Small $x$ Contributions to Structure Functions using $k_\perp$ Factorization

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1 Introduction

- In the small $x$ range a **novel** behaviour of nucleon structure functions is expected. Among the possible dynamical effects are: Screening and effects due to non strong $k_\perp$ ordering.

- This requires to consider rather:

$$ \int \! dK^2 \frac{\partial xG(x, K^2)}{\partial K^2} $$

than

$$ xG(x, \mu^2) $$

(CF. S. DREUILLAT)

(1)

(2)

- To obtain results which are **consistent with perturbative** QCD a consistent factorization relation has to be derived.

- Furthermore, approximations of the $x$ behaviour of the $k_\perp$ dependent coefficient functions or (additional) low $Q^2$ 'cuts' bear the danger to obtain partly unphysical and misleading results (e.g. negative structure functions).

- The goal of the present paper is to find a consistent solution of the above problems and to extend earlier investigations (LEVIN AND RYSKIN 1991, ASKEW ET AL. 1993).

- We aim on a **general** formulation of the gluon contribution to the structure functions which offers the possibility to **unfold** the $k_\perp$ dependence of the gluon at small $x$. Theoretical predictions of its small $x$ behaviour can thus be confronted with the data with respect to the $k_\perp$ dependence **directly**.

- **LARGE $x$:** Do we obtain the **collinear** picture? DGLAP; $\hat{\sigma}(k_{1i}^2) \gg \hat{\sigma}(0)^2$ -- $\rightarrow$ **NUM. ANALYSIS**.
HOW TO DESCRIBE A $k^2$ DEPENDENT GLUON DISTRIBUTION?

RELATION TO HIGHER ORDER CORRECTIONS IN FIXED ORDER PERTURBATION THEORY.
2 Kinematics

\[ q + k = p_1 + p_2 \]  \hspace{1cm} (3)

The 2 \to 2 process is described by the invariants:

\[ \hat{s} = (q + k)^2 = (p_1 + p_2)^2 \]
\[ \hat{t} = (k - p_1)^2 = (q - p_2)^2 \]
\[ \hat{u} = (q - p_1)^2 = (k - p_2)^2 \]  \hspace{1cm} (4)

with

\[ \hat{s} + \hat{t} + \hat{u} = -Q^2 - K^2 + m_1^2 + m_2^2 \]  \hspace{1cm} (5)

and \( Q^2 = -q^2, K^2 = -k^2, q' = q + xP. \)

\[ k_\mu = \xi q' + \eta P_\mu + k_{\perp \mu} \]  \hspace{1cm} (6)

with \( k_{\perp}.q' = k_{\perp}.P = 0 \) and

\[ x = \frac{Q^2}{2P.q} \]  \hspace{1cm} (7)

\[ \xi = \frac{2k.P}{2q'.P} \]  \hspace{1cm} (8)

\[ \eta = \frac{2k.q'}{2q'.P} \]  \hspace{1cm} (9)
\( \gamma g - CMS \)

\[ P \quad (P \text{ carries } p_\perp) \]

( \rightarrow \text{ Derive the } x \text{ dependent coefficient functions.} )

\text{Earlier: Ciafaloni et al. } 0^{th} \text{ moment}

\[ k = z_3 P + z_4 q' + k'_\perp \]

\[ q = z_1 P + z_2 q' + q'_\perp \]

One integration to be done using the 0\textsuperscript{th} moment, otherwise nontrivial in this ref. frame.

High energy limit: \( z_2 \gg z_4 \)

\( z_2 \gg z_1 \)
and

\[ K^2 = k^2 - \xi \eta \frac{Q^2}{x} \]  \hspace{1cm} (10)

with \( k_\perp^2 = -k^2 \). between \( P \) and \( k \) is needed. We will assume that

\[ \xi \ll \eta \]  \hspace{1cm} (11)

which implies

\[ \xi \approx 0 \quad P.k \approx 0 \quad k_\mu \approx \eta P_\mu + k_\perp \mu \quad K^2 \approx k^2 \]  \hspace{1cm} (12)

