3}(N-1)+28S_{-1,3}(N+1)-8S_{-1,3}(N)\\
&-\frac{4}{5}S_1(N-3)\zeta_2+\frac{4}{5}S_1(N-2)\zeta_2-20S_1(N-1)\zeta_2+20S_1(N+1)\zeta_2-\frac{156}{5}S_1(N+2)\zeta_2\\
&+\frac{36}{5}S_1(N+3)\zeta_2+24S_1(N)\zeta_2-\frac{8}{5}S_{1,-2}(N-3)+\frac{8}{5}S_{1,-2}(N-2)-40S_{1,-2}(N-1)\\
&+40S_{1,-2}(N+1)-\frac{312}{5}S_{1,-2}(N+2)+\frac{72}{5}S_{1,-2}(N+3)+48S_{1,-2}(N)-8S_2(N-1)\zeta_2\\
&-32S_2(N+1)\zeta_2+40S_2(N)\zeta_2-16S_{2,-2}(N-1)-64S_{2,-2}(N+1)+80S_{2,-2}(N)\Big\}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
&c_{2,\mathrm{q}}^{(2),-\mathrm{ns}}(N)= \qquad (198)\\
&\delta(N-2)\Big\{\Big(\frac{5327}{540}-\frac{172}{9}\zeta_2-\frac{64}{5}\zeta_2^2+\frac{238}{5}\zeta_3\Big)C_F\Big(C_F-\frac{C_A}{2}\Big)\Big\}+\theta(N-3)\,(-1)^N\times\\
&\Big[C_F\Big(C_F-\frac{C_A}{2}\Big)\Big\{-8\zeta_2-\frac{64}{5}\zeta_2^2-12S_{-4}(N-1)-12S_{-4}(N+1)+8S_{-3}(N-1)\\
&+80S_{-3}(N+1)-\frac{168}{5}S_{-3}(N+2)-\frac{72}{5}S_{-3}(N+3)-40S_{-3}(N)+8S_{-3,1}(N-1)\\
&+8S_{-3,1}(N+1)+\frac{8}{5}S_{-2}(N-2)-\Big(\frac{74}{5}+16\zeta_2\Big)S_{-2}(N-1)-\Big(\frac{74}{5}+32\zeta_2\Big)S_{-2}(N+1)\\
&-\frac{72}{5}S_{-2}(N+2)+\Big(\frac{132}{5}+24\zeta_2\Big)S_{-2}(N)-8S_{-2,1}(N-1)-8S_{-2,1}(N+1)\\
&+16S_{-2,1}(N)+\frac{4}{5}S_{-1}(N-3)\zeta_2+\Big(\frac{8}{5}-\frac{4}{5}\zeta_2\Big)S_{-1}(N-2)+\Big(\frac{154}{5}+20\zeta_2\Big)S_{-1}(N-1)\\
&-\Big(\frac{154}{5}+28\zeta_2\Big)S_{-1}(N+1)+\Big(\frac{72}{5}+\frac{84}{5}\zeta_2\Big)S_{-1}(N+2)+\frac{36}{5}S_{-1}(N+3)\zeta_2\\
&-\Big(16+16\zeta_2\Big)S_{-1}(N)+16S_{-1,1}(N-1)-16S_{-1,1}(N+1)-\frac{4}{5}S_1(N-3)\zeta_2\\
&+\frac{4}{5}S_1(N-2)\zeta_2-\Big(16\zeta_2-20\zeta_3\Big)S_1(N-1)-\Big(40\zeta_2-36\zeta_3\Big)S_1(N+1)+\frac{84}{5}S_1(N+2)\zeta_2\\
&+\frac{36}{5}S_1(N+3)\zeta_2+\Big(32\zeta_2-24\zeta_3\Big)S_1(N)+48S_{1,-3}(N-1)+80S_{1,-3}(N+1)\\
&-48S_{1,-3}(N)-\frac{8}{5}S_{1,-2}(N-3)+\frac{8}{5}S_{1,-2}(N-2)-32S_{1,-2}(N-1)-80S_{1,-2}(N+1)\\
&+\frac{168}{5}S_{1,-2}(N+2)+\frac{72}{5}S_{1,-2}(N+3)+64S_{1,-2}(N)-16S_{1,-2,1}(N-1)\\
&-16S_{1,-2,1}(N+1)-24S_{1,1}(N-1)\zeta_2-56S_{1,1}(N+1)\zeta_2+48S_{1,1}(N)\zeta_2\\
&-48S_{1,1,-2}(N-1)-112S_{1,1,-2}(N+1)+96S_{1,1,-2}(N)+16S_2(N-1)\zeta_2
\end{aligned}
$$

