QCD Evolution of the Structure Functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ at Small x

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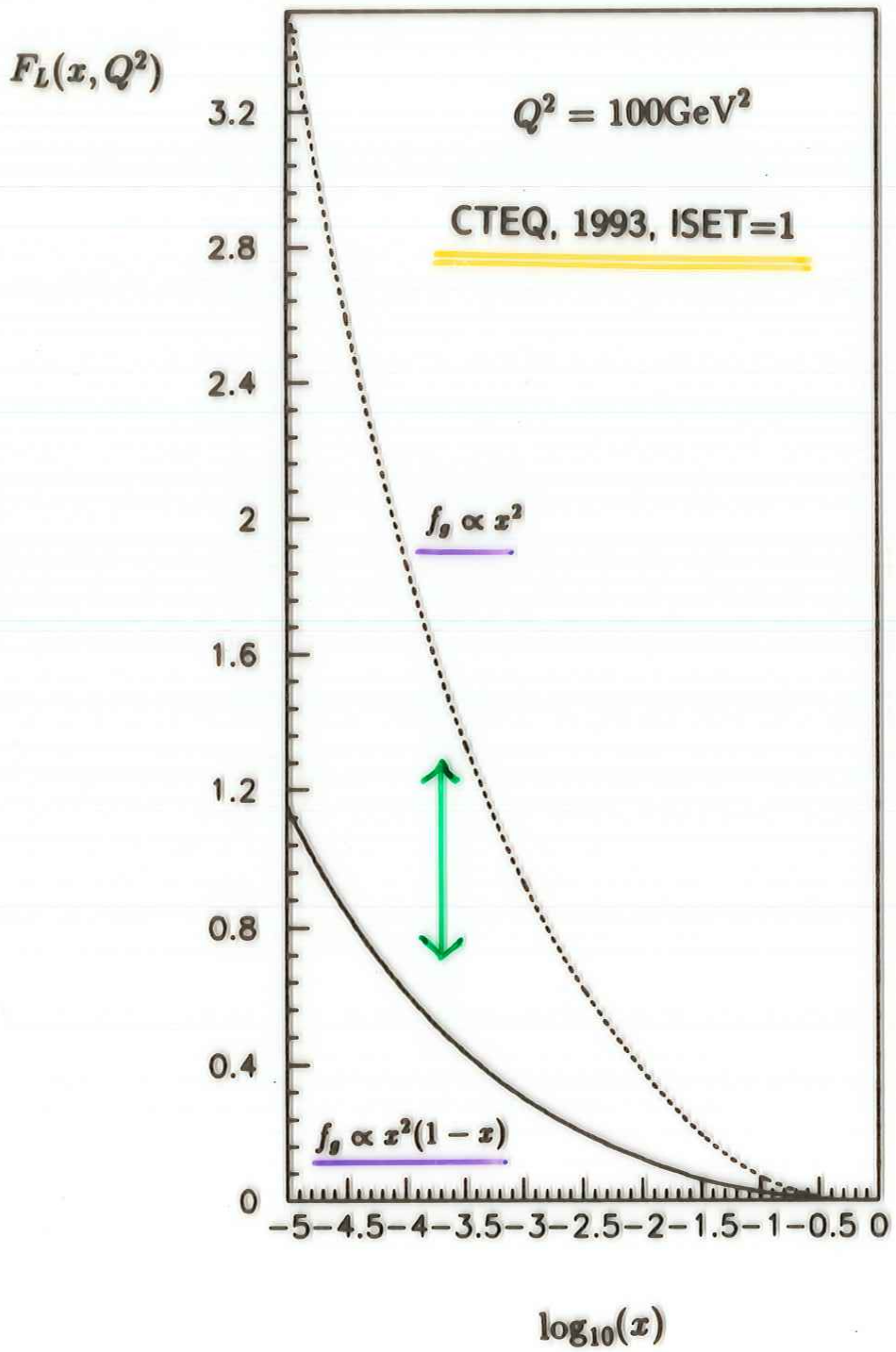
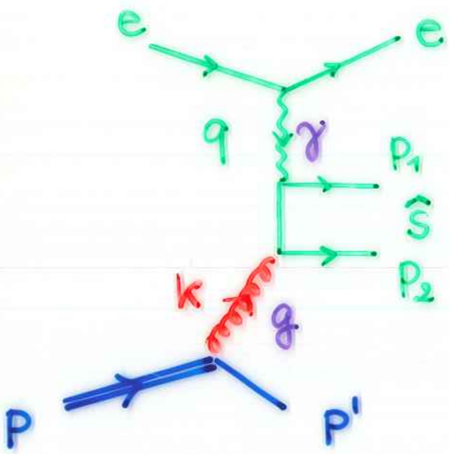
1. **Motivation**
2. **Phase Space**
3. **$k_\perp$ Factorization in the DIS Scheme**
4. **The Structure Functions**
5. **Conclusions**

