

**The Evolution of the Singlet Structure
Functions
of the Proton and the Photon at Small x**

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1. Basic Formalism
2. Non-Singlet Combinations
3. $F_2^p(x, Q^2)$ and $F_L^p(x, Q^2)$
4. $F_2^\gamma(x, Q^2)$
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1. Basic Formalism

$$q_{NS,i}^{\pm} = q_i \pm \bar{q}_i - \frac{1}{N_f} \sum_{r=1}^{N_f} (q_r \pm \bar{q}_r),$$

$$q_{NS}^{\text{val}} = \sum_{r=1}^{N_f} (q_r - \bar{q}_r),$$

$$q_S = \begin{pmatrix} \Sigma \\ g \end{pmatrix}, \quad \Sigma \equiv \sum_{r=1}^{N_f} (q_r + \bar{q}_r).$$

$$F_i(x, Q^2) = \sum_{r=1}^{2N_f} a_{ir} c_{i,r}(x, Q^2) \otimes q_r(x, Q^2) + a_{ig} c_{i,g}(x, Q^2) \otimes g(x, Q^2),$$

$$\frac{\partial q_{NS}^{\pm}(x, Q^2)}{\partial \ln Q^2} = P_{NS}^{\pm}(x, \alpha_s) \otimes q_{NS}^{\pm}(x, Q^2),$$

$$\frac{\partial q_S(x, Q^2)}{\partial \ln Q^2} = P_S(x, \alpha_s) \otimes q_S(x, Q^2).$$

$$\frac{da_s}{d \ln Q^2} = - \sum_{k=0}^{\infty} a_s^{k+2} \beta_k.$$

$$P^{\pm}(x, a_s) = \sum_{l=0}^{\infty} a_s^{l+1} P_l^{\pm}(x),$$

$$P(x, a_s) \equiv \begin{pmatrix} P_{qq}(x, a_s) & P_{qg}(x, a_s) \\ P_{gq}(x, a_s) & P_{gg}(x, a_s) \end{pmatrix} = \sum_{l=0}^{\infty} a_s^{l+1} P_l(x),$$

$$c_{i,j}(x, Q^2) = \delta(1-x) \delta_{jq} + \sum_{l=1}^{\infty} a_s^l c_{i,j,l}(x).$$

$$\int_0^1 dx P_l^-(x) = 0,$$

$$\int_0^1 dx x \sum_i P_{i,j,l}^{\text{unpol}}(x) = 0,$$

$$\gamma_L(N, a_s) = -2 \begin{pmatrix} 0 & 0 \\ C_F/C_A & 1 \end{pmatrix} \gamma_L(N, \alpha_s).$$

$$\gamma(N, a_s) = -2 \int_0^1 dx x^{N-1} P(x, a_s).$$

$$\rho \equiv \frac{N}{\alpha_s} = 2\psi(1) - \psi(\gamma_L) - \psi(1 - \gamma_L) \equiv \chi[\gamma_L],$$

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eq.

$$\gamma_L(N, a_s) = \sum_{k=1}^{\infty} g_k^{(0)} \left(\frac{\bar{\alpha}_s}{N} \right)^k.$$

$$\gamma_{NL}(N, \alpha_s) = -2 \begin{pmatrix} \frac{C_F}{C_A} [\gamma_{NL} - \frac{8}{3} a_s T_F] & \gamma_{NL} \\ \gamma_{gg,NL} & \gamma_{gg,NL} \end{pmatrix},$$

$$\begin{aligned} \gamma_{NL}^{\text{DIS}}(N, a_s) &= \gamma_{NL}^{Q_0}(N, a_s) R(\gamma_L) = T_F \frac{\alpha_s}{3\pi} \frac{2 + 3\gamma_L - 3\gamma_L^2}{3 - 2\gamma_L} \frac{[B(1 - \gamma_L, 1 + \gamma_L)]^3}{B(2 + 2\gamma_L, 2 - 2\gamma_L)} R(\gamma_L) \\ &= \frac{\alpha_s}{3\pi} T_F \sum_{k=1}^{\infty} g_k^{gg,(1)} \left(\frac{\bar{\alpha}_s}{N} \right)^k, \end{aligned}$$

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with $B(x, y)$ denoting the Beta function and

$$R(\gamma) = \left[\frac{\Gamma(1 - \gamma)\chi(\gamma)}{\Gamma(1 + \gamma)\{-\gamma\chi'(\gamma)\}} \right]^{1/2} \exp \left[\gamma\psi(1) + \int_0^\gamma d\zeta \frac{\psi'(1) - \psi'(1 - \zeta)}{\chi(\zeta)} \right].$$

k	$g_k^{(0)}$	$g_k^{qq,1}$	c_k^L
1	1.000000000E+0	1.000000000E+0	1.000000000E+0
2	0.000000000E+0	2.166666666E+0	-3.333333333E-1
3	0.000000000E+0	2.299510377E+0	2.132843711E+0
4	2.404113806E+1	8.271090891E+0	2.272314949E+0
5	0.000000000E+0	1.492492689E+1	0.434344619E+0
6	2.073855510E+1	2.922678142E+1	2.026429511E+1
7	1.733928958E+1	1.028116363E+2	2.303149346E+1
8	2.016698555E+0	1.948865992E+2	3.464487971E+1
9	3.988627732E+1	4.851003221E+2	2.650042403E+2
10	1.687465187E+2	1.524438954E+3	3.300383179E+2
11	6.998811177E+1	3.114512279E+3	8.503712548E+2
12	6.612526445E+2	8.583754389E+3	3.908491758E+3
13	1.945314152E+3	2.475708601E+4	5.674329386E+3
14	1.717675538E+3	5.474354877E+4	1.776800580E+4
15	1.064326485E+4	1.561954918E+5	6.219822925E+4
16	2.556678709E+4	4.269798325E+5	1.070279253E+5
17	3.678133240E+4	1.011107922E+6	3.514745865E+5
18	1.716848319E+5	2.893976929E+6	1.050584301E+6
19	3.753792748E+5	7.690417370E+6	2.103406452E+6
20	7.360245729E+5	1.919194565E+7	6.807468863E+6

k	$a_{gg,k}^{qq,(1)}$	$a_{gg,k}^{qq,(2)}$	\hat{r}_k
1	-1.000000000E+0	0.000000000E+0	0.000000000E+0
2	-3.833333333E+0	0.000000000E+0	0.000000000E+0
3	-2.299510376E+0	0.000000000E+0	0.000000000E+0
4	-5.065605818E+0	3.205485075E+0	9.616455224E+0
5	-3.523670351E+1	8.568702514E+0	-3.246969702E+0
6	-3.218245315E+1	1.835447655E+1	2.281241061E+1
7	-1.060268680E+2	8.632838009E+1	1.654162989E+2
8	-4.853159484E+2	1.924088636E+2	-2.469139930E+0
9	-5.806186371E+2	4.962344972E+2	7.458249428E+2
10	-2.176371931E+3	1.794742819E+3	2.784859262E+3
11	-7.553679737E+3	4.023320193E+3	1.505001272E+3
12	-1.158215080E+4	1.136559381E+4	1.818320928E+4
13	-4.328579102E+4	3.589638820E+4	4.899274185E+5
14	-1.269309428E+5	8.412529889E+4	6.109247725E+5
15	-2.392549581E+5	2.456097133E+5	3.984470167E+5
16	-8.469557573E+5	7.168572021E+6	9.205515787E+5
17	-2.262541206E+6	1.764587230E+6	1.783326920E+6
18	-4.974873276E+6	5.167844173E+6	8.347774614E+6
19	-1.648990863E+7	1.443009883E+7	1.842662795E+7
20	-4.222994214E+7	3.702246358E+7	4.535538189E+7

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Table 1: The expansion coefficients in eqs. (29,34,38) and (40).