$$-\Big(\frac{17761}{135}+4\zeta_2-66\zeta_3\Big)S_{-1}(N+1)-\Big(\frac{36}{5}-\frac{78}{5}\zeta_2\Big)S_{-1}(N+2)-\frac{18}{5}S_{-1}(N+3)\zeta_2$$
$$+\Big(\frac{4051}{27}-72\zeta_3\Big)S_{-1}(N)+\Big(\frac{16}{9}+4\zeta_2\Big)S_{-1,1}(N-1)-\Big(\frac{740}{9}-28\zeta_2\Big)S_{-1,1}(N+1)$$
$$+\Big(\frac{724}{9}-32\zeta_2\Big)S_{-1,1}(N)-\frac{22}{3}S_{-1,1,1}(N-1)-\frac{22}{3}S_{-1,1,1}(N+1)+\frac{44}{3}S_{-1,1,1}(N)$$
$$-4S_{-1,1,2}(N-1)-4S_{-1,1,2}(N+1)+8S_{-1,1,2}(N)+\frac{22}{3}S_{-1,2}(N-1)+\frac{22}{3}S_{-1,2}(N+1)$$
$$-\frac{44}{3}S_{-1,2}(N)+4S_{-1,2,1}(N-1)+4S_{-1,2,1}(N+1)-8S_{-1,2,1}(N)-4S_{-1,3}(N-1)$$
$$-28S_{-1,3}(N+1)+32S_{-1,3}(N)+\frac{2}{5}S_1(N-3)\zeta_2-\frac{2}{5}S_1(N-2)\zeta_2+10S_1(N-1)\zeta_2$$
$$-10S_1(N+1)\zeta_2+\frac{78}{5}S_1(N+2)\zeta_2-\frac{18}{5}S_1(N+3)\zeta_2-12S_1(N)\zeta_2+\frac{4}{5}S_{1,-2}(N-3)$$
$$-\frac{4}{5}S_{1,-2}(N-2)+20S_{1,-2}(N-1)-20S_{1,-2}(N+1)+\frac{156}{5}S_{1,-2}(N+2)$$
$$-\frac{36}{5}S_{1,-2}(N+3)-24S_{1,-2}(N)+4S_2(N-1)\zeta_2+16S_2(N+1)\zeta_2-20S_2(N)\zeta_2$$
$$+8S_{2,-2}(N-1)+32S_{2,-2}(N+1)-40S_{2,-2}(N)\Bigg\}$$
$$+\ C_Fn_f\Bigg\{(-1)^N\Big(\frac{457}{36}-\frac{247}{27}S_1(N-1)-\frac{58}{9}S_{1,1}(N-1)-\frac{8}{3}S_{1,1,1}(N-1)+\frac{8}{3}S_{1,2}(N-1)$$
$$+\frac{38}{3}S_2(N-1)+\frac{16}{3}S_{2,1}(N-1)-\frac{20}{3}S_3(N-1)\Big)+\frac{10}{3}S_{-3}(N-1)+\frac{10}{3}S_{-3}(N+1)$$
