3-Loop Heavy Flavor Corrections to DI Scattering

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The deep–inelastic process

- Kinematic invariants:
  \[ Q^2 = -q^2, \quad x = \frac{Q^2}{2P \cdot q} \]

- The cross section factorizes into leptonic and hadronic tensor:
  \[ \frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W_{\mu\nu} \]

- The hadronic tensor can be expressed through structure functions:
  \[ W_{\mu\nu} = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | \left[ J^e_m(\xi), J^e_m(\xi) \right] | P \rangle \]
  \[ = \frac{1}{2x} \left( g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left( P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2 g_{\mu\nu}} \right) F_2(x, Q^2) \]
  \[ + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho(q \cdot PS^\sigma - q \cdot SP^\sigma)}{(q \cdot P)^2} g_2(x, Q^2) \]

- \( F_L, F_2, g_1 \) and \( g_2 \) contain contributions from both, charm and bottom quarks.
Why are Heavy Flavor Contributions important?

- They form a significant contribution to all structure functions particularly at small x and high $Q^2$.
- Concise 3-loop corrections are needed to determine $\alpha_s(M_Z)$, $m_c$ and perhaps $m_b$.
- The accuracy of measurements at the LHC reaches a level of precision requiring 3-loop VFNS matching.

**NNLO:** [S. Alekhin, J. Blümlein, S. Moch and R. Placakyte (Phys. Rev. D96 (2017))]

\[ \alpha_s(M_Z^2) = 0.1147 \pm 0.0008 \]

\[ m_c(m_c) = 1.252 \pm 0.018 (exp) \quad +0.03 \quad -0.02 \quad (scale), \quad +0.00 \quad -0.07 \quad (thy) \text{GeV} \]

Yet approximate NNLO treatment for $F_2$ (non-negligible error) [H. Kawamura et al. (Nucl. Phys. B864 (2012))]

NS and PS corrections are already final [J. Ablinger et al. (Nucl. Phys. B 844 (2011), B886 and B890 (2014))]

EIC: many more high precision data ahead for various detailed unpolarized and polarized precision measurements.
Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

\[ F_{(2,L)}(x, Q^2) = \sum_j C_{j,(2,L)} \left( x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes f_j(x, \mu^2) \]

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs). Wilson coefficients:

\[ C_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right). \]

At \( Q^2 \gg m^2 \) the heavy flavor part

\[ H_{j,(2,L)} \left( N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left( N, \frac{Q^2}{\mu^2} \right) A_{ij} \left( \frac{m^2}{\mu^2}, N \right) \]

[Buza, Matiounine, Smith, van Neerven (Nucl.Phys.B (1996))] factorizes into the light flavor Wilson coefficients \( C \) and the massive operator matrix elements (OMEs) of local operators \( O_i \) between partonic states \( j \).

For \( F_2(x, Q^2) : \) at \( Q^2 \gtrsim 10 m^2 \) the asymptotic representation holds at the 1% level.
Status of OME calculations

Unpolarized and Polarized:

Leading Order:
1976 [Witten] and others later.

Next-to-Leading Order:
1992 full $m^2/Q^2$ dependence (numeric) [Laenen, van Neerven, Riemersma, Smith]
1996 analytic asymptotic [Buza, Matiounine, Smith, Migneron, van Neerven], [Bierenbaum, Blümlein, Klein (2007)]

Next-to-Next-to-Leading Order:
2009 a series of moments: [Bierenbaum, Blümlein, Klein]
2010 the $N_F$ terms
2014 the $A_{gq,Q}$, $NS$, $PS$, $A_{gq,Q}$, $A_{gg,Q}$ and all unpolarized log. terms; CC $@O(a_s^2)$
2015 $A_{gg}$ nearly complete, HQ corrections to $g_1$, $xF_3$
2016 HQ $F_{L,2}^W$, HQ sum rules 2- and 3-loop, binomial topologies $A_{Qg}$
2017 all non-elliptic terms to $A_{Qg}$ completed, unpolarized 2-mass terms: $NS$, $A_{gg,Q}$, $PS$
unp. 3-loop anom. dims $\propto T_F$
2018 2-mass $A_{gg,Q}$ unpol.
2019 2-mass $A_{qq}^{PS}$ pol., pol. 3-loop anom. dims $\propto T_F$; complete analytic 2-loop PS pol+unpol WC
2020 2-mass $A_{gq,Q}$ pol., single mass $A_{qq}^{PS}$ pol.
2021 $A_{gq,Q}$ pol., all pol. logarithmic contributions.
Calculation of the 3-loop operator matrix elements