In the scaling limit, \( \xi = 0 \) and \( k_\perp = 0 \), the gluon momentum is

\[ k_\mu = \eta P_\mu \]  \hspace{1cm} (13)

and the relations given below simplify accordingly. The general representation of the particle momenta is thus given by:

\[ k = (K_0, 0, 0, |\vec{k}|) \]
\[ q = (Q_0, 0, 0, -|\vec{k}|) \]
\[ P = E_p(1, \sin \beta, 0, \cos \beta) \]
\[ p_1 = (E_1, q_1 \sin \theta \cos \varphi, q_1 \sin \theta \sin \varphi, q_1 \cos \theta) \]
\[ p_2 = (E_2, -q_1 \sin \theta \cos \varphi, -q_1 \sin \theta \sin \varphi, -q_1 \cos \theta) \]  \hspace{1cm} (14)

with

\[ K_0 = \mathcal{E} (\hat{s}, -K^2, -Q^2) \]
\[ Q_0 = \mathcal{E} (\hat{s}, -Q^2, -K^2) \]
\[ E_1 = \mathcal{E} (\hat{s}, m_1^2, m_2^2) \]
\[ E_2 = \mathcal{E} (\hat{s}, m_2^2, m_1^2) \]
\[ |\vec{k}| = \mathcal{P} (\hat{s}, -Q^2, -K^2) \]
\[ q_1 = \mathcal{P} (\hat{s}, m_1^2, m_2^2) \]
\[ \cos \theta = \frac{2 K_0 E_1 + K^2 - m_1^2 + \hat{t}}{2 |\vec{k}| q_1} \]
\[ E_p = \mathcal{E}(\hat{s}, 0, t_{qP'}) = \frac{Q^2}{2x\sqrt{\hat{s}}} \]
\[ \cos \beta = \frac{K_0}{|\hat{k}|} = \frac{1 - \zeta/2}{\sqrt{1 - x\zeta}} \] (15)

where
\[ \zeta = \frac{4K^2x}{Q^2} \] (16)
\[ \mathcal{E}(a, b, c) = \frac{a + b - c}{2\sqrt{a}} \]
\[ \mathcal{P}(a, b, c) = \sqrt{\frac{\lambda(a, b, c)}{4a}} \] (17)
\[ \lambda(a, b, c) = (a - b - c)^2 - 4bc \] (18)
\[ t_{qP'} = (q - P')^2 = \hat{s} - Q^2/x \] (19)

\( \varphi \) is defined by
\[ \cos \varphi = \left. \frac{(q \times P) \cdot (q \times p_1)}{|q \times P| |q \times p_1|} \right|_{p_1 = -p_2} \] (20)

In the limit \( m_{1,2} \to 0 \) the 3-particle phase space is described by
\[
dPS^{(3)} = \frac{1}{128\pi^3} \frac{d\varphi_p}{2\pi} \frac{d\hat{s}d\hat{t}dK^2}{\lambda^{1/2}(\hat{s}, -K^2, -Q^2)\lambda^{1/2}(s_\gamma, 0, -Q^2)/2\pi} \times \Theta\{-G(s_\gamma, -K^2, \hat{s}, 0, -Q^2, 0)\} \times \Theta\{-G(\hat{s}, \hat{t}, 0, -K^2, -Q^2, 0)\}\Theta\{\hat{s}\} \] (21)

with \( G(x, y, z, u, v, w) \) the Caley determinants.

\[
\int dPS^{(3)} = \frac{1}{128\pi^3} \int_{\eta_{\text{max}}}^{\eta_{\text{min}}} d\eta \int_{K_{\text{max}}^{(\eta)}(\eta)}^{K_{\text{min}}^{(\eta)}(\eta)} dK^2 \int_0^{2\pi} \frac{d\varphi_P}{2\pi} \frac{1}{2} \int_{-1}^{1} d\cos \theta \int_0^{2\pi} \frac{d\varphi}{2\pi} \] (22)
The variables in (22) are bounded by:

\[ K_{\text{min}}^2 = 0 \]
\[ K_{\text{max}}^2 = Q^2 \eta - x \]
\[ \eta_{\text{min}} = x \]
\[ \eta_{\text{max}} = 1 \]  \hspace{1cm} (23)
3 Factorization

- The $k_\perp$ integration starts at $K^2 = 0$. However, a physical definition of $\partial x G(x, K^2) / \partial K^2$ is ONLY possible for $K^2 \geq Q_0^2$!