SUBLEADING TERMS:

$$\gamma_{99,1}^{DIS} = \frac{123.3}{N_f = 4} + 405.9$$

$$\gamma_{98,1}^{DIS} = - \frac{277.3}{N} + 846.2$$

$$\gamma_{89,1}^{DIS} = \frac{91.26}{N} - 453.5$$

$$\gamma_{88,1}^{DIS} = \frac{245.3}{N} - 988.3$$

MODEL:

$$A: \gamma_{ij} \rightarrow \gamma_{ij} - \delta_{ij} \gamma_{ij} (1, a_s)$$

$$B: \gamma_{ij} \rightarrow \gamma_{ij} \cdot (1-N)$$

$$C: \gamma_{ij} \rightarrow \gamma_{ij} (1-N)^2$$

$$D: \gamma_{ij} \rightarrow \gamma_{ij} (1-2N+N^3).$$

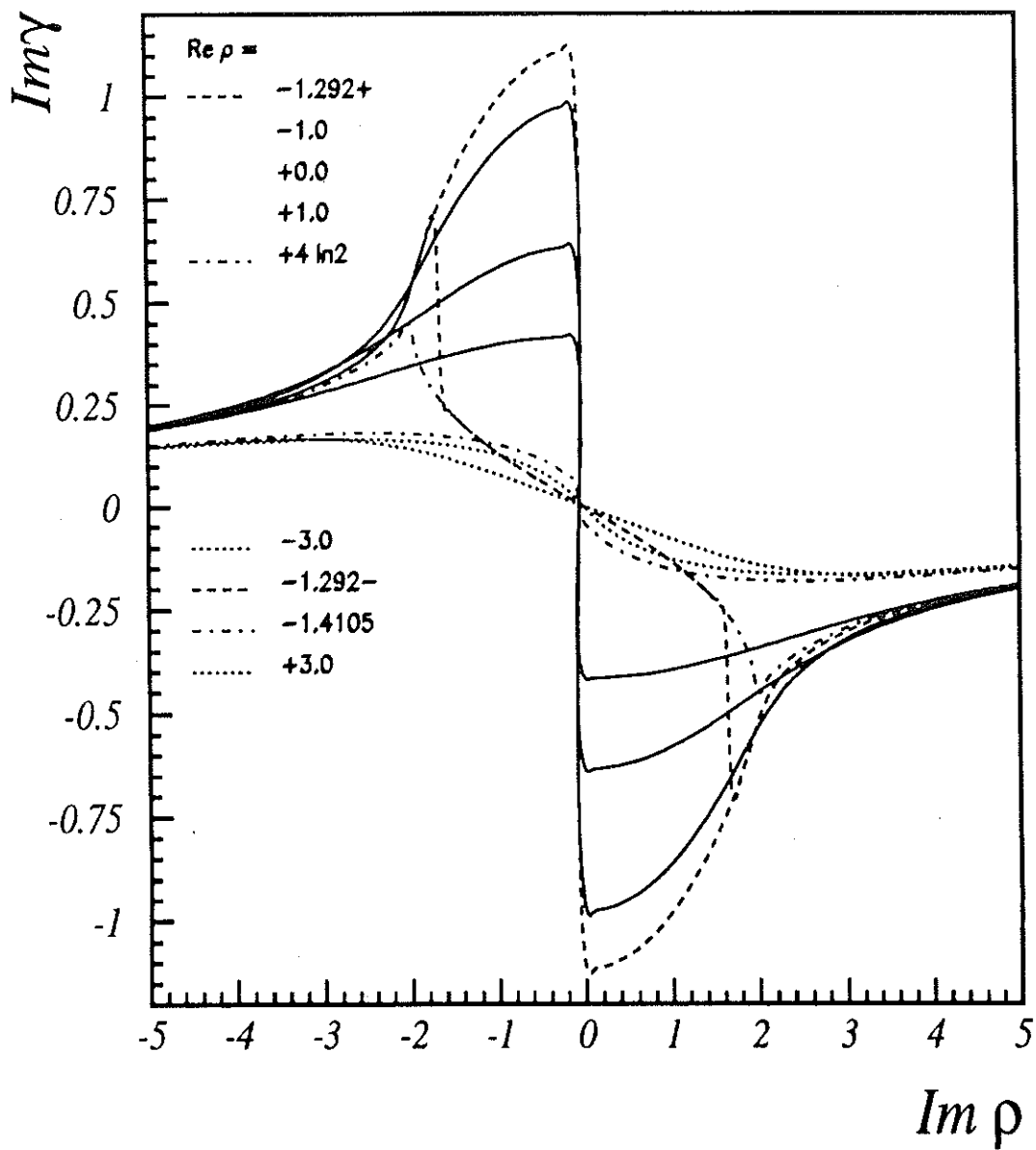


Figure 1b: Imaginary part of $\gamma_L(\rho)$ for $-3 \leq \text{Re} \rho \leq 3$ and $-5 \leq \text{Im} \rho \leq 5$. The notations are the same as in figure 1a.

ALL THE ABOVE QUANTITIES ARE FUNCTIONS OF $\gamma_L(N)$.

