Structural Relations between Harmonic Sums up to w=6

Johannes Blümlein
DESY

- Introduction
- Algebraic Relations
- Structural Relations
- Representation of some Observables
- Factorial Series
- The Basis
- Conclusions

1. Introduction

- Single scale processes in massless Quantum Field Theories, or being considered in the limit $m^2/Q^2 \rightarrow 0$, exhibit significant simplifications when calculated in Mellin space.
- This is, to some extent, due to structure of Feynman parameter integrals which possess a Mellin symmetry.
- Harmonic sums form the appropriate language to derive compact expressions in the respective calculations.
- We will line out the relations of the harmonic sums, resp. their continuations to $N \epsilon \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

x-space results :

Nielsen-type integrals, resp. harmonic polylogarithms (E. Remiddi and J. Vermaseren (1999)

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{\Gamma(n)p!q!} \int_0^1 \frac{dz}{z} \ln^{(n-1)}(z) \ln^p(1-zx) \ln^q(1+zx)$$

2 Loop Wilson Coefficients: x space

Order α_s^2 contributions to the deep inelastic Wilson coefficient

W.L. van Neerven and E.B. Zijlstra

Instituut-Lorentz, University of Leyden, P.O. Box 9506, NL-2300 RA Leyden, The Netherlands

Volume 272, number 1,2 PHYSICS LETTERS B 28 November 1991

$$
\begin{aligned}
C_2^{(2),(+)}(x,1) = C_F^2\Bigg[&\frac{1+x^2}{1-x}\{4\ln^3(1-x)-(14\ln x+9)\ln^2(1-x)\\
&-[4\,\mathrm{Li}_2(1-x)-12\ln^2 x-12\ln x+16\zeta(2)+\tfrac{27}{2}]\ln(1-x)-\tfrac{4}{3}\ln^3 x-\tfrac{3}{2}\ln^2 x\\
&+[-24\,\mathrm{Li}_2(-x)+24\zeta(2)+\tfrac{61}{2}]\ln x+12\,\mathrm{Li}_3(1-x)-12S_{1,2}(1-x)\\
&+48\,\mathrm{Li}_3(-x)-6\,\mathrm{Li}_2(1-x)+32\zeta(3)+18\zeta(2)+\tfrac{51}{4}\}\\
&+(1+x)\{2\ln x\ln^2(1-x)+4[\mathrm{Li}_2(1-x)-\ln^2 x]\ln(1-x)\\
&-4[\mathrm{Li}_2(1-x)+\zeta(2)]\ln x+\tfrac{5}{3}\ln^3 x-4\,\mathrm{Li}_3(1-x)\}\\
&+\left(40+8x-48x^2-\tfrac{72}{5}x^3+\frac{8}{5x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]\\