- The naive $k_\perp$ factorization fails.

$$F_i(x, Q^2) = \int_0^1 \int_0^1 dx_1 dx_2 \delta(x - x_1 x_2) \int d^2 k f_i(x_1, K^2/Q^2) \frac{\partial x_2 G(x_2, K^2)}{\partial k^2}$$

(If the $k_\perp$ distribution of the gluon is infrared finite there is no problem!)

One observes:

\[\text{for: } Q_0^2 \ll Q^2\]

one has for $K^2 \leq Q_0^2$: $f_i^G(K^2/Q^2, z) \Rightarrow f_i^G(K^2/Q^2 \rightarrow 0, z)$

The $k_\perp$ Factorization Relation is thus:

$$F_L^q(x, Q^2) = \int_z^1 \frac{dz}{z} f_L^q(z) \frac{x}{z} G(z, Q_0^2)$$

$$+ \int_z^1 \frac{dz}{z} \int_{Q_0^2}^{K_{\text{max}}^2} dK^2 f_L^q(z, \frac{K^2}{Q^2}) \theta(K_{\text{max}}^2 - Q_0^2) \frac{\partial G(x/z, K^2)}{\partial K^2}$$

$$F_2^q(x, Q^2) = F_2^q,\text{coll}(x, Q^2)$$

$$+ \int_z^1 \frac{dz}{z} \int_{Q_0^2}^{K_{\text{max}}^2} dK^2 \left\{ f_2^q(z, \frac{K^2}{Q^2}) - f_2^q(z, \frac{\Lambda^2}{Q^2}) \theta(Q^2 - K^2) \right\}$$

$$\times \left( \frac{x}{z} \frac{\partial G(x/z, K^2)}{\partial K^2} \right) \theta(K_{\text{max}}^2 - Q_0^2)$$

The Scale of $\alpha_s$:

For the present leading order calculation we have choosen: $Q^2$.

Other characteristic scales would do as well.
4 The Structure Functions in the $k_\perp$ Factorization Scheme in $\mathcal{O}(\alpha_s)$

\[
\frac{d^2\sigma}{dQ^2dy} = 2\pi\alpha^2 \frac{Ms}{(s-M^2)^2} \frac{1}{Q^4} L_{\mu\nu} W_{\mu\nu} \tag{26}
\]

\[
L_{\mu\nu} = 2 \left[ l_{\mu} l_{\nu} + l_{\mu} l_{\nu} - g_{\mu\nu} l_{\nu} l_{\nu} \right] \]

\[
W_{\mu\nu} = \frac{1}{4\pi} \sum_n \langle P | J^{em}_\mu(0) | n \rangle \langle n | J^{em}_\nu(0) | P \rangle (2\pi)^4\delta(4)(P + q - p_n) \tag{27}
\]

\[
W_{\mu\nu} = \left( -g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{q^2} \right) W_1(x, Q^2) + \frac{1}{M^2} \left[ \left( P_{\mu} - \frac{P \cdot q}{q^2} q_{\mu} \right) \left( P_{\nu} - \frac{P \cdot q}{q^2} q_{\nu} \right) \right] W_2(x, Q^2) \tag{28}
\]

\[
F_2(x, Q^2) = x T^1_{\mu\nu} W_{\mu\nu} = x \left( -g_{\mu\nu} + \frac{12x^2}{Q^2} P_{\mu} P_{\nu} \right) W_{\mu\nu} \]

\[
F_L(x, Q^2) = x T^2_{\mu\nu} W_{\mu\nu} = \frac{8x^3}{Q^2} P_{\mu} P_{\nu} W_{\mu\nu} \tag{29}
\]

\[
-g_{\mu\nu} \tilde{W}_{\mu\nu} = 32\pi\alpha_s e^2_q \left\{ \frac{(p_1 \cdot P)^2 + (p_2 \cdot P)^2}{t \bar{u}} - \frac{Q^2}{K^2} \left[ \frac{p_1 \cdot P}{t} - \frac{p_2 \cdot P}{\bar{u}} \right]^2 \right\} \tag{30}
\]