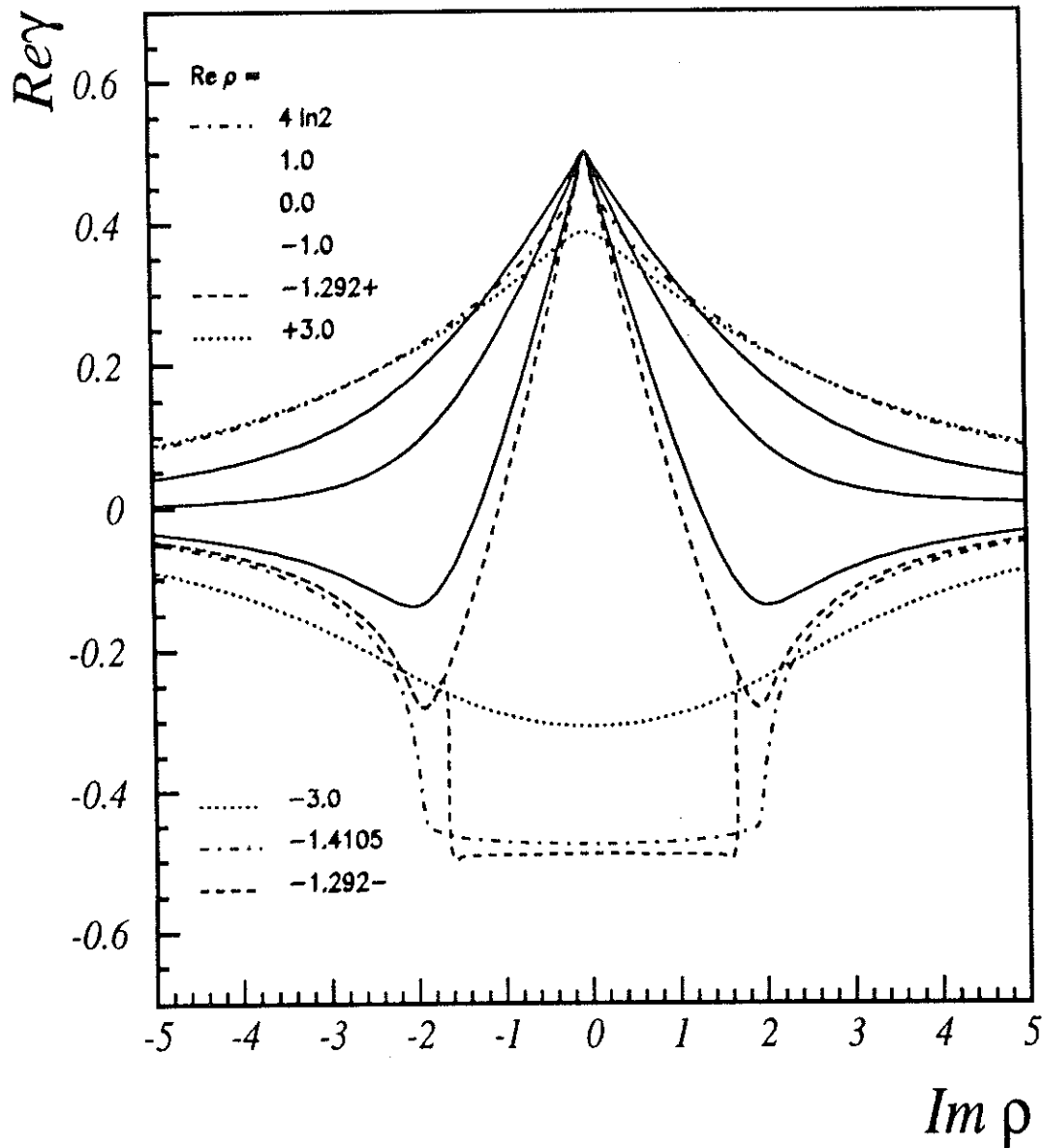


Figure 1a: Real part of $\gamma_L(\rho)$ for $-3 \leq \text{Re} \rho \leq 3$ and $-5 \leq \text{Im} \rho \leq 5$. The labels of the curves $\gamma_L|_{\text{Re} \rho}$ refer to the order of their values for $|\text{Im} \rho| \sim 0$. The dash-dotted lines show the behavior of $\text{Re} \gamma_L$ at the singularities of the perturbative branch, $\text{Re} \rho = 4 \ln 2$ and -1.4105 . The branch points manifest by the 'roof' at $\gamma_L = 0.5$ and the corner points of $\gamma_L(\text{Re} \rho = -1.4105)$. The dashed lines mark a transition in γ_L for smaller values of $\text{Im} \rho$ at $\text{Re} \rho = 1.292$. The dotted lines illustrate the behavior of $\text{Re} \gamma_L$ outside the range of the singularities.

SOLUTION OF THE EVOLUTION eqs.

$$\begin{aligned} \frac{\partial q(a_s, N)}{\partial a} &= -\frac{1}{\beta_0 a_s} \left[P_0(N) + a_s \left(P_1(N) - \frac{\beta_1}{\beta_0} P_0(N) \right) \right. \\ &\quad \left. + a_s^2 \left(P_2(N) - \frac{\beta_1}{\beta_0} P_1(N) + \left\{ \left(\frac{\beta_1}{\beta_0} \right)^2 - \frac{\beta_2}{\beta_0} \right\} P_0(N) \right) + \dots \right] q(a_s, N) \\ &\equiv -\frac{1}{a_s} \left[R_0(N) + \sum_{k=1}^{\infty} a_s^k R_k(N) \right] q(a_s, N). \end{aligned}$$

$$R_0 = \frac{1}{\beta_0} P_0,$$

$$R_k = \frac{1}{\beta_0} P_k - \sum_{i=1}^k \frac{\beta_i}{\beta_0} R_{k-i}$$

$$[R_{k \geq 1}, R_0] \neq 0$$

$$q^{LO}(a_s, N) = \left(\frac{a_s}{a_0} \right)^{-R_0(N)} q(a_0, N) \equiv L(a_s, a_0, N) q(a_0, N),$$

$$\begin{aligned} q(a_s, N) &= U(a_s, N) L(a_s, a_0, N) U^{-1}(a_0, N) q(a_0, N) \\ &= \left[1 + \sum_{k=1}^{\infty} a_s^k U_k(N) \right] L(a_s, a_0, N) \left[1 + \sum_{k=1}^{\infty} a_0^k U_k(N) \right]^{-1} q(a_0, N). \end{aligned}$$

1st STEPS:
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$$\begin{aligned}
 [U_1, R_0] &= R_1 + U_1, \\
 [U_2, R_0] &= R_2 + R_1 U_1 + 2U_2, \\
 &\vdots \\
 [U_k, R_0] &= R_k + \sum_{i=1}^{k-1} R_{k-i} U_i + k U_k \equiv \tilde{R}_k + k U_k.
 \end{aligned} \tag{52}$$

These equations can be solved recursively by applying the eigenvalue decomposition of the LO splitting function matrix, completely analogous to the NLO case of only U_1 in ref. [4]. One writes

$$R_0 = r_- e_- + r_+ e_+, \tag{53}$$

where r_- (r_+) stands for the smaller (larger) eigenvalue of R_0 ,

$$r_{\pm} = \frac{1}{2\beta_0} \left[P_{qq}^{(0)} + P_{gg}^{(0)} \pm \sqrt{(P_{qq}^{(0)} - P_{gg}^{(0)})^2 + 4P_{qg}^{(0)} P_{gq}^{(0)}} \right]. \tag{54}$$

The matrices e_{\pm} denote the corresponding projectors,

$$\begin{aligned}
 e_- &= \frac{1}{r_- - r_+} [R_0 - r_+ I], \\
 e_+ &= \frac{1}{r_+ - r_-} [R_0 - r_- I],
 \end{aligned} \tag{55}$$

with I the 2×2 unit matrix. Hence the LO evolution operator in eq. (50) can be written as

$$L(a, a_0, N) = e_-(N) \left(\frac{a}{a_0} \right)^{-r_-(N)} + e_+(N) \left(\frac{a}{a_0} \right)^{-r_+(N)}. \tag{56}$$

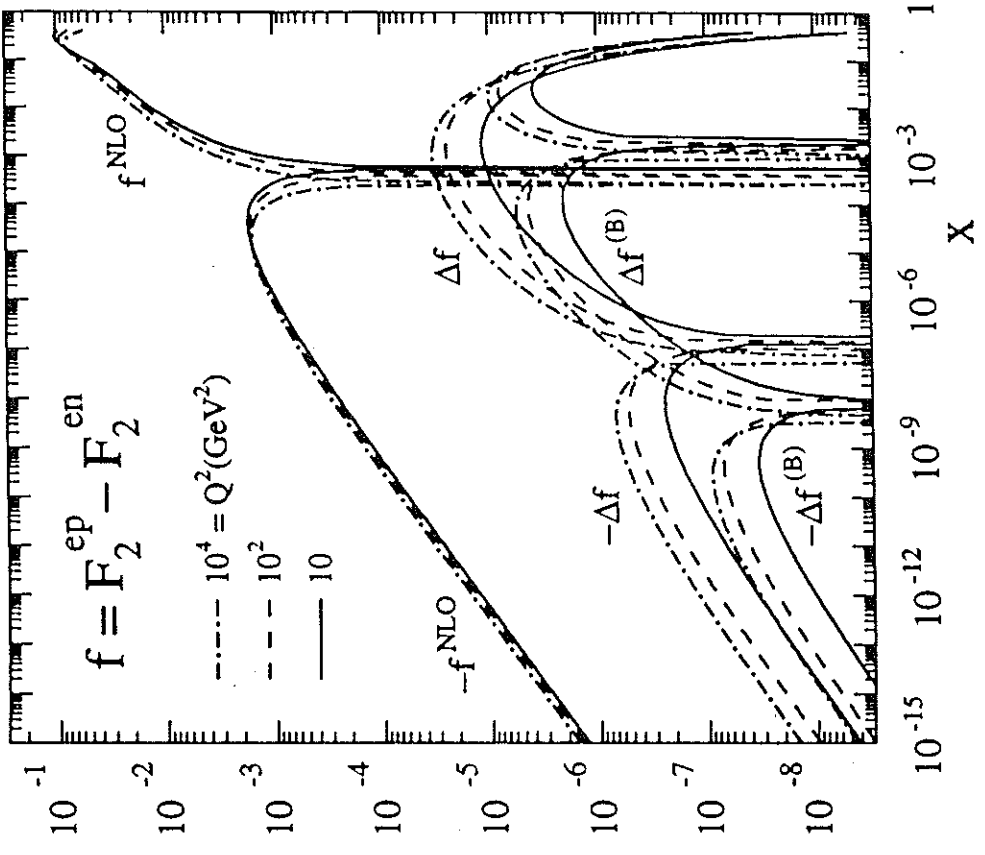
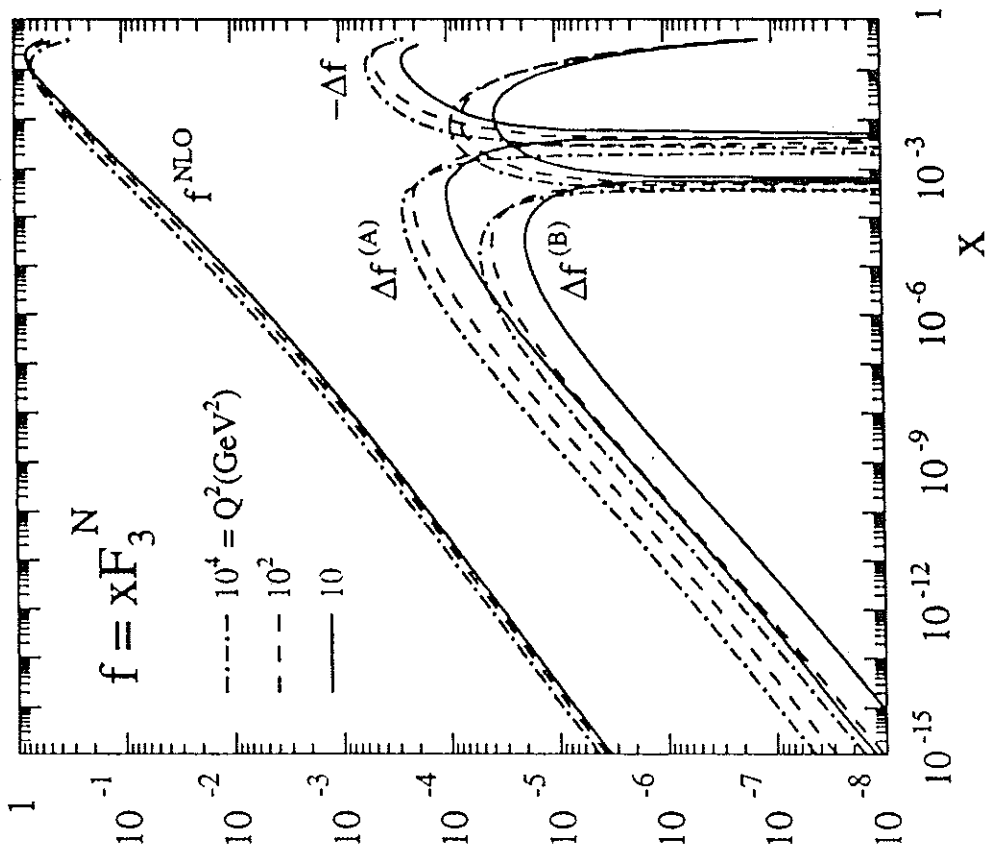
Inserting the obvious identity