&+(-8+40x)[\ln x\,\mathrm{Li}_2(-x)+S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)]+(5+9x)\ln^2(1-x)\\
&+\tfrac{1}{2}(-91+141x)\ln(1-x)-(28+44x)\ln x\ln(1-x)-(14+30x)\,\mathrm{Li}_2(1-x)\\
&+(\tfrac{29}{2}+\tfrac{25}{2}x+24x^2+\tfrac{36}{5}x^3)\ln^2 x+\frac{1}{10}\left(13-407x+144x^2-\frac{16}{x}\right)\ln x+(-10+6x-48x^2-\tfrac{72}{5}x^3)\zeta(2)\\
&+\tfrac{407}{20}-\tfrac{1917}{20}x+\tfrac{72}{5}x^2+\frac{8}{5x}+[6\zeta(2)^2-78\zeta(3)+69\zeta(2)+\tfrac{331}{8}]\delta(1-x)\Bigg]\\
+C_A C_F\Bigg[&\frac{1+x^2}{1-x}\{-\tfrac{11}{3}\ln^2(1-x)+[4\,\mathrm{Li}_2(1-x)+2\ln^2 x+\tfrac{44}{3}\ln x-4\zeta(2)+\tfrac{367}{18}]\ln(1-x)\\
&-\ln^3 x-\tfrac{55}{6}\ln^2 x+[4\,\mathrm{Li}_2(1-x)+12\,\mathrm{Li}_2(-x)-\tfrac{239}{6}]\ln x-12\,\mathrm{Li}_3(1-x)+12S_{1,2}(1-x)-24\,\mathrm{Li}_3(-x)\\
&+\tfrac{22}{3}\mathrm{Li}_2(1-x)+2\zeta(3)+\tfrac{22}{3}\zeta(2)-\tfrac{3155}{108}\}\\
&+4(1+x)[\mathrm{Li}_2(1-x)+\ln x\ln(1-x)]+\left(-20-4x+24x^2+\tfrac{36}{5}x^3-\frac{4}{5x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]\\
&+(4-20x)[\ln x\,\mathrm{Li}_2(-x)+S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)]+(\tfrac{133}{6}-\tfrac{1113}{18}x)\ln(1-x)\\
&+(-2+2x-12x^2-\tfrac{18}{5}x^3)\ln^2 x+\frac{1}{30}\left(13+1753x-216x^2+\frac{24}{x}\right)\ln x+(-2-10x+24x^2+\tfrac{36}{5}x^3)\zeta(2)\\
&-\tfrac{9687}{540}+\tfrac{59157}{540}x-\tfrac{36}{5}x^2-\frac{4}{5x}+[\tfrac{71}{5}\zeta(2)^2+\tfrac{140}{3}\zeta(3)-\tfrac{251}{3}\zeta(2)-\tfrac{5465}{72}]\delta(1-x)\Bigg]\\
+n_f C_F\Bigg(&\frac{1+x^2}{1-x}[\tfrac{2}{3}\ln^2(1-x)-(\tfrac{8}{3}\ln x+\tfrac{29}{9})\ln(1-x)-\tfrac{4}{3}\mathrm{Li}_2(1-x)+\tfrac{5}{3}\ln^2 x+\tfrac{19}{3}\ln x-\tfrac{4}{3}\zeta(2)+\tfrac{247}{54}]\\
&+\tfrac{1}{3}(1+13x)\ln(1-x)-\tfrac{1}{3}(7+19x)\ln x-\tfrac{23}{18}-\tfrac{27}{2}x+[\tfrac{4}{3}\zeta(3)+\tfrac{38}{3}\zeta(2)+\tfrac{457}{36}]\delta(1-x)\Bigg),
\end{aligned}
\qquad (9)
$$