\[
P_{\mu\nu} P^\mu P^\nu \tilde{W}_{\mu\nu} = 64\pi\alpha_s e^2_q \frac{1}{K^2} \left\{ -2 \frac{(p_1 \cdot P)^2(p_2 \cdot P)^2}{t \bar{u}} + \frac{(p_1 \cdot P)^3(p_2 \cdot P)}{t^2} + \frac{(p_1 \cdot P)(p_2 \cdot P)^3}{\bar{u}^2} \right\} \tag{31}
\]

\[
f^g_L(z, \frac{K^2}{Q^2}) = \frac{2}{\pi} \alpha_s(Q^2) \sum_{q=1}^{N_f/2} \left( e_{q_u}^2 + e_{q_d}^2 \right) \left\{ \frac{1}{64z} \left( \frac{Q^2}{K^2} \right)^2 G_{1L}(\omega, \beta) \right. \]
\[
+ \left. \frac{1}{16K^2} \left[ G_{2L}(\omega, \beta) + G_{3L}(\omega, \beta) \log \frac{1 - \omega}{1 + \omega} \right] \right\} \tag{32}
\]
with

\[
\omega = \sqrt{1 - 4K^2z/Q^2} \tag{33}
\]

\[
f^g_L \left( z, \frac{K^2}{Q^2} \rightarrow 0 \right) = \frac{2}{\pi} \alpha_s(Q^2) \sum_{q=1}^{N_f/2} \left( e_{q_u}^2 + e_{q_d}^2 \right) \frac{N_f/2}{1} \Xi^2(1 - \Xi) \]
\[
G_{1L}(\omega, \beta) = \sum_{k=0}^{4} \frac{1}{\omega^k} g_{1k}^{(L)}(\beta)
\]
(34)
\[
G_{2L}(\omega, \beta) = \sum_{k=2}^{4} \frac{1}{\omega^k} g_{2k}^{(L)}(\beta)
\]
(35)
\[
G_{3L}(\omega, \beta) = \sum_{k=1}^{5} \frac{1}{\omega^k} g_{3k}^{(L)}(\beta)
\]
(36)

and

\[
\begin{align*}
g_{10}^{(L)}(\beta) &= \frac{5}{2} + 3 \cos^2 \beta - \frac{3}{2} \cos^4 \beta \\
g_{11}^{(L)}(\beta) &= 4 \cos \beta - 12 \cos^3 \beta \\
g_{12}^{(L)}(\beta) &= 3 - 18 \cos^2 \beta + 15 \cos^4 \beta \\
g_{13}^{(L)}(\beta) &= -12 \cos \beta + 20 \cos^3 \beta \\
g_{14}^{(L)}(\beta) &= -\frac{3}{2} + 15 \cos^2 \beta - \frac{35}{2} \cos^4 \beta \\
g_{20}^{(L)}(\beta) &= 4 - 12 \cos^2 \beta \\
g_{21}^{(L)}(\beta) &= -24 \cos \beta + 40 \cos^3 \beta \\
g_{22}^{(L)}(\beta) &= -3 + 30 \cos^2 \beta - 35 \cos^4 \beta \\
g_{30}^{(L)}(\beta) &= \frac{3}{4} + \frac{1}{2} \cos^2 \beta + \frac{3}{4} \cos^4 \beta \\
g_{31}^{(L)}(\beta) &= 2 \cos \beta - 6 \cos^3 \beta \\
g_{32}^{(L)}(\beta) &= \frac{7}{2} - 15 \cos^2 \beta + \frac{15}{2} \cos^4 \beta \\
g_{33}^{(L)}(\beta) &= -18 \cos \beta + 30 \cos^3 \beta \\
g_{34}^{(L)}(\beta) &= -\frac{9}{4} + \frac{45}{2} \cos^2 \beta - \frac{105}{4} \cos^4 \beta \\
\end{align*}
\]  
(37)

\[
\cos \beta = \frac{1 - \frac{2k^2 \bar{\tau}}{Q^2}}{\sqrt{1 - \frac{4k^2 \bar{\tau}^2}{Q^2}}}
\]
\[
\omega = \sqrt{1 - \frac{4k^2 \bar{\tau}^2}{Q^2}}
\]
5 Numerical Results

• CONSIDER $F_2(x, Q^2)$ AS AN EXAMPLE.