$$U_k = e_- U_k e_- + e_- U_k e_+ + e_+ U_k e_- + e_+ U_k e_+ \tag{57}$$

into eq. (52), one finally obtains the expansion coefficients in eq. (51) as

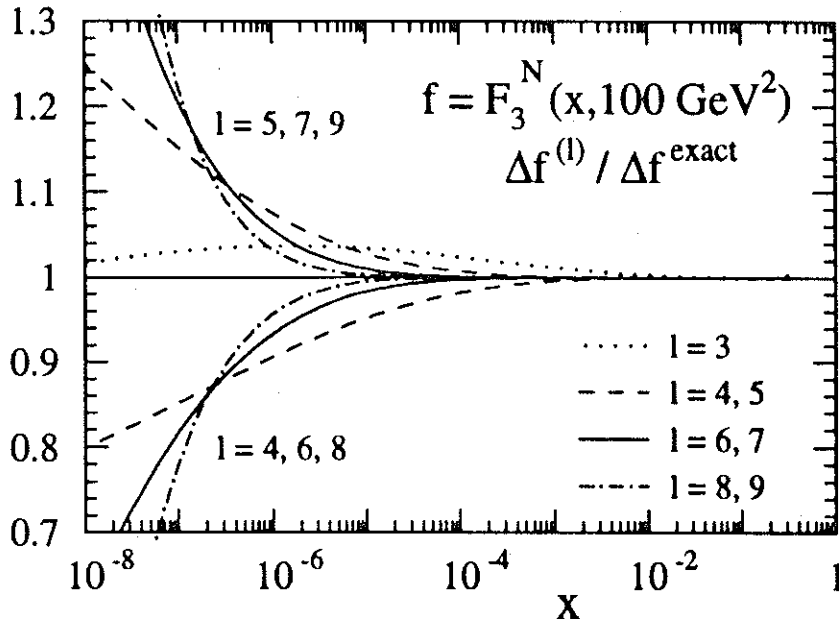
$$U_k = -\frac{1}{k} [e_- \tilde{R}_k e_- + e_+ \tilde{R}_k e_+] + \frac{e_+ \tilde{R}_k e_-}{r_- - r_+ - k} + \frac{e_- \tilde{R}_k e_+}{r_+ - r_- - k}. \tag{58}$$

2. Non-Singlet Combinations

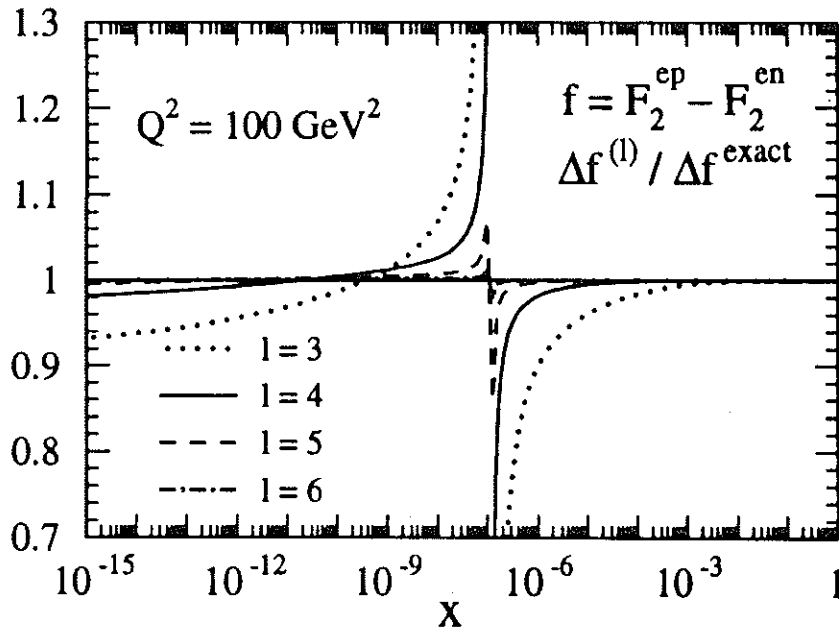


CONVERGENCE OF THE SERIES

IN α_s :