where C_A, C_F denote the colour factors and n_f stands for the number of flavours. Here we have put $\mu^2=Q^2$. The more general case ($\mu^2\neq Q^2$) can be easily derived using renormalization group methods (see ref. [14]). In the above expression the terms of the type $\ln^i(1-x)/(1-x)$ have to be understood in the distributional sense [12]. The latter and the coefficient of the delta function can be derived from eq. (16) in ref. [13]. The second part in (8) is given by

$$
\begin{aligned}
C_2^{(2),G}(x,1) = n_f C_F\Bigg[&8(1+x)^2\\
&\times[-4S_{1,2}(-x)-4\ln(1+x)\,\mathrm{Li}_2(-x)-2\zeta(2)\ln(1+x)-2\ln x\ln^2(1+x)+\ln^2 x\ln(1+x)]\\
&+4(1-x)^2\{\tfrac{5}{6}\ln^3(1-x)-(2\ln x+\tfrac{13}{4})\ln^2(1-x)+[2\,\mathrm{Li}_2(1-x)+2\ln^2 x+4\ln x+\tfrac{7}{2}]\ln(1-x)-\tfrac{5}{12}\ln^3 x\\
&+[\mathrm{Li}_2(1-x)-4\,\mathrm{Li}_2(-x)+3\zeta(2)]\ln x-4\,\mathrm{Li}_3(1-x)-S_{1,2}(1-x)+12\,\mathrm{Li}_3(-x)+13\zeta(3)+\tfrac{13}{2}\zeta(2)\}\\
&+x^2\{\tfrac{10}{3}\ln^3(1-x)-12\ln x\ln^2(1-x)+[16\ln^2 x-16\zeta(2)]\ln(1-x)-5\ln^3 x\\
&+[12\,\mathrm{Li}_2(1-x)+20\zeta(2)]\ln x-8\,\mathrm{Li}_3(1-x)+12S_{1,2}(1-x)\}\\
&+\left(48+\tfrac{64}{3}x+\tfrac{96}{5}x^3+\frac{8}{15x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]+(14x-23x^2)\ln^2(1-x)\\
&+(-12x+10x^2)\ln(1-x)+(-24x+56x^2)\ln x\ln(1-x)+64x\,\mathrm{Li}_3(-x)+(-10+24x)\mathrm{Li}_2(1-x)\\
&+(-\tfrac{3}{2}+\tfrac{22}{3}x-36x^2-\tfrac{48}{5}x^3)\ln^2 x+\frac{1}{15}\left(-236+339x-648x^2-\frac{8}{x}\right)\ln x+(64x+36x^2)\zeta(3)\\
&+(-\tfrac{20}{3}x+46x^2+\tfrac{96}{5}x^3)\zeta(2)-\tfrac{647}{15}+\tfrac{239}{5}x-\tfrac{36}{5}x^2+\frac{8}{15x}\Bigg]\\
+n_f C_A\Bigg\{&4(1+x)^2[S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)+4S_{1,2}(-x)-2\ln x\,\mathrm{Li}_2(1-x)+4\ln(1+x)\,\mathrm{Li}_2(-x)\\
&+2\ln x\,\mathrm{Li}_2(-x)+2\zeta(2)\ln(1+x)+2\ln x\ln^2(1+x)+\ln^2 x\ln(1+x)]\\
&+8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)-\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right)-\ln(1-x)\,\mathrm{Li}_2(-x)-\ln x\ln(1-x)\ln(1+x)\right]\\
&+\left(-24+\tfrac{80}{3}x^2-\frac{16}{3x}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]+x^2[-4S_{1,2}(1-x)+16\,\mathrm{Li}_3(-x)+8\ln x\,\mathrm{Li}_2(1-x)\\
&+8\ln^2 x\ln(1+x)]+\tfrac{2}{3}(1-2x+2x^2)\ln^3(1-x)+(24x-8x^2)\ln x\ln^2(1-x)\\
&+\left(-2+36x-\tfrac{122}{3}x^2+\frac{8}{3x}\right)\ln^2(1-x)+(-4-32x+8x^2)\ln^2 x\ln(1-x)\\
&+(8-144x+148x^2)\ln x\ln(1-x)+(4+40x-8x^2)\ln(1-x)\,\mathrm{Li}_2(1-x)\\
&+(-20+24x-32x^2)\zeta(2)\ln(1-x)+\frac{1}{9}\left(-186-1362x+1570x^2+\frac{104}{x}\right)\ln(1-x)\\
&+(-4-72x+8x^2)\,\mathrm{Li}_3(1-x)+\frac{1}{3}\left(12-192x+176x^2+\frac{16}{x}\right)\mathrm{Li}_2(1-x)+\tfrac{1}{3}(10+28x)\ln^3 x\\
&+(-1+88x-\tfrac{194}{3}x^2)\ln^2 x+(-48x+16x^2)\zeta(2)\ln x+(58+\tfrac{584}{3}x-\tfrac{2090}{9}x^2)\ln x-(10+12x+12x^2)\zeta(3)\\
&+\frac{1}{3}\left(12-240x+268x^2-\frac{32}{x}\right)\zeta(2)+\tfrac{239}{9}+\tfrac{1072}{9}x-\tfrac{4493}{27}x^2+\frac{344}{27x}\Bigg\},
\end{aligned}
\qquad (5)
$$

W.L. van Neerven et al.: 79 functions   80 objects would be maximal.

2. Algebraic Relations

cf. J.Blümlein, Comput. Phys. Commun. **159** (2004) 19

Number of harmonic sums up to weight w : 3^{w-1}.
Harmonic sums form a quasi-shuffle algebra through $\sqcup\!\sqcup$. (M. Hoffman)

$$\begin{aligned} S_{a_1,a_2} \sqcup\!\sqcup S_{a_3,a_4} &= S_{a_1,a_2,a_3,a_4} + S_{a_1,a_3,a_2,a_4} + S_{a_1,a_2,a_4,a_2} \\ &+ S_{a_3,a_4,a_1,a_2} + S_{a_3,a_1,a_4,a_2} + S_{a_3,a_1,a_2,a_4} \quad etc. \end{aligned}$$

Solve all the linear equations possible for the harmonic sums $\Longrightarrow$ algebraic basis.