• $\frac{\partial x G(x, k^2)}{\partial \log k^2}$ = $D^-$

DO'\ 
D^-

SMOOTHED BY
MRSA
CTEQ2M
GRV HO

NEW FIT!

• $x$ DEPENDENCE
• $Q^2$ DEPENDENCE OF THE $k_1$ EFFECTS
• $Q^2_0$ DEPENDENCE ($?\!?$)

⇒ KIN. EFFECT TAKING INTO ACCOUNT THE $k_1^2$
LIMITS CORRECTLY.
$K_1$ DEPENDENT GLUON DISTRIBUTION

**How to describe it?**

$K_1$ dependence should depend on the dynamics accounted for:

- AP
- BFKL

- Unfold $F_{g^2}(x_1, K_1^2, Q_1^2, P_2^2)$ experimentally from 2 jet cross sections, knowing $\hat{\sigma}(x_1, Q_1^2, K_1^2)$!

- Which gluong distributions are covered?

  → **Single gluon distributions**

  Factorized by:

  $$
  \sum_{\lambda} E_{\nu}^*(\lambda) E_{\nu}^0(\lambda) = \frac{P_\nu P_{\lambda}}{k^2}.
  $$

  → Includes both the cases of the Lipatov and the AP dynamics.
A. GLAP EQUATIONS:

(moments)

\[
\left( \frac{\partial \Sigma_n}{\partial K^2} \right) = \frac{\alpha_s(K^2)}{2\pi K^2} \begin{pmatrix} P_{qq}^{(n)} & P_{qqG}^{(n)} \\ P_{Gq}^{(n)} & P_{GG}^{(n)} \end{pmatrix} \begin{pmatrix} \Sigma_n \\ G_n \end{pmatrix}
\]

**GLUONS:**

\[
\frac{\partial G_n}{\partial K^2} \approx \frac{\alpha_s}{2\pi} \frac{2C_6}{K^2} \left[ \frac{1}{n-1} + \frac{1}{2C_6} \left( \frac{P_{GG}^{(n)}}{n-1} - \frac{2C_6}{n-1} \right) \right] G_n(K^2)
\]

\[+ \frac{\alpha_s N_c}{2\pi} \frac{1}{k^2} \left( \frac{2}{n-1} + \ldots \right) \Sigma_n(K^2)\]

\[
\frac{\partial G_n}{\partial K^2} \approx \frac{\alpha_s}{K^2} \frac{1}{n-1} \frac{G_n}{G_n} \quad \text{(DOUBLE LOG)}
\]

\[
\overline{\alpha_s} = \frac{3\alpha_s}{\pi}.
\]

\[
\text{(ONE EXPECTS } \propto \text{ CONST. SLOPES FOR } G_n \text{ VS } Q^2). \quad \text{AT } x = \text{ const.}
\]

"""
$d^2 x G(x, Q^2) / d log Q^2$

$x = 10^{-4}$

- GRV HO
- D" prime
- CTEQ2M
- MRSA
- D0 prime
$dxG(x,Q^2)/d\log Q^2$

$x = 10^{-3}$

GRV HO
CTEQ2M
D$^+$ prime
MRSA
D0 prime
GLAP DYNAMICS:

\[ F_g(x, K^2) = \frac{\partial \Theta G(x, K^2)}{\partial K^2} \quad \text{WITH} \]

\[ \int_0^{K^2} dK'^2 \frac{\partial \Theta G(x, K'^2)}{\partial K'^2} = \Theta G(x, K^2) \quad \text{FOR} \quad K^2 \text{ IN THE PERTURBATIVE RANGE.} \]

\[ \rightarrow \quad F_L(x, Q^2) = \int_L^{Q^2} f_L^0(x) \otimes \Theta G(x, Q_0^2) \]

\[ + \int_{Q^2}^{K_{\text{max}}} \frac{dK^2}{K^2} \int_{Q_0^2}^{K^2} f_L^0(\xi, \frac{K^2}{Q_0^2}) \frac{\partial \Theta G(x/\xi, K^2)}{\partial K^2} \Theta(K_{\text{max}} - Q_0^2). \]

HOWEVER: GOING TO LARGE $x \gtrsim O(0.1)$

$\partial G(x, K^2)/\partial K^2$ BECOMES NEGATIVE

(FORTUNATELY IT IS ALSO SMALL!)