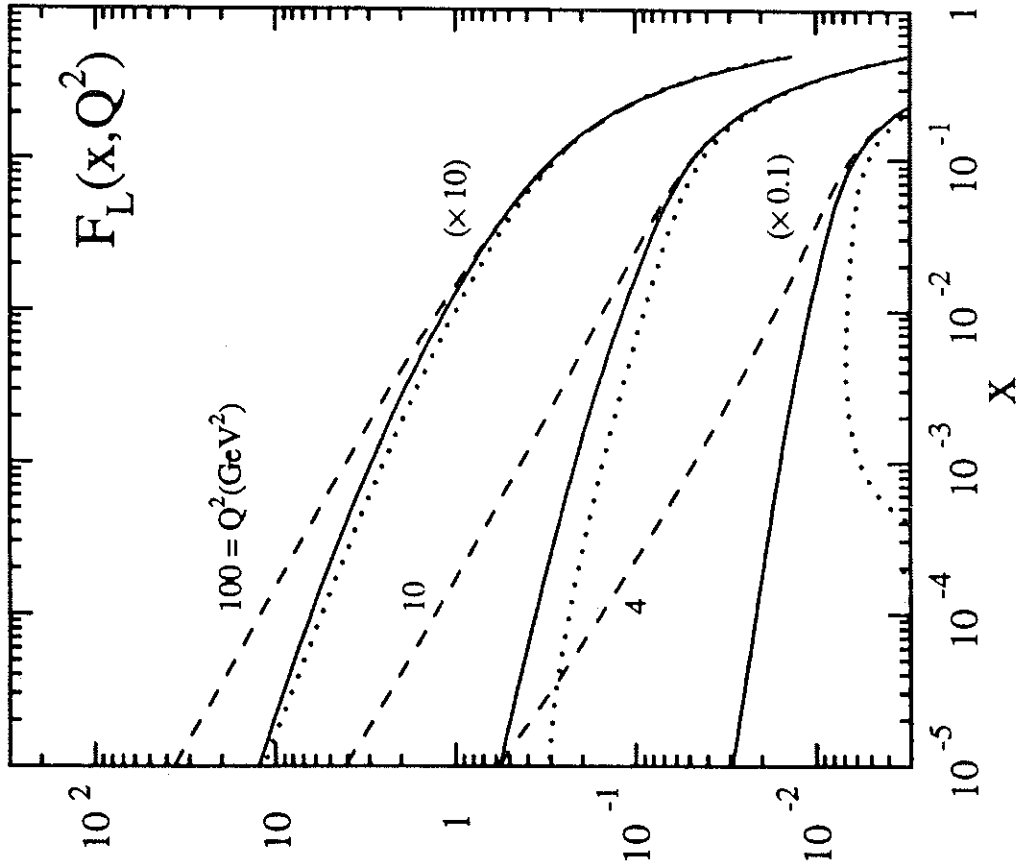
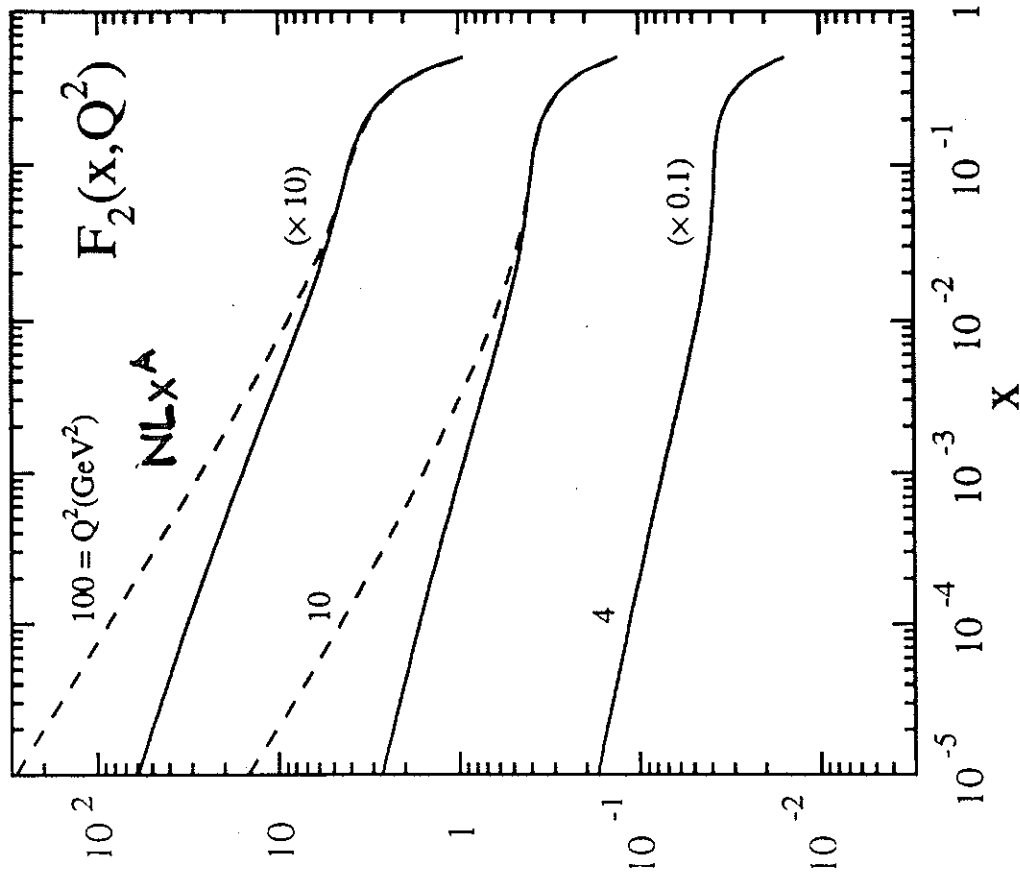


ASYMPTOTIC
SERIES



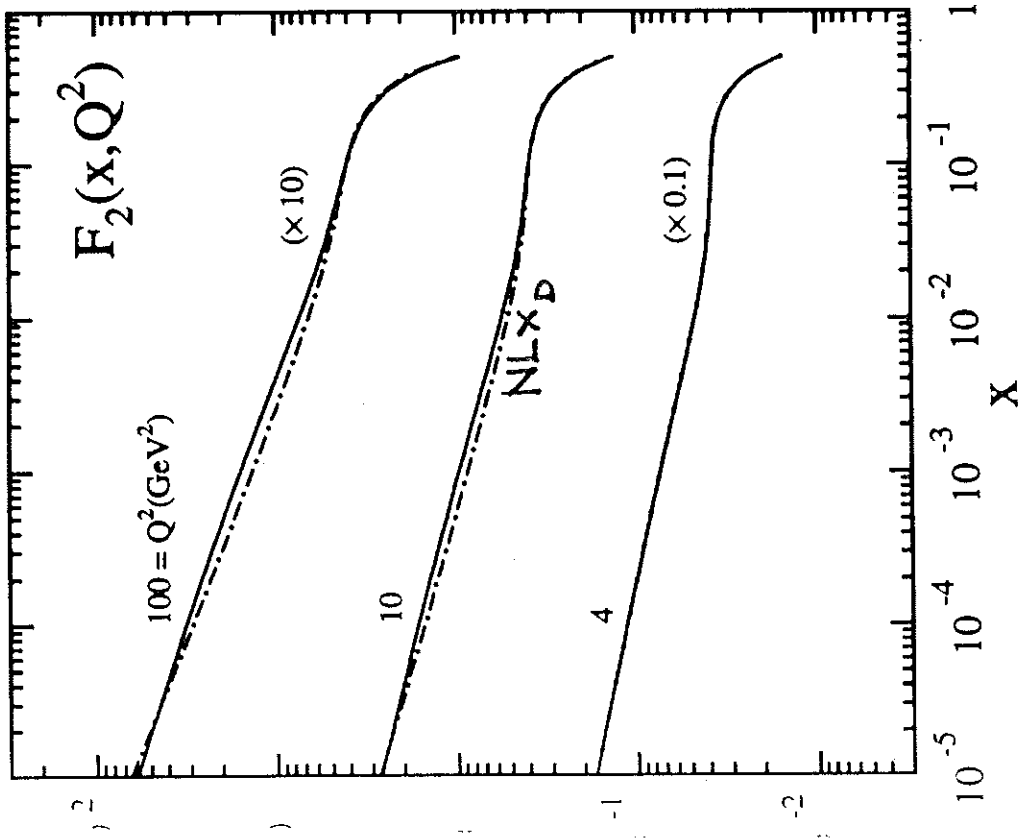
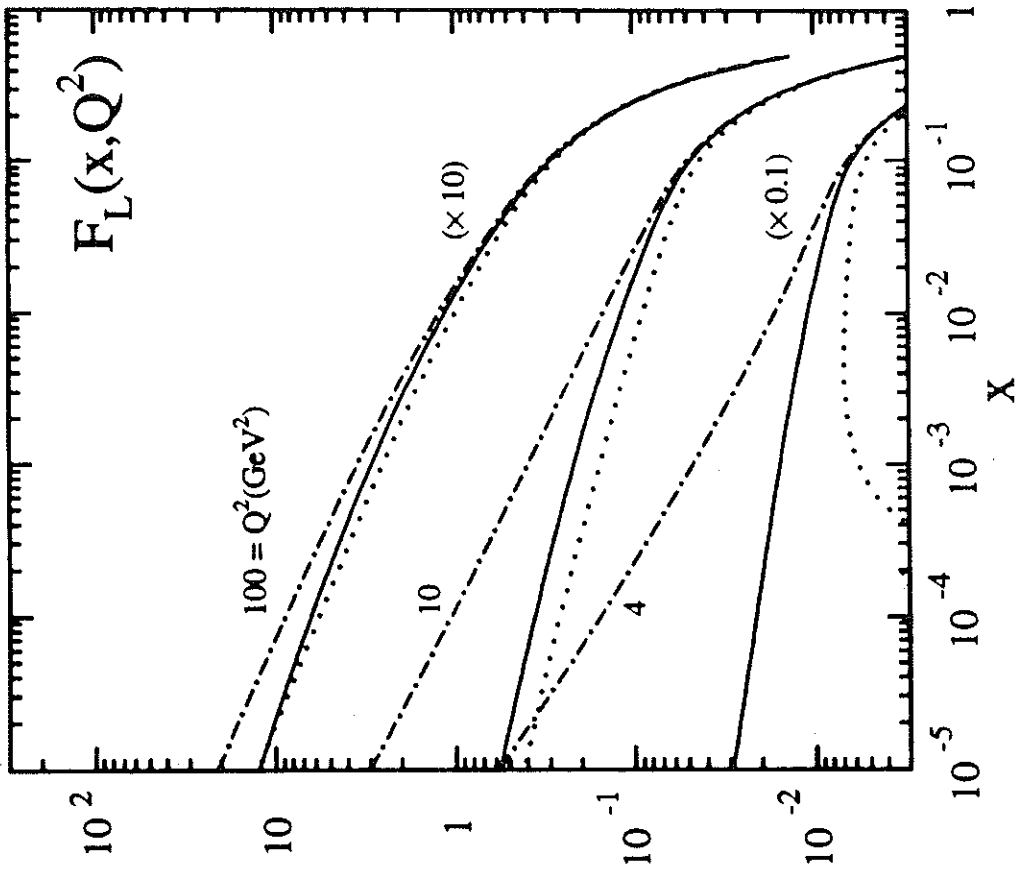
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3. $F_2^P(x, Q^2)$, $F_L^P(x, Q^2)$