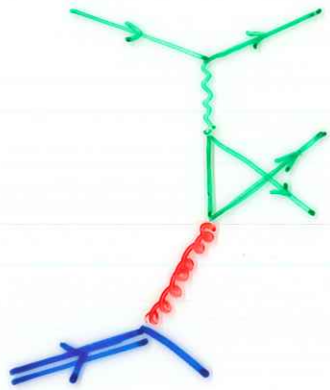
An Example:

$$F_L(x,Q^2) = \frac{\alpha_s(Q^2)}{2\pi} x^2 \int_x^1 \frac{dy}{y^3} \left\{ \frac{8}{3} F_2(y,Q^2) + 2 \left(\sum_{i=1}^{2f} e_i^2 \right) \left(1 - \frac{x}{y} \right) yG(y,Q^2) \right\}$$



2 Phase space





$$\underline{q + k = p_1 + p_2}$$ BB- CMS

$$\begin{aligned}\hat{s} &= (q+k)^2 = (p_1+p_2)^2 \\ \hat{t} &= (k-p_1)^2 = (q-p_2)^2 \\ \hat{u} &= (q-p_1)^2 = (k-p_2)^2\end{aligned}$$

$$\hat{s} + \hat{t} + \hat{u} = -Q^2 - K^2 + m_1^2 + m_2^2$$

$$k_\mu = \xi q'_\mu + \eta P_\mu + k_{\perp\mu}$$

$$\begin{aligned}\xi &= \frac{2k.P}{2q'.P} \\ \eta &= \frac{2k.q'}{2q'.P}\end{aligned}$$

$$x = \frac{Q^2}{2P.q}$$

$$K^2 = \boldsymbol{k}^2 - \xi\eta\frac{Q^2}{x}$$

$$y = \frac{P.q}{P.l_e} = \frac{Q^2}{sx}$$

$$\xi \ll \eta$$

$$\xi \approx 0 \qquad P.k \approx 0 \qquad k_\mu \approx \eta P_\mu + k_{\perp\mu} \qquad \underline{K^2 \approx \boldsymbol{k}^2}$$

The following relations may be derived with these definitions

$$\hat{s} = Q^2\left(\frac{\eta}{x}-1\right) - K^2$$
$$\eta = x\frac{\hat{s}+K^2+Q^2}{Q^2}$$
$$\hat{t} = m_1^2 - K^2 - z(\hat{s}+K^2+Q^2)$$
$$s_\gamma \equiv W^2 \approx Q^2\frac{1-x}{x}$$

The representation of the particle momenta is thus given by:

$$k = (K_0, 0, 0, |\vec{k}|)$$
$$q = (Q_0, 0, 0, -|\vec{k}|)$$
$$P = E_p(1, \sin\beta, 0, \cos\beta)$$
$$p_1 = (E_1, q_1\sin\theta\cos\varphi, q_1\sin\theta\sin\varphi, q_1\cos\theta)$$
$$p_2 = (E_2, -q_1\sin\theta\cos\varphi, -q_1\sin\theta\sin\varphi, -q_1\cos\theta)$$

with

$$K_0 = \mathcal{E}(\hat{s}, -K^2, -Q^2)$$
$$Q_0 = \mathcal{E}(\hat{s}, -Q^2, -K^2)$$
$$E_1 = \mathcal{E}(\hat{s}, m_1^2, m_2^2)$$
$$E_2 = \mathcal{E}(\hat{s}, m_2^2, m_1^2)$$
$$|\vec{k}| = \mathcal{P}(\hat{s}, -Q^2, -K^2)$$
$$q_1 = \mathcal{P}(\hat{s}, m_1^2, m_2^2)$$
$$\cos\theta = \frac{2K_0E_1 + K^2 - m_1^2 + \hat{t}}{2|\vec{k}|q_1}$$
$$E_p = \mathcal{E}(\hat{s}, 0, t_{qP'}) = \frac{Q^2}{2x\sqrt{\hat{s}}}$$
$$\cos\beta = \frac{K_0}{|\vec{k}|}$$