\[ \leftrightarrow \quad F_L \text{ STILL IS POSITIVE. (AKMS $\rightarrow$ OTHER APPROACH, CUT AT $Q_0^2$ WOULD OBTAIN NEGATIVE VALUES).} \]

\[ \rightarrow \quad F_g \text{ HAS TO BE DIFFERENT FROM $\frac{\partial \Theta G}{\partial K^2}$.} \]
\[ \frac{dG(x, Q^2)}{d \log Q^2} \]

\[ Q^2 = 20 \text{ GeV}^2 \]

- GRV HO
- D\(^\prime\) prime
- CTEQ2M
- D0 prime

$x$
B. BFKL EQUATION:

Consider again:
\[
\frac{\partial G_n^{(\alpha_s)}}{\partial k^2} = \frac{\partial G_n^{(\alpha_s)}}{\partial k^2} \left( \frac{1}{n-1} \right) G_n(n^2) = F_{g}^{\alpha_s}(2k^2, n^2)
\]

For \( \alpha_s \ll 1 \) this is the 1st expansion term of
\[
F_{g}^{\alpha_s}(k^2, n^2) = \gamma(\tilde{\alpha}_s) \frac{\tilde{\alpha}_s}{k^2} \left( \frac{k^2}{\mu^2} \right) \tilde{G}(\mu^2)
\]
(COLLINS, ELLIS)

\[
\begin{align*}
n-1 - \tilde{\alpha}_s \chi(\gamma(\tilde{\alpha}_s)) &= 0 \\
\chi(\beta) &= 2\psi(1) - \psi(\beta) - \psi(1-\beta)
\end{align*}
\]

Implicit equation for the BFKL 'anomalous' dimension: More precisely one should call this BFKL resummation of the \((\alpha_s/N)^n\) terms in the matrix element \(\gamma_{GG}\) of the twist-2 anomalous dimension.

In this way information of the BFKL equation is incorporated into \(F_{g}^{\alpha_s}(k^2, n^2)\).

Representation:
\[
F_{g}^{\alpha_s}(k^2, n^2) = \gamma(\tilde{\alpha}_s) \frac{1}{k^2} \left( \frac{k^2}{\mu^2} \right) \tilde{G}(\mu^2)
\]
\[
\tilde{G}(\mu^2) = \left[ \frac{\ln(k^2)}{\ln(\mu^2)} \right] d_{T(x)}^{(\alpha_s)} \exp \left[ \frac{1}{b(1-1)} \right] d\chi \gamma \frac{dX(\chi)}{d\gamma} \right] \Theta(\chi)
\]
\[
d_{T(x)} = c - \frac{1}{b(1-1)}
\]
MORE GENERALLY ONE HAS TO ACCOUNT FOR
SFKL & DGLAP EFFECTS IN A COMBINED
EVOLUTION EQU.

- G. MARCHESINI) 'QCD AT 200 TEV', (PLENUM, NY, 1992), P. 183

\[
\begin{align*}
\tilde{F}(x, \alpha_t, \overline{Q}) &= 8(1-x) \delta(\alpha_t) \Delta_s(\overline{Q}, Q_s) \\
&\quad + \int \Delta_s(\overline{Q}, zq) \theta(Q - zq) \frac{dz}{z} \frac{d^2q}{Q^2} \theta(1-z - \frac{Q^2}{9}) \cdot \tilde{P}_g(zq, Q_t) \tilde{F} \left( \frac{z}{z_t}, \alpha_t - (1-z)q, q \right)
\end{align*}
\]

\[
\tilde{P}_g(zq, Q_t) = \left[ \frac{\alpha_s}{1-z} + \frac{\alpha_s}{z} \Delta_{ns}(zq, Q_t) \right]
\]

\[
\Delta_{ns}(zq, Q_t) = \exp \left[ - \alpha_s \int \frac{dz}{z} \int \frac{dq^2}{Q^2} \right]
\]

\[
\Delta_s(Q, zq) = \exp \left[ - \int \frac{dq^2}{(2q^2)} \int \frac{dx^2}{Q^2} \frac{dx}{1-z} \right]
\]

\[
\rightarrow \tilde{F} \text{ DESCRIBES AN EVOLUTION KERNEL;}
\]