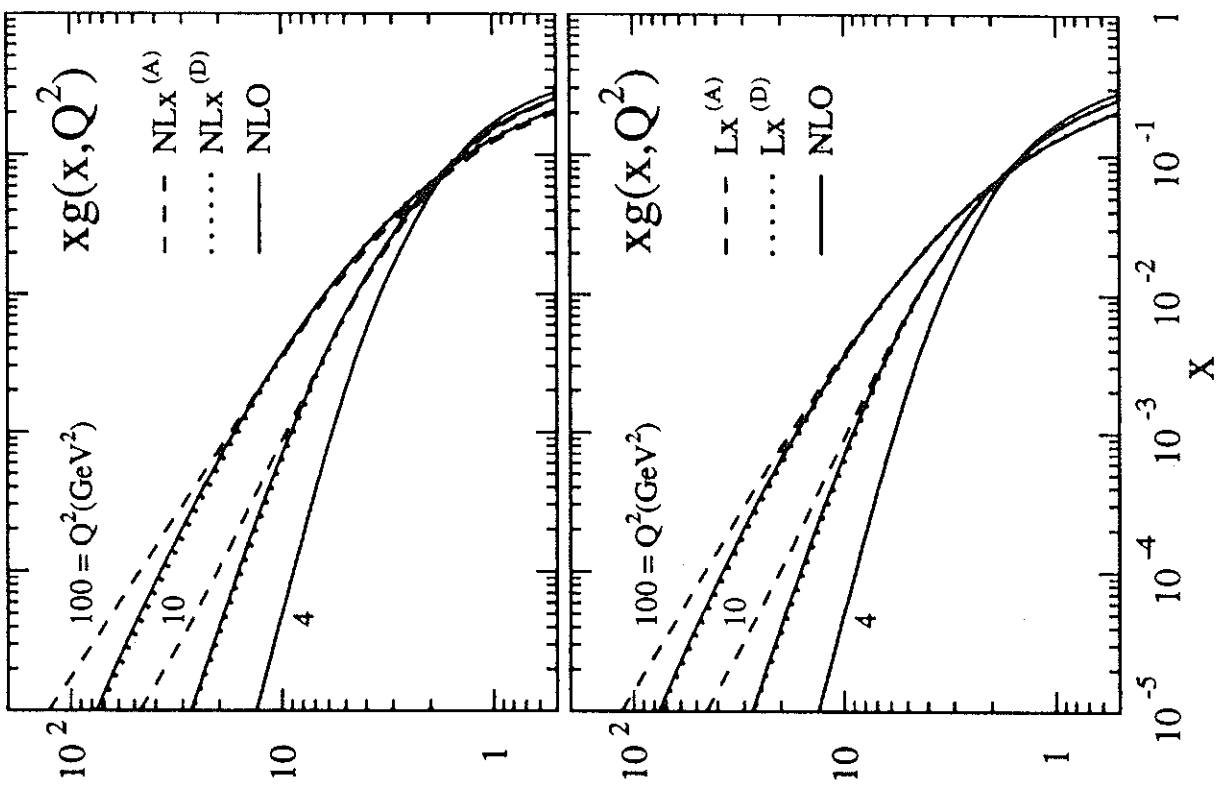
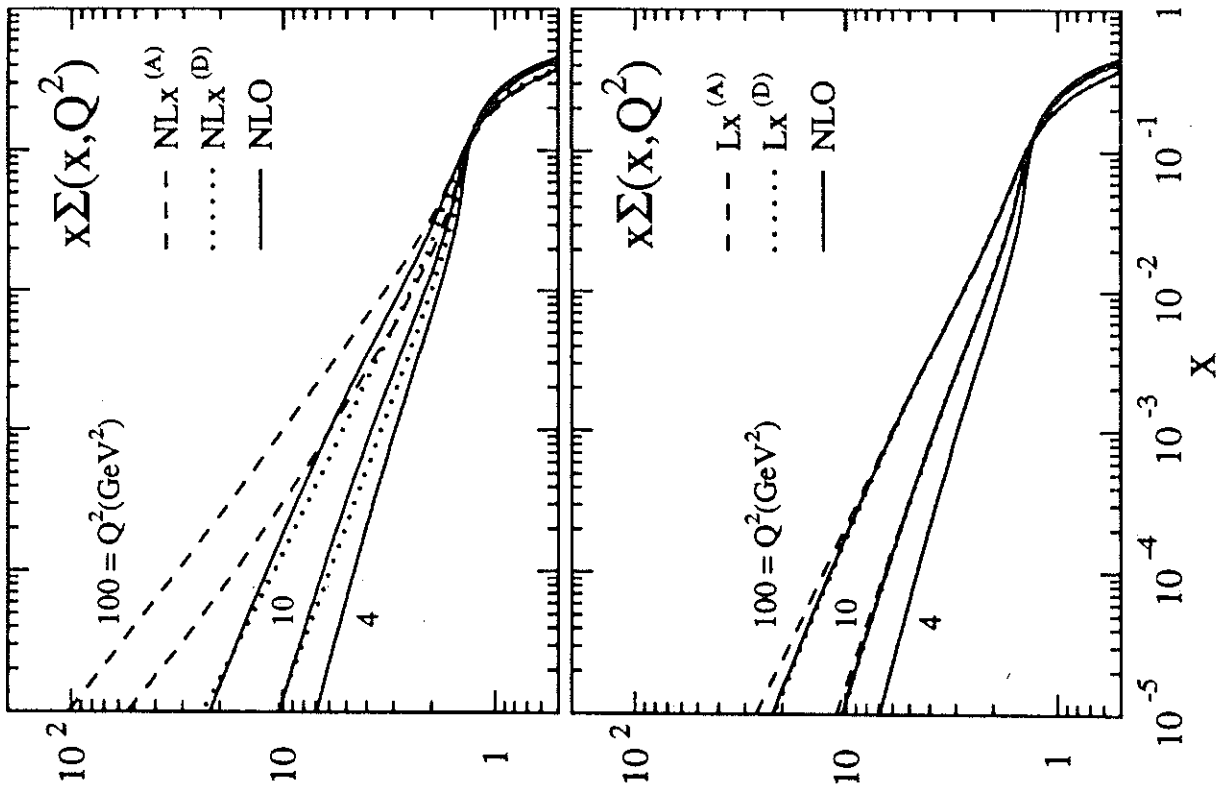