Let $\{a, a, a, ..., b, b, ..., ..., z, z\}$ a set of n_1 a's, n_2 b's etc. The number of basis elements corresponding to all words formed by ALL the above letters is:

$$l_n(n_1, ..., n_q) = \frac{1}{n} \sum_{d|n_i} \mu(d) \frac{(n/d)!}{(n_1/d)!...(n_d/d)!}, \quad \sum_i n_i = n$$

(E. Witt, 1937) $\Longrightarrow$ # Lyndon words

w	1	2	3	4	5	6
$\#_c$	2	8	26	80	242	728
$\#_r$	0	1	7	23	69	183

Algebraic Relations

Observation in Quantum Field Theory :

At least up to $O(\alpha_s^3)$ the contributing harmonic sums never exhibit any index $a_k = -1$ applying a compact representation.

The number of sums of this type is

$$N_{\neg\{-1\}}(w) = \frac{1}{2}\left[\left(1-\sqrt{2}\right)^w + \left(1+\sqrt{2}\right)\right]$$

$$N^{\rm basic}_{\neg\{-1\}}(\mathrm{w}) \;=\; \frac{2}{\mathrm{w}} \sum_{d|\mathrm{w}} \mu\left(\frac{\mathrm{w}}{d}\right) N^{\rm basic}_{\neg\{-1\}}(d)\ .$$

w	1	2	3	4	5	6
$\#_c$	1	4	11	28	69	168
$\#_r$	1	3	7	14	30	60

Algebraic Relations

Side Remark:

Harmonic, Generalized Harmonic Polylogarithms and Multiple Polylogarithms also form shuffle algebras. As shuffle algebras are sub-sets of the quasi-shuffle algebra studied above, the respective algebraic relations can be derived directly.

- Form the index alphabet.
- Solve the shuffle-relations $\Longrightarrow$ Basis

As the relations in J.Blümlein, Comput. Phys. Commun. **159** (2004) 19 are of arbitrary weight (general alphabet) and depth $d \leq 6$ the corresponding relations can be read off there.

Algorithms to extend this scenario are available and can be run.

3. Structural Relations

w = 1:

$$\frac{1}{1-x} \qquad \& \qquad \frac{1}{1+x}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$$

$$\mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right]\left(\frac{N}{2}\right) = \mathbf{M}\left[\left(\frac{1}{1-x}\right)_+\right](N) + \mathbf{M}\left[\frac{1}{1+x}\right](N) + \ln(2)$$

$$-\psi\left(\frac{N}{2}\right) - \gamma_E = -\psi(N) - \gamma_E + \beta(N) + \ln(2)$$

$$\beta(N) = \frac{1}{2}\left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right)\right]$$

- $S_{-1}(N)$ depends on $S_1(N)$ for $N \epsilon \mathbf{Q}$

Structural Relations

$N \epsilon \mathbf{R}$:

$$S_2(N) = -\frac{d}{dN} S_1(N) + \zeta_2 \quad \text{(etc.)}$$

For $N \epsilon \mathbf{R}$: only one independent single sum occurs.

$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} = \psi(N+1) + \gamma_E$$

w = 2:

$$\mathbf{M}\left[\frac{\ln(1-x)}{1+x}\right](N) = -\mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) - [\psi(N) + \gamma_E + \ln(2)]\beta)N) + \beta'(N)$$

See also relations for Nielsen´s ξ, η, ξ_1 and ξ_2 functions.

$$F_1(N) := \mathbf{M}\left[\frac{\ln(1+x)}{1+x}\right](N) \rightarrow S_{1,-1}(N)$$

Structural Relations

The Reduction for $\mathrm{Li}_k(-x)/(x \pm 1)$:

$$\frac{1}{2^{k-2}} \frac{\mathrm{Li}_k(x^2)}{1-x^2} = \frac{\mathrm{Li}_k(x)}{1-x} + \frac{\mathrm{Li}_k(x)}{1+x} + \frac{\mathrm{Li}_k(-x)}{1-x} + \frac{\mathrm{Li}_k(-x)}{1+x} \rightarrow \frac{\mathrm{Li}_k(-x)}{1-x}$$