$$-\frac{20}{3}S_{-3}(N)-\frac{26}{3}S_{-2}(N-1)-\frac{38}{3}S_{-2}(N+1)+\frac{64}{3}S_{-2}(N)-\frac{8}{3}S_{-2,1}(N-1)$$
$$-\frac{8}{3}S_{-2,1}(N+1)+\frac{16}{3}S_{-2,1}(N)+\frac{158}{27}S_{-1}(N-1)+\frac{488}{27}S_{-1}(N+1)-\frac{646}{27}S_{-1}(N)$$
$$+\frac{32}{9}S_{-1,1}(N-1)+\frac{68}{9}S_{-1,1}(N+1)-\frac{100}{9}S_{-1,1}(N)+\frac{4}{3}S_{-1,1,1}(N-1)$$
$$+\frac{4}{3}S_{-1,1,1}(N+1)-\frac{8}{3}S_{-1,1,1}(N)-\frac{4}{3}S_{-1,2}(N-1)-\frac{4}{3}S_{-1,2}(N+1)+\frac{8}{3}S_{-1,2}(N)\Bigg\}$$
$$+\ C_F^2\Bigg\{(-1)^N\Big(\frac{331}{8}+8\zeta_2+\frac{64}{5}\zeta_2^2-72\zeta_3+24S_{-2}(N-1)\zeta_2+48S_{-2,-2}(N-1)$$
$$-\Big(\frac{51}{2}-16\zeta_3\Big)S_1(N-1)-\Big(27-32\zeta_2\Big)S_{1,1}(N-1)+36S_{1,1,1}(N-1)$$
$$+48S_{1,1,1,1}(N-1)-64S_{1,1,2}(N-1)-36S_{1,2}(N-1)-48S_{1,2,1}(N-1)+24S_{1,3}(N-1)$$
$$+\Big(61-24\zeta_2\Big)S_2(N-1)-24S_{2,1}(N-1)-56S_{2,1,1}(N-1)+48S_{2,2}(N-1)$$
$$+6S_3(N-1)+48S_{3,1}(N-1)-16S_4(N-1)\Big)+18S_{-4}(N-1)+18S_{-4}(N+1)$$
$$-36S_{-4}(N)-32S_{-3}(N-1)-76S_{-3}(N+1)+\frac{312}{5}S_{-3}(N+2)-\frac{72}{5}S_{-3}(N+3)$$
$$+60S_{-3}(N)-32S_{-3,1}(N-1)-32S_{-3,1}(N+1)+64S_{-3,1}(N)+\frac{8}{5}S_{-2}(N-2)$$
$$-\Big(\frac{154}{5}-8\zeta_2\Big)S_{-2}(N-1)-\Big(\frac{284}{5}-32\zeta_2\Big)S_{-2}(N+1)-\frac{72}{5}S_{-2}(N+2)$$
$$+\Big(\frac{502}{5}-40\zeta_2\Big)S_{-2}(N)+40S_{-2,1}(N-1)+56S_{-2,1}(N+1)-96S_{-2,1}(N)$$
$$+32S_{-2,1,1}(N-1)+32S_{-2,1,1}(N+1)-64S_{-2,1,1}(N)-28S_{-2,2}(N-1)$$