where

$$\mathcal{E}(a,b,c) = \frac{a+b-c}{2\sqrt{a}}$$
$$\mathcal{P}(a,b,c) = \sqrt{\frac{\lambda(a,b,c)}{4a}}$$

$$\cos\varphi = \left.\frac{(q\times P).(q\times p_1)}{|q\times P||q\times p_1|}\right|_{p_1=-p_2}$$

$$\cos\beta = \frac{\hat{s}-K^2+Q^2}{[(\hat{s}-K^2+Q^2)^2+4K^2\hat{s}]^{1/2}} = \frac{1-\rho}{\sqrt{1-2\rho x/\eta}}$$

$$\rho = \frac{2K^2x}{Q^2\eta}$$

$$\approx 1-\rho\left(1-\frac{x}{\eta}\right)$$

$$\rho \ll 1$$

$$
\begin{aligned}
dPS^{(3)} &= \frac{1}{128\pi^3}\frac{d\varphi_P}{2\pi}\frac{d\hat{s}d\hat{t}dK^2}{\lambda^{1/2}(\hat{s},-K^2,-Q^2)\lambda^{1/2}(s_\gamma,0,-Q^2)}\frac{d\varphi}{2\pi} \\
&\times \Theta\{-\mathcal{G}(s_\gamma,-K^2,\hat{s},0,-Q^2,0)\}\Theta\{-\mathcal{G}(\hat{s},\hat{t},0,-K^2,-Q^2,0)\}\Theta\{\hat{s}\}
\end{aligned}
$$

$$
\int dPS^{(3)} = \frac{1}{128\pi^3}\int_{\eta_{min}}^{\eta_{max}} d\eta \int_{K^2_{min}(\eta)}^{K^2_{max}(\eta)} dK^2 \int_0^{2\pi}\frac{d\varphi_P}{2\pi}\frac{1}{2}\int_{-1}^{1} d\cos\theta \int_0^{2\pi}\frac{d\varphi}{2\pi}
$$

$$
\begin{aligned}
K^2_{min} &= 0 \\
K^2_{max} &= Q^2\frac{\eta - x}{x} \\
\eta_{min} &= x \\
\eta_{max} &= 1
\end{aligned}
$$

$$
dPS^{(2)} = \frac{1}{8\pi}\frac{1}{2}\int_{-1}^{1} d\cos\theta \int_0^{2\pi}\frac{d\varphi}{2\pi}
$$

5 Conclusions

- $k_\perp$ DEPENDENT SPLITTING FUNCTIONS HAVE BEEN DERIVED
- THEIR BEHAVIOUR FOR $k_\perp^2 \to 0$ ALLOWS THE FACTORIZATION OF THE INPUT DISTRIBUTION IN A WELL DEFINED WAY.
- NO $x \ll 1$ APPROXIMATION WAS MADE IN THE SENSE OF A 0^{th} MOMENT etc.
- CRUCIAL: USE OF THE BOSON-BOSON CMS
- THE GLAP - SPLITTING FUNCTIONS ARE FOUND FOR $k_\perp^2/Q^2 \to 0$.

$$
\begin{aligned}
g_{01}^{(2)}(\beta) &= -\frac{3}{2} + \frac{1}{2}\cos^2\beta \\
g_{11}^{(2)}(\beta) &= 2\cos\beta \\
g_{21}^{(2)}(\beta) &= \frac{1}{2} - \frac{3}{2}\cos^2\beta \\
g_{02}^{(2)}(\beta) &= -1 - \cos^2\beta \\
g_{12}^{(2)}(\beta) &= 4\cos\beta \\
g_{22}^{(2)}(\beta) &= 1 - 3\cos^2\beta \\
g_{03}^{(2)}(\beta) &= -3 + \cos^2\beta \\
g_{13}^{(2)}(\beta) &= 0 \\
g_{23}^{(2)}(\beta) &= 1 - 3\cos^2\beta \\
g_{04}^{(2)}(\beta) &= 0 \\
g_{14}^{(2)}(\beta) &= 0 \\
g_{24}^{(2)}(\beta) &= 4 - 12\cos^2\beta
\end{aligned}
$$