- There always exists another IBP relation to express also $\mathrm{Li}_k(-x)/(1+x)$
- At even k there exists an algebraic relation which yields an additional relation for $\mathrm{Li}_k(x)/(1+x)$.
- Applying differential operators one may show :

For $N \epsilon \mathbf{R}$ double harmonic sums can always be represented by one basic function for even weight and two basic functions for odd weight).

w = 3:

$$\rightarrow \frac{\mathrm{Li}_2(x)}{x \pm 1}, \qquad \frac{\ln^2(1+x)}{x \pm 1}$$

Structural Relations

w = 4; $i \neq -1, \rightarrow$

$$\frac{\mathrm{Li}_3(x)}{x+1}, \qquad \frac{S_{1,2}(x)}{x \pm 1}$$

w = 5; $i \neq -1, \rightarrow$

$$\frac{\mathrm{Li}_4(x)}{x \pm 1} \qquad \frac{S_{1,3}(x)}{x+1} \qquad \frac{S_{2,2}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2^2(x)}{x+1} \qquad \frac{S_{2,2}(x) - \mathrm{Li}_2^2(x)/2}{x \pm 1}$$

w = 6; $i \neq -1, \rightarrow$

$$\frac{\mathrm{Li}_5(x)}{x+1} \qquad \frac{S_{3,2}(x)}{x \pm 1} \qquad \frac{S_{2,3}(x)}{x \pm 1} \qquad \frac{S_{1,4}(x)}{x \pm 1} \qquad \frac{\mathrm{Li}_2(x)\mathrm{Li}_3(x)}{x \pm 1}$$

$$\frac{\mathrm{Li}_2(-x)\mathrm{Li}_3(-x)}{x-1} \qquad \frac{S_{3,2}(-x)}{x-1} \qquad \frac{A_1(x)}{x+1} \qquad \frac{A_1(-x)}{x-1} \qquad \frac{A_2(x)}{x+1}, + \ldots$$

Structural Relations

New numerator functions :

$$A_1(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_2^2(y)$$

$$A_2(x) = \int_0^x \frac{dy}{y} [\mathrm{Li}_4(1-y) - \zeta_4], \ldots$$

Representation of some Observables

- Unpolarized and Polarized Drell-Yan an Higgs-Boson Production Cross Section $O(\alpha_s^2)$ w = 4 J.B. and V. Ravindran, Nucl. Phys. **B716** (2005) 128.
- Unpolarized and Polarized Time-like Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^2)$ w = 4 J.B. and V. Ravindran, Nucl. Phys. **B749** (2006) 1.
- Anomalous Dimensions and Wilson Coefficients $O(\alpha_s^3)$ w = 5, 6 from: S. Moch, J. Vermaseren, A. Vogt, Nucl. Phys. **B688** (2004) 101; **691** (2004) 129; **B724** (2005) 3 $\longrightarrow$ J.B., DESY 07-042
- Polarized and Unpolarized Wilson Coefficients $O(\alpha_s^2)$ w = 4 J.B. and S. Moch, to appear
- Polarized and Unpolarized asymptotic Heavy Flavor Wilson Coefficients $O(\alpha_s^{2(3)})$ w = 4 J.B., A. de Freitas, W. van Neerven, S. Klein, Nucl. Phys. **B755** (2006) 272; I. Bierenbaum, J.B., S. Klein, DESY 07-026, J.B. and S. Klein, DESY 07-027
- Virtual and soft corrections to Bhabha Scattering $O(\alpha^2)$ w = 4 J.B. and S. Klein