C. SCHEME-INVARIANT EVOLUTION EQS.

USUAL APPROACH:

FIT: $q(x),\ \bar{q}(x),\ G(x) \quad \Big|\ Q_0^2$

& Λ_{QCD}

$$\frac{d\Delta_{NS}}{dt} = P_{NS} \otimes \Delta_{NS}$$

$$\frac{d\vec{\Sigma}}{dt} = \mathbb{P} \otimes \vec{\Sigma} \quad , \quad \vec{\Sigma} = \begin{pmatrix} \Sigma \\ G \end{pmatrix}$$

$$\Sigma = \sum (q_i + \bar{q}_i)$$

SCHEME-INVARIANT APPROACH:

CHOOSE OBSERVABLES : $\left\{ F_2,\ \frac{\partial F_2}{\partial \log Q^2} \right\}$

$\left\{ F_2,\ F_L \right\}$

→ 'FIX' THE INPUT BY A PRECISE MEASUREMENT OF THESE OBSERVABLES AT A SCALE Q_0^2.

→ Λ REMAINS AS THE ONLY PARAMETER TO BE FITTED TO THE EVOLUTION. OF (F_A, F_B).

NEW EVOLUTION KERNELS (ANOM. DIMENSIONS)

FURMANSKI, PETRONZIO

$$F_A(Q^2) = F_2^{N(S)}(Q^2), \qquad F_B(Q^2) = \frac{\partial}{\partial t} F_2^{N(S)}(Q^2)$$

$$\Gamma_{22}^{(0)} = 0 \qquad \Gamma_{2d}^{(0)} = -4$$

$$\Gamma_{d2}^{(0)} = \frac{1}{4}\left(\gamma_{qq}^{(0)}\gamma_{gg}^{(0)} - \gamma_{qg}^{(0)}\gamma_{gq}^{(0)}\right) \qquad \Gamma_{dd}^{(0)} = \gamma_{qq}^{(0)} + \gamma_{gg}^{(0)}$$

$$\begin{aligned}
\Gamma_{22}^{(1)} &= 0 \\
\Gamma_{2d}^{(1)} &= 0 \\
\Gamma_{d2}^{(1)} &= \frac{1}{4}\left[\gamma_{gg}^{(0)}\gamma_{qq}^{(1)} + \gamma_{gg}^{(1)}\gamma_{qq}^{(0)} - \gamma_{qg}^{(1)}\gamma_{gq}^{(0)} - \gamma_{qg}^{(0)}\gamma_{gq}^{(1)}\right] \\
&\quad - \frac{\beta_1}{2\beta_0}\left(\gamma_{qq}^{(0)}\gamma_{gg}^{(0)} - \gamma_{gq}^{(0)}\gamma_{qg}^{(0)}\right) + \frac{\beta_0}{2} C_{2q}^{N(1)}\left(\gamma_{qq}^{(0)} + \gamma_{gg}^{(0)} - 2\beta_0\right) \\
&\quad - \frac{\beta_0}{2}\frac{C_{2g}^{N(1)}}{\gamma_{qg}^{(0)}}\left[(\gamma_{qq}^{(0)})^2 - \gamma_{qq}^{(0)}\gamma_{gg}^{(0)} + 2\gamma_{qg}^{(0)}\gamma_{gq}^{(0)} - 2\beta_0\gamma_{qq}^{(0)}\right] \\
&\quad - \frac{\beta_0}{2}\left(\gamma_{qq}^{(1)} - \frac{\gamma_{qq}^{(0)}\gamma_{qg}^{(1)}}{\gamma_{qg}^{(0)}}\right) \\
\Gamma_{dd}^{(1)} &= \gamma_{qq}^{(1)} + \gamma_{gg}^{(1)} - \frac{\beta_1}{\beta_0}\left(\gamma_{qq}^{(0)} + \gamma_{gg}^{(0)}\right) \\
&\quad - \frac{2\beta_0}{\gamma_{qg}^{(0)}}\left[C_{2g}^{N(1)}\left(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 2\beta_0\right) - \gamma_{qg}^{(1)}\right] + 4\beta_0 C_{2q}^{N(1)} - 2\beta_1
\end{aligned}$$