$$
\begin{aligned}
g_{01}^{(2)}(\beta) &= -\frac{3}{2}+\frac{1}{2}\cos^2\beta \\
g_{11}^{(2)}(\beta) &= 2\cos\beta \\
g_{21}^{(2)}(\beta) &= \frac{1}{2}-\frac{3}{2}\cos^2\beta \\
g_{02}^{(2)}(\beta) &= -1-\cos^2\beta \\
g_{12}^{(2)}(\beta) &= 4\cos\beta \\
g_{22}^{(2)}(\beta) &= 1-3\cos^2\beta \\
g_{03}^{(2)}(\beta) &= -3+\cos^2\beta \\
g_{13}^{(2)}(\beta) &= 0 \\
g_{23}^{(2)}(\beta) &= 1-3\cos^2\beta \\
g_{04}^{(2)}(\beta) &= 0 \\
g_{14}^{(2)}(\beta) &= 0 \\
g_{24}^{(2)}(\beta) &= 4-12\cos^2\beta
\end{aligned}
$$

$$
\begin{aligned}
g_{01}^{(L)}(\beta) &= -\frac{1}{8}+\frac{1}{4}\cos\beta-\frac{1}{4}\cos^3\beta+\frac{1}{8}\cos^4\beta \\
g_{02}^{(L)}(\beta) &= -\frac{1}{4}+2\cos\beta-\cos^2\beta-3\cos^3\beta+\frac{9}{4}\cos^4\beta \\
g_{03}^{(L)}(\beta) &= -\frac{1}{4}+6\cos\beta-\frac{9}{2}\cos^2\beta-10\cos^3\beta+\frac{35}{4}\cos^4\beta \\
g_{11}^{(L)}(\beta) &= \cos\beta-\frac{3}{4}\cos^2\beta-\frac{3}{2}\cos^3\beta+\frac{5}{4}\cos^4\beta \\
g_{12}^{(L)}(\beta) &= \frac{1}{4}+\frac{13}{2}\cos\beta-\frac{15}{2}\cos^2\beta-\frac{21}{2}\cos^3\beta+\frac{45}{4}\cos^4\beta \\
g_{13}^{(L)}(\beta) &= 1+18\cos\beta-24\cos^2\beta-30\cos^3\beta+35\cos^4\beta \\
g_{21}^{(L)}(\beta) &= \frac{3}{16}+\frac{9}{8}\cos\beta-\frac{9}{4}\cos^2\beta-\frac{15}{8}\cos^3\beta+\frac{45}{16}\cos^4\beta \\
g_{22}^{(L)}(\beta) &= \frac{5}{4}+\frac{27}{4}\cos\beta-15\cos^2\beta-\frac{45}{4}\cos^3\beta+\frac{75}{4}\cos^4\beta \\
g_{23}^{(L)}(\beta) &= \frac{7}{2}+18\cos\beta-42\cos^2\beta-30\cos^3\beta+\frac{105}{2}\cos^4\beta \\
g_{31}^{(L)}(\beta) &= \frac{3}{16}+\frac{6}{16}\cos\beta-\frac{15}{8}\cos^2\beta-\frac{5}{2}\cos^3\beta+\frac{35}{4}\cos^4\beta \\
g_{32}^{(L)}(\beta) &= \frac{9}{8}+\frac{9}{4}\cos\beta-\frac{45}{4}\cos^2\beta-\frac{15}{4}\cos^3\beta+\frac{105}{8}\cos^4\beta \\
g_{33}^{(L)}(\beta) &= 3+6\cos\beta-30\cos^2\beta-10\cos^3\beta+35\cos^4\beta \\
g_{41}^{(L)}(\beta) &= \frac{3}{64}-\frac{15}{32}\cos^2\beta+\frac{35}{64}\cos^4\beta \\
g_{42}^{(L)}(\beta) &= \frac{9}{32}-\frac{45}{16}\cos^2\beta+\frac{105}{32}\cos^4\beta \\
g_{43}^{(L)}(\beta) &= \frac{3}{4}-\frac{15}{2}\cos^2\beta+\frac{35}{4}\cos^4\beta
\end{aligned}
$$