The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:

\[
\delta^i(\Delta p_j)(\Delta p)^{N-1}, \quad N \geq 1
\]

\[
g^2\delta^i(\Delta p_1^a)(\Delta p_2^b)(\Delta p_3^c)(\Delta p_1^d)(\Delta p_2^e)(\Delta p_3^f)(\Delta p_4^g)(\Delta p_5^h)\]

\[
\left[ (t^{12})_i, (\Delta p_1 + \Delta p_2 + \Delta p_3 + \Delta p_4 + \Delta p_5) \right] \left[ (t^{12})_j, (\Delta p_1 + \Delta p_2 + \Delta p_3 + \Delta p_4 + \Delta p_5) \right] \left[ (t^{12})_k, (\Delta p_1 + \Delta p_2 + \Delta p_3 + \Delta p_4 + \Delta p_5) \right] \]

\[
N \geq 3
\]

\[
\gamma_+ = 1, \quad \gamma_- = \gamma_5
\]
Method of calculation

- Resummation of operator insertion into propagator structure ($\Delta \cdot \Delta = 0$):
  \[
  \sum_{N=0}^{\infty} t^N (\Delta \cdot k)^N = \frac{1}{1 - t \Delta \cdot k}
  \]

- Reduction to master integrals using IBP relations implemented in Reduze2 [Manteuffel, Studerus (2012)].

- Solution of master integrals obtained by different methods:
  - direct integration using Mellin-Barnes and $pF_q$-techniques
  - differential equations in resummation variable $t$
  - method of arbitrary high moments, i.e. reconstructing all-$N$ solution from a large number of fixed moments

⇒ All these methods use the packages Sigma, EvaluateMultiSums [Schneider (2007-)] and HarmonicSums [Ablinger et al. (2010-)] which have been developed with these calculations.
The Wilson Coefficients at large $Q^2$

\[
L_{q, (2, L)}^{\text{NS}}(N_F + 1) = a_s^2 [A_{qq,Q}^{(2), \text{NS}}(N_F + 1) \delta_2 + \tilde{C}_{q, (2, L)}^{(2), \text{NS}}(N_F)] + a_s^3 [A_{qq,Q}^{(3), \text{NS}}(N_F + 1) \delta_2 + A_{qg, Q}^{(2), \text{NS}}(N_F + 1) C_{q, (2, L)}^{(1), \text{NS}}(N_F + 1) + \tilde{C}_{q, (2, L)}^{(3), \text{NS}}(N_F)]
\]

\[
L_{q, (2, L)}^{\text{PS}}(N_F + 1) = a_s^3 [A_{qq,Q}^{(3), \text{PS}}(N_F + 1) \delta_2 + N_F A_{qg, Q}^{(2), \text{NS}}(N_F) \tilde{C}_{q, (2, L)}^{(1), \text{NS}}(N_F + 1) + N_F \tilde{C}_{q, (2, L)}^{(3), \text{PS}}(N_F)]
\]

\[
L_{g, (2, L)}^{S}(N_F + 1) = a_s^2 A_{gg, Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g, (2, L)}^{(2)}(N_F + 1) + a_s^2 [A_{qg, Q}^{(3)}(N_F + 1) \delta_2 + A_{gg, Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g, (2, L)}^{(2)}(N_F + 1)
\]