F_2, F_L NLO JB, VOGT

To next to leading order in $a_s(Q^2)$, one finds

$$\begin{aligned}
\Gamma_{22}^{(1)} &= \gamma_{qq}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qq}^{(0)} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\left(\gamma_{qg}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{(0)}\right) + \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)}\gamma_{qq}^{(0)} \\
&\quad - \left[\frac{C_{Lq}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} + \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2 C_{2g}^{\prime N(1)} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}}\right]\gamma_{qg}^{(0)} + C_{2g}^{\prime N(1)}\gamma_{gq}^{(0)} \\
&\quad - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)}\gamma_{gg}^{(0)} + 2\beta_0\left(C_{2q}^{\prime N(1)} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)}\right) \qquad (27)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{2L}^{(1)} &= \gamma_{qg}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{(0)} - C_{2g}^{\prime N(1)}(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)}) + 2\beta_0 C_{2g}^{\prime N(1)} \\
&\quad + \left(C_{2q}^{\prime N(1)} + \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)} - \frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}}\right)\gamma_{qg}^{(0)} \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{L2}^{(1)} &= \gamma_{gq}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{gq}^{(0)} + \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\left(\gamma_{qq}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qq}^{(0)}\right) \\
&\quad - \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2\left(\gamma_{qg}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{(0)}\right) - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\left(\gamma_{gg}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{gg}^{(0)}\right) \\
&\quad + \left[\frac{C_{Lq}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2q}^{\prime N(1)} + \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2 C_{2g}^{\prime N(1)}\right]\gamma_{qq}^{(0)} \\
&\quad - \left[\left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^3 C_{2g}^{\prime N(1)} + 2\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\frac{C_{Lq}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} - \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2\frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}}\right. \\
&\quad \left. - \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2 C_{2q}^{\prime N(1)}\right]\gamma_{qg}^{(0)} + \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)} - C_{2q}^{\prime N(1)} + \frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}}\right)\gamma_{gq}^{(0)} \\
&\quad - \left[\frac{C_{Lq}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} + \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2 C_{2g}^{\prime N(1)} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2q}^{\prime N(1)}\right]\gamma_{gg}^{(0)} \\
&\quad + 2\beta_0\left(\frac{C_{Lq}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}}\right) \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{LL}^{(1)} &= \gamma_{gg}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{gg}^{(0)} + \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\left(\gamma_{qg}^{(1)} - \frac{\beta_1}{\beta_0}\gamma_{qg}^{(0)}\right) \\
&\quad - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)}\gamma_{qq}^{(0)} + \left[\frac{C_{Lq}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} - \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} + \left(\frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}\right)^2 C_{2g}^{\prime N(1)}\right]\gamma_{qg}^{(0)} \\
&\quad - C_{2g}^{\prime N(1)}\gamma_{gq}^{(0)} + \frac{C_{Lq}^{\prime N(1)}}{C_{Lg}^{\prime N(1)}}C_{2g}^{\prime N(1)}\gamma_{gg}^{(0)} + 2\beta_0\frac{C_{Lg}^{\prime N(2)}}{C_{Lg}^{\prime N(1)}} \qquad (30)
\end{aligned}$$

8. CONCLUSIONS

1. WILSON COEFFICIENTS & SPLITTING FUNCTIONS IN MASSLESS FIELD THEORIES AS QED AND QCD ($m \to 0$) ARE STRUCTURED BY MELLIN CONVOLUTIONS.

2. THESE QUANTITIES CAN BE BUILT AS POLYNOMS ($\otimes$) OUT OF 24 BASIC FUNCTIONS @ $O(\alpha^2)$.

3. THE CORRESPONDING RING CAN BE REPRESENTED BY THE FINITE HARMONIC SUMS. $O(\alpha^2) \leq 4$ FOLD H-SUMS.

4. A DICTIONARY OF MELLIN TRANSFORMS IS PROVIDED.

5. THE LINEAR REPRESENTATIONS OF H-SUMS CAN BE SIGNIFICANTLY REDUCED APPLYING ALGEBRAIC RELATIONS DUE TO COMPLETE AND PARTIAL PERMUTATION SYMMETRY.

6. THE RESULTS OF THE ANALYSIS ARE APPLIED IN SCHEME-INDEP. EVOLUTION PROGRAMS, STRUCTURAL INV., CALCUL. OF 3-LOOP ANOM. DIMENSIONS.