(TO BE CONVOLUTED WITH AN INPUT DISTRIBUTION.)
\[ F_{L}^{g_{10}}(x, Q^2) = f_{L}^{g_{10}}(x, Q^2) \otimes G(x, Q^2) \]

\[ F_{L}^{g_{15}}(x, Q^2) = f_{L}^{g_{10}}(x, Q^2) \otimes \Theta(x, Q^2) \]

\[ F_{L}^{g_{1kL}}(x, Q^2) = f_{L}^{g_{10}}(x) \otimes G(x, Q_0^2) \]

\[ + \int \frac{dx}{x} \int dK^2 f_{L}^{g}(x) \frac{K_{L}}{Q_0^2} \frac{\partial G(x, z, K^2)}{\partial K^2} \Theta(K_{max} - Q^2). \]
\( Q^2 = 20 \text{ GeV}^2 \)
\( Q_0^2 = 3 \text{ GeV}^2 \)

CTEQLO

mass factorization, \( \mu^2 = Q^2 \)

mass factorization, \( \mu^2 = Q^2(1-z)/z \)

\( k_t \) factorization
$Q_0^2 = 3 \text{ GeV}^2$

CTEQLO

mass factorization, $\mu^2 = Q^2$

mass factorization, $\mu^2 = Q^2(1-z)/z$

$k_t$ factorization
The document contains a graph and mathematical expressions. The graph is labeled $F_L^g(x, Q^2)$ and appears to show a plot of $Q^2$ dependence. The mathematical expressions include:

$$F_L^g(x, Q^2)$$

And:

$$\frac{\alpha}{\alpha_0}(x, 0) + \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \delta(x - \mathbf{k}^2)$$

The expressions suggest a study of the gluon structure function in quantum chromodynamics (QCD).
Dependence on the infrared cutoff $k_0$:

- Joint effect of the special treatment of the BFKL evol. & fact. formula.
$F_L(x, Q^2)$

$q^2 = 20 \text{ GeV}^2$

mass factorization, $\mu_r = q^2$

LO, CTEQLO
NTLO, CTEQ2M

gluons --

van Neerven, MSbar

$F_L(x, Q^2)$ vs. $x$
$Q^2 = 20 \text{ GeV}^2$

van Neerven, MSbar
mass factorization, $\mu^2 = Q^2$
full : $g : O(\alpha_s^2)$ CTEQ2M

dotted : $g : O(\alpha_s)$ CTEQLO

dashed : $g : k_T$ fact. $O(\alpha_s)$ CTEQLO

dash–dotted : full $F_L O(\alpha_s^2)$ CTEQ2M

$F_L(x, Q^2)$

$0 \quad 10^{-4} \quad 10^{-3} \quad 10^{-2} \quad 10^{-1}$

$x$
6 Conclusions

1. A derivation of $k_\perp$ factorization which is consistent with perturbative QCD has been given.

2. The gluon contribution to the structure functions has been calculated using $k_\perp$ factorization without approximations of the Mellin convolution and the $x$ dependence of the coefficient functions contrasting earlier investigations. The obtained contributions to the structure functions are positive in the whole $x$ range.

3. The derived coefficient functions approach those found using mass factorization in the limit $K^2 \to 0$.

4. The numerical result obtained for $F_1(x, Q^2)$ in $k_\perp$ factorization for suitably 'large' values of $x$ approaches the result obtained ignoring the $k_\perp$ dependence of the coefficient functions. (This is an expectation in the parton model.)

5. The numerical results behave very stable against the choice of the scale $Q_0^2$. There is essentially no change for all $x$, as long as the condition $Q^2 \gg Q_0^2$ is met. (As we do not provide a thorough inclusion of higher twist effects this is a natural choice.)

6. There is no fixed onset (e.g. $x \sim 10^{-2}$) of small $x$ effects observed. Deviations from the $k_\perp \to 0$ results become smaller with rising $Q^2$ at $x = \text{const}$.

7. The $k_\perp$ dependence of the coefficient functions and gluon distribution results into smaller values of the structure functions in the small $x$ range. A similar behaviour has been observed recently by Van Neerven et al. in a higher order calculation using mass factorization.