Example: Bhabha s+v

$$
\begin{aligned}
T_0 \; = \;& \frac{248+15\,N^2+N^4}{2(N-2)(N-1)N(N+1)(N+2)} S_{1,1,1,1}(N) + \frac{-2}{(N-1)(N+1)} \mathbf{S_{2,1,1}}(N) \\
&+ \frac{-340+120\,N+17\,N^2+18\,N^3-31\,N^4}{2(N-2)(N-1)N(N+1)(N+2)} S_{3,1}(N) + \frac{1344-502\,N-69\,N^2-2\,N^3+57\,N^4}{8(N-2)(N-1)N(N+1)(N+2)} S_4(N) \\
&+ \frac{304-328\,N-500\,N^2+330\,N^3-6\,N^4+6\,N^5-2\,N^6+4\,N^7}{(N-2)^2(N-1)^2N^2(N+1)(N+2)} \mathbf{S_{2,1}}(N) \\
&+ \frac{-112-4\,N^2-4\,N^4}{(N-2)(N-1)N(N+1)(N+2)} S_{2,1}(N)\mathbf{S_1}(N) + \frac{-48+8\,N+6\,N^2+7\,N^3}{(N-1)N(N+1)(N+2)} S_3(N)S_1(N) \\
&+ \frac{-1840+292\,N+5532\,N^2+827\,N^3-1978\,N^4-274\,N^5+36\,N^6+19\,N^7-22\,N^8}{4(N-2)^2(N-1)^2N^2(N+1)^2(N+2)} S_{1,1,1}(N) \\
&+ \frac{128-56\,N-252\,N^2+54\,N^3+177\,N^4-91\,N^5+19\,N^6+9\,N^7}{2(N-2)(N-1)^2N^2(N+1)^2(N+2)} S_3(N) \\
&+ \frac{4032-2048\,N-14200\,N^2+5036\,N^3+23610\,N^4+2521\,N^5-12342\,N^6}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)} S_{1,1}(N) \\
&+ \frac{-3365\,N^7+2148\,N^8+903\,N^9+14\,N^{10}-167\,N^{11}+50\,N^{12}}{4(N-2)^3(N-1)^3N^3(N+1)^3(N+2)} S_{1,1}(N) \\
&+ \frac{-124+16\,N+24\,N^2-4\,N^3-14\,N^4}{(N-2)(N-1)N(N+1)(N+2)} S_{1,1}(N)\zeta(2) + \frac{424-118\,N+9\,N^2-2\,N^3+23\,N^4}{4(N-2)(N-1)N(N+1)(N+2)} S_2(N)S_{1,1}(N) \\
&+ \frac{224+144\,N-1216\,N^2-56\,N^3+1786\,N^4+641\,N^5-406\,N^6}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)} S_2(N) \\
&+ \frac{+17\,N^7-308\,N^8+141\,N^9-56\,N^{10}+N^{11}}{4(N-2)^2(N-1)^3N^3(N+1)^3(N+2)} S_2(N) + \frac{58+21\,N+N^2+15\,N^3+10\,N^4}{(N-2)(N-1)N(N+1)(N+2)} S_2(N)\zeta(2)
\end{aligned}
$$

Example: Bhabha s+v

$$+\frac{232-384N^2-17N^3+286N^4-128N^5-14N^6+N^7}{4(N-2)(N-1)^2N^2(N+1)^2(N+2)}S_2(N)S_1(N)$$
$$+\frac{-560-26N-31N^2-10N^3-33N^4}{8(N-2)(N-1)N(N+1)(N+2)}S_2(N)^2$$
$$+\frac{576+1088N-3280N^2-5136N^3+11764N^4+20392N^5-17385N^6-30114N^7}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)$$
$$+\frac{+5984N^8+17228N^9-1228N^{10}-2754N^{11}-112N^{12}-8N^{13}+33N^{14}-24N^{15}}{4(N-2)^3(N-1)^4N^4(N+1)^4(N+2)}S_1(N)$$
$$+\frac{-56+336N+522N^2+424N^3-53N^4-500N^5+60N^6+28N^7-5N^8}{2(N-2)^2(N-1)^2N^2(N+1)^2(N+2)}S_1(N)\zeta(2)$$
$$+\frac{64+6N^2+N^3}{(N-2)(N-1)N(N+1)}S_1(N)\zeta(3)+\frac{2112+608N+76N^2-140N^3+107N^4}{10(N-2)(N-1)N(N+1)(N+2)}\zeta(2)^2$$
$$+\frac{-224-136N+1688N^2+1290N^3-1998N^4-1997N^5+198N^6}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)$$
$$+\frac{+405N^7+376N^8-119N^9+56N^{10}+5N^{11}}{2(N-1)^3N^3(N-2)^2(N+2)(N+1)^3}\zeta(2)$$
$$+\frac{-552+144N+1654N^2-370N^3-361N^4+19N^5+35N^6-25N^7}{2(N-2)^2(N-1)^2N^2(N+1)^2}\zeta(3)$$
$$+\frac{320-64N-1920N^2+1600N^3+6524N^4-14872N^5-19036N^6+31543N^7-43960N^8-13935N^9}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}$$
$$+\frac{+65372N^{10}+26822N^{11}-44576N^{12}-9558N^{13}+9840N^{14}+339N^{15}+428N^{16}-371N^{17}+128N^{18}}{16(N-1)^5(N+1)^5(N-2)^3N^5(N+2)}$$
$$+4\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,2}+(-2)\frac{N^4-N^2+12}{(N-2)(N-1)N(N+1)(N+2)}f_{0,1}^2$$

z-space: A. Penin, (2005)