\[
+ A_{gg, Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g, (2, L)}^{(1)}(N_F + 1) + A_{qg, Q}^{(1)}(N_F + 1) N_F \tilde{C}_{q, (2, L)}^{(3), \text{PS}}(N_F + 1) + N_F \tilde{C}_{g, (2, L)}^{(3), \text{PS}}(N_F)]
\]

\[
H_{q, (2, L)}^{\text{PS}}(N_F + 1) = a_s^2 [A_{qg}^{(2), \text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q, (2, L)}^{(2), \text{PS}}(N_F + 1)]
\]

\[
+ a_s^3 [A_{qg}^{(3), \text{PS}}(N_F + 1) \delta_2 + A_{gq}^{(2)}(N_F + 1) \tilde{C}_{g, (1, L)}^{(2)}(N_F + 1) + A_{qg}^{(2), \text{PS}}(N_F + 1) \tilde{C}_{q, (2, L)}^{(1), \text{NS}}(N_F + 1) + \tilde{C}_{q, (2, L)}^{(3), \text{PS}}(N_F + 1)]
\]

\[
H_{g, (2, L)}^{S}(N_F + 1) = a_s [A_{qg}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g, (2, L)}^{(1)}(N_F + 1)]
\]

\[
+ a_s^2 [A_{qg}^{(2)}(N_F + 1) \delta_2 + A_{qg}^{(1)}(N_F + 1) \tilde{C}_{q, (2, L)}^{(1)}(N_F + 1) + A_{gq}^{(1)}(N_F + 1)]
\]

\[
+ a_s^3 [A_{qg}^{(3)}(N_F + 1) \delta_2 + A_{qg}^{(2)}(N_F + 1) \tilde{C}_{q, (2, L)}^{(1)}(N_F + 1) + A_{gq}^{(1)}(N_F + 1)]
\]

\[
+ A_{qg}^{(1)}(N_F + 1) \tilde{C}_{q, (2, L)}^{(2)}(N_F + 1) + A_{gq}^{(1)}(N_F + 1) \tilde{C}_{g, (2, L)}^{(1)}(N_F + 1) + \tilde{C}_{q, (2, L)}^{(3), \text{PS}}(N_F + 1)]
\]

- All first order factorizable contributions and $O(1000)$ fixed moments of $A_{qg}^{(3)}$ are known.
- The case for two different masses obeys an analogous representation.
The variable flavor number scheme

Matching conditions for parton distribution functions: important for LHC physics

\[ f_k(N_F + 2) + f_{\bar{k}}(N_F + 2) = A_{qq, q}^{NS} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot [f_k(N_F) + f_{\bar{k}}(N_F)] + \frac{1}{N_F} A_{qq, \bar{q}}^{PS} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) \]
\[ + \frac{1}{N_F} A_{qg, q} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F), \]

\[ f_{\bar{q}}(N_F + 2) + f_{q}(N_F + 2) = A_{qq, \bar{q}}^{PS} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) + A_{qg} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F), \]

\[ \Sigma(N_F + 2) = \left[ A_{qq, q}^{NS} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{qg, q}^{PS} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{qq, \bar{q}}^{PS} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot \Sigma(N_F) \]
\[ + \left[ A_{qg, q} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) + A_{qg} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \right] \cdot G(N_F), \]

\[ G(N_F + 2) = A_{gq, q} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot \Sigma(N_F) + A_{gg, q} \left( N_F + 2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2} \right) \cdot G(N_F). \]

\[ m_c/m_b \sim 3. \] Decoupling of these two flavors at once; two mass OMEs @ 3 loops.
The NC PS contributions to $F_2(x, Q^2)$ and $F_L(x, Q^2)$

Figure 1: The ratios $R^{(1)}_{2,q}$ (left) and $R^{(1)}_{L,q}$ (right) as a function of $\chi = Q^2/m^2$ for different values of $z$ gradually improved with $\kappa$ suppressed terms. Dotted lines: asymptotic result; dashed lines: $O(m^2/Q^2)$ improved; solid lines: $O((m^2/Q^2)^2)$ improved.
Polarized OMEs