... $C_L^{NLO} + C_L^{\mathcal{O}(\alpha_s^2)} \cdot (1-2N)$

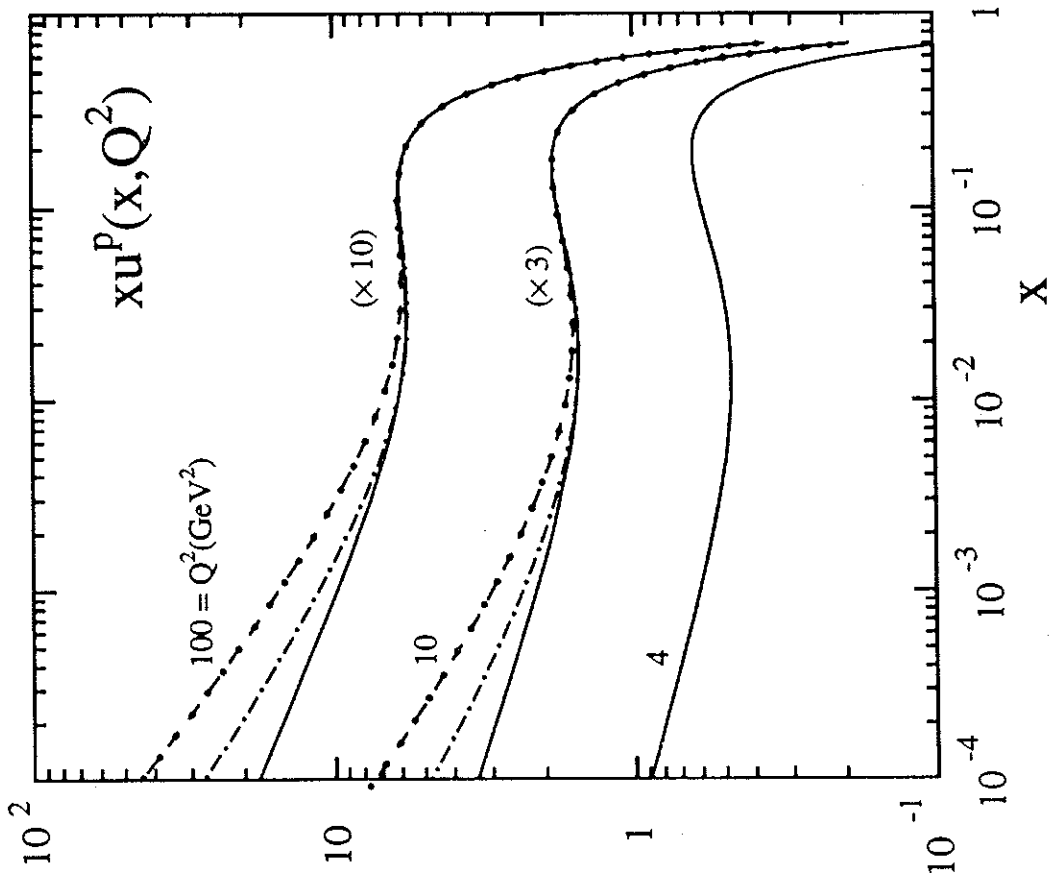
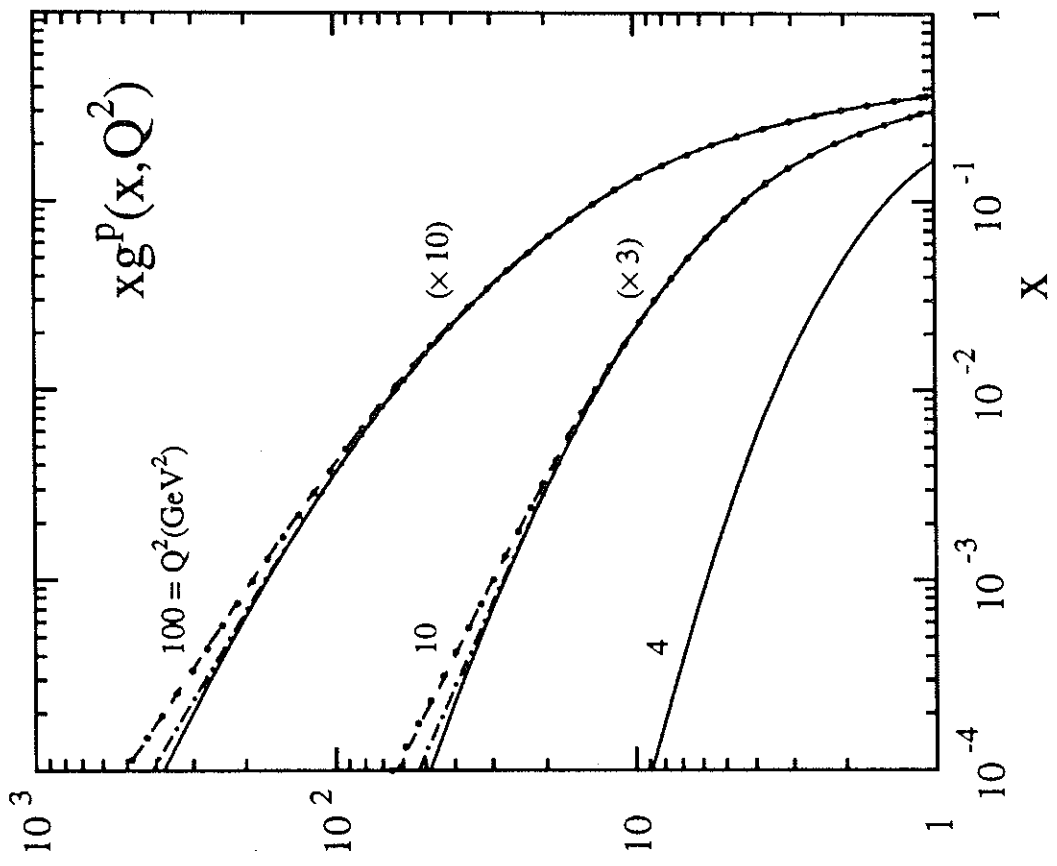
F_L IS DRIVEN BY THE COEFFICIENT FUNCTION (NOT BY THE PDF'S).



$$\gamma_0(1 - 2N + N^2)$$



..... + Ciaffaloni/
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4. $F_2^{\gamma}(x, Q^2)$

$$\begin{pmatrix} \dot{q}^{\gamma} \\ \dot{\bar{q}}^{\gamma} \\ \dot{G}^{\gamma} \\ \dot{\Gamma}^{\gamma} \end{pmatrix} = \frac{\alpha}{2\pi} \begin{pmatrix} P_{qq} \\ P_{q\bar{q}} \\ P_{Gq} \\ P_{\gamma q} \end{pmatrix} \otimes \Gamma^{\gamma} + \begin{cases} \frac{\alpha_s}{2\pi} P_{ij} \otimes P_j \\ \frac{\alpha}{2\pi} P_{\gamma j} \otimes P_j \end{cases} \quad i, j = 1, 3.$$

$$\Gamma^{\gamma,0} = \delta(1-x) + O(\alpha). \quad q^{\gamma} = \bar{q}^{\gamma}$$

SOLVE :

$$\dot{q}^{\gamma} = \frac{\alpha}{2\pi} K_q + \frac{\alpha_s}{2\pi} \left[\sum_{k=1}^{N_f} (P_{qq} + P_{q\bar{q}}) \otimes q^{\gamma} + P_{qg} \otimes G^{\gamma} \right]$$

$$\dot{G}^{\gamma} = \frac{\alpha}{2\pi} K_g + \frac{\alpha_s}{2\pi} \left[\sum_{k=1}^{N_f} (P_{gq} + P_{g\bar{q}}) \otimes q^{\gamma} + P_{gg} \otimes G^{\gamma} \right]$$

INHOM. EVOLUTION EQUUS. $K_i = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^k K_{i,k}$

$$P_{ij} = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^k P_{ij}^{(k)}$$

→ EXPRESS F_2^{γ} & EXPAND.