$\Longrightarrow$ 3 basic sums only; no alternating sums.

4. Factorial Series

Consider

$$\Omega(z) = \int_0^1 dt\, t^{z-1}\, \varphi(t); \qquad \varphi(1-t) = \sum_{k=0}^{\infty} a_k t^k$$

$$Re(z) > 0, \qquad \Omega(z) = \sum_{k=0}^{\infty} \frac{a_{k+1} k!}{z(z+1)\ldots(z+k)}$$

- $\Omega(z)$ is meromorphic in $z \epsilon \mathbf{C}$, obeys a recursion $z \rightarrow z+1$ and has an analytic asymptotic representation.

Example:

$$\begin{aligned}
F_5(z) &= \mathbf{M}\left[\frac{\mathrm{Li}_2(z)}{1+z}\right](z) \\
F_5(z+1 &= -F_5(z) + \frac{1}{z}\left[\zeta_2 - \frac{\psi(z+1)+\gamma_E}{z}\right] \\
\mathrm{Asymp.\ ser.}: \mathrm{Li}_2(z) &\rightarrow \mathrm{Li}_2(1-z) \\
\mathbf{M}\left[\frac{\mathrm{Li}_2(1-z)}{1+z}\right](N) &\propto \frac{1}{2N^2} + \frac{1}{4N^3} - \frac{7}{24}\frac{1}{N^4} - \frac{1}{3}\frac{1}{N^5} + \frac{73}{120}\frac{1}{N^6}\ldots
\end{aligned}$$

5. The Basis

$$
\begin{array}{llll}
w=1 & 1/(x-1)_+ & & \\
w=2 & \ln(1+x)/(x+1) & & \\
w=3 & \mathrm{Li}_2(x)/(x\pm1) & & \\
w=4 & \mathrm{Li}_3(x)/(x+1) & S_{1,2}(x)/(x\pm1) & \\
w=5 & \mathrm{Li}_4(x)/(x\pm1) & S_{1,3}(x)/(x+1) & S_{2,2}(x)/(x\pm1) \\
 & \mathrm{Li}_2^2(x)/(x+1) & [S_{2,2}(-x)-\mathrm{Li}_2^2(-x)/2]/(x\pm1) & \\
w=6 & \mathrm{Li}_5(x)/(x+1) & S_{1,4}(x)/(x\pm1) & S_{2,3}(x)/(x\pm1) \\
 & S_{3,2}(x)/(x\pm1) & \ldots &
\end{array}
$$

- $O(\alpha)$ Wilson Coefficients/anom. dim. $\#1$
- $O(\alpha^2)$ Anomalous Dimensions $\#2$
- $O(\alpha^2)$ Wilson Coefficients $\# \leq 5$
- $O(\alpha^3)$ Anomalous Dimensions $\#15$
- $O(\alpha^3)$ Wilson Coefficients $\#29+$

6. Conclusions

- The single-scale quantities in Quantum Field Theories to 3 Loop Order $\Leftrightarrow$ w = 6 can be represented in a polynomial ring spanned by a few Mellin transforms of the above basic functions, which are the same for all known processes. This points to their general nature.
- The basic Mellin transforms are meromorphic functions with single poles at the non-positive integers.
- The total amount of harmonic sums reduces due to algebraic relations (index structure), and structural relations N ϵ $\mathbf{Q}$, N ϵ $\mathbf{R}$.
- They can be represented in terms of factorial series up to simple "soft components". This allows an exact analytic continuation.
- Up to w = 6 physical (pseudo-) observables are free of harmonic sums with index = {-1}. Up to w = 5 all numerator functions are Nielsen integrals.