• SCHEME INDEP. REQUESTS :

$$R_0 = i\nu.$$

$$R_0 C_1 + R_1 - \delta \cdot P_0 C_{\gamma 1} = i\nu.$$

$$R_2 + C_2 R_0 + C_1 R_1 - \delta \cdot (P_0 C_{\gamma 2} + P_1 C_{\gamma 1} + [C_1, P_0] C_{\gamma 1})$$

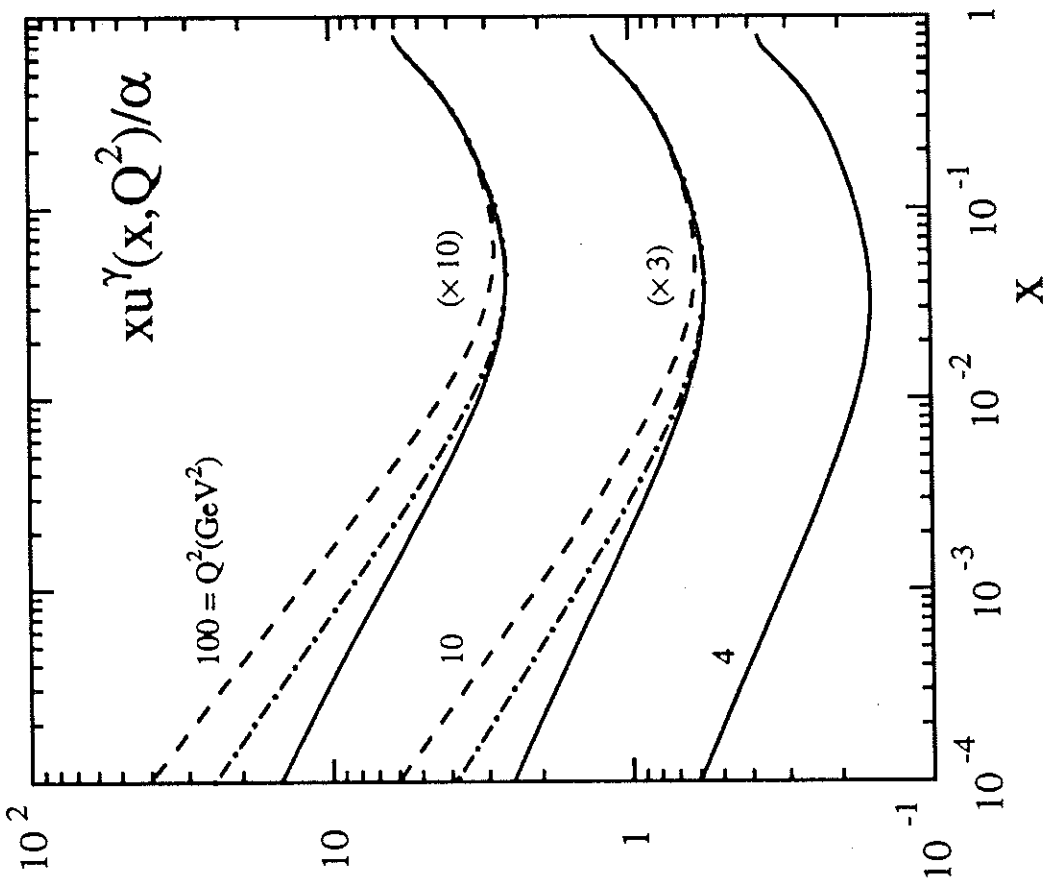
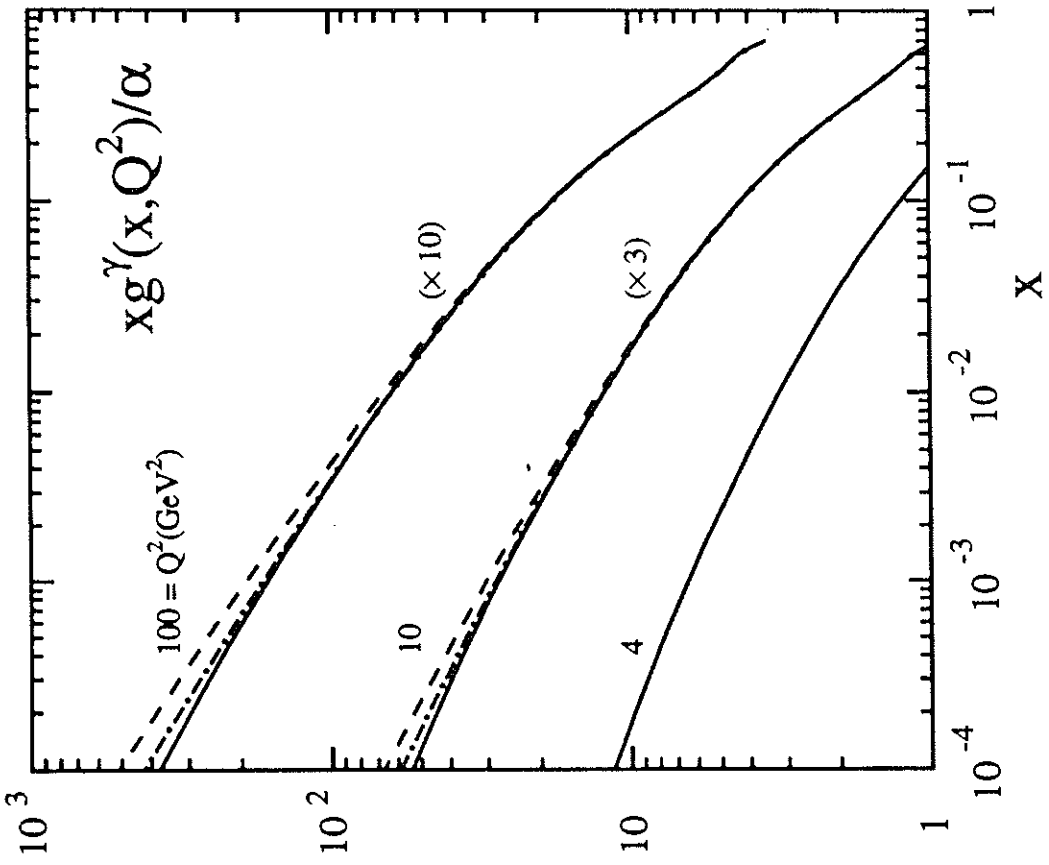
$$+ \delta \beta_0 C_1 C_{\gamma 1} - \delta \beta_0 C_{\gamma 2} = i\nu$$

etc.

Dis γ : NO NLX terms in R_n ($C_i \sim \frac{1}{(N-1)^{i-1}}$)

$R_{\text{Dis. \gamma in}} = 0 \mid \text{NLX}$. TREAT IN NLO ONLY.

4. $F_2^\gamma(x, Q^2)$



--- NLX
 -.- NLX₂ [$\cdot(1-N)$]

5. Conclusions

- AN UPDATE OF THE SMALL x RESUMMATION FOR F_2 AND F_L WAS PERFORMED.
- SUBLEADING TERMS IN $\left(\frac{\alpha_s}{N}\right)^k (\alpha_s N)^a$ ARE AS IMPORTANT. \rightarrow NEED TO BE CALCULATED.
 \rightarrow ANOMALOUS DIMENSIONS & COEFFICIENT FCTS.
- DIS SCHEME: F_L DEPENDS ON C_L^g MOSTLY,
 g - 'RATHER' STABLE!

$$\delta C_L^g \leftrightarrow \delta F_L !$$

- THE TERMS $\propto N_f$ IN γ_{gg} WERE INCLUDED. THEIR NUMERICAL EFFECT IS SMALL ($< 1\%$). STILL TO WAIT UNLESS THE COMPLETE NLO RESUMMED γ_{gg} & γ_{gq} BECOME AVAILABLE.
- THE SMALL x CONTRIBUTIONS TO F_2^{γ} WERE CALCULATED.

THE POINTLIKE TERM DOES NOT RECEIVE RESUMMED CORRECTIONS IN NLX IN THE DIS SCHEME.

THE CORRECTIONS ARE DUE TO THE HADRONIC PARTS.