- Polarized OMEs for heavy flavor production at 2-loop order have been calculated before.  
  [Buza et al. (1996), Klein (2009), Hasselhuhn (2013)]

- Calculation of OMEs relied on tensor-decomposition in order to arrive at Larin scheme, i.e.
  \[ \gamma_5 \rightarrow \frac{i}{24} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma. \]

- We found out that a change of projector can accomplish the same:
  \[ \epsilon_{\mu\nu\rho\sigma} \text{tr} \left[ \not p \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma G_q \right] \rightarrow \epsilon_{\mu\nu\rho\sigma} p^\rho \Delta^\sigma \text{tr} \left[ \not p \gamma^\mu \gamma^\nu G_q \right] \]

⇒ this allows to apply all technologies for the unpolarized OMEs directly to the polarized ones, i.e.

1. the terms \( \sim T_F \) of the polarized 3-loop anomalous dimensions from a massive calculation  
   [Behring et al. (Nucl. Phys. B948 (2020))]
2. \( A^{(3),PS}_{qq,Q} \), \( A^{(3),PS}_{Qq} \), \( A^{(3)}_{gq,Q} \) (single and 2-mass contributions)  

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also $\gamma_{qq}^{(2),PS}$ complete. Agree with [Moch, Vermaseren, Vogt (2014)] + all other terms $\propto T_F$. 

> Polarized anomalous dimensions

\[
\gamma_{qq}^{(2)} = C_A N_F T_F \left\{ -\frac{5N^2 + 8N + 10}{N(N+1)(N+2)} S_{-3} - \frac{64P_4}{9N(N+1)^2(N+2)^2} S_1 \right\} - \frac{64P_4}{9N(N+1)(N+2)^2} S_3 + \frac{27N(N+1)^2(N+2)}{S_4}
\]

\[
+ g_T(N) \left\{ \frac{32}{9} S_{-3} - \frac{32}{3} S_{21} \right\} + \frac{64}{9} S_t + \frac{128}{3} S_{-3} + \frac{128}{9} S_{21} \right\} \right\} 
\]

\[
+ C_A N_F T_F \left\{ \frac{5N^2 + 3N + 2}{N(N+1)(N+2)} S_{-3} + \frac{10N^3 + 13N^2 + 29N + 64}{N^2(N+1)(N+2)^2} S_2 
\]

\[
+ \frac{27N^2(N+1)^2(N+2)}{2} S_{-2} + \frac{27(N-1)N^2(N+1)(N+2)^2}{2} S_2 + \frac{16P_9(N)}{3(N-1)N(N+1)(N+2)} S_{11} \right\} 
\]

\[
+ \frac{704}{3} S_{-2} + 128 S_{21} + 512 S_{23} \right\} S_{-2} - 96 S_{23}^2 + 96 S_{-4} + 448 S_{-2} - 128 S_{21} + 512 S_{23}^2 - 768 S_{-2,1} + 192 S_{23} \right\} + \frac{96(N-2)(N+3)P_4}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2}
\]

\[
+ \frac{8P_4}{(N-1)N^2(N+1)^2(N+2)^2} S_1 + \frac{64P_4}{9N(N+1)^2(N+2)^2} S_2 + \frac{3(N-1)N^2(N+1)(N+2)}{2} S_2 
\]

\[
+ \frac{8P_4}{(N-1)N^2(N+1)^2(N+2)^2} S_1 + \frac{96(N-1)N(N+1)(N+2)}{S_{11}} \right\} 
\]

\[
+ C_F N_F T_F \left\{ \frac{64P_4}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2} - \frac{9(N-1)N^2(N+1)(N+2)^2}{16P_4} S_1 \right\} 
\]

\[
+ \frac{3(N-1)N^2(N+1)(N+2)^2}{8P_4} S_{11} + \frac{5N^2(N+1)^2(N+2)}{4} S_1 + \frac{3(N-1)N(N+1)(N+2)}{8P_4} S_{2,1} \right\} 
\]

\[
+ \frac{128}{N(N+1)^2(N+2)} S_{11} - 128 S_{21} + 256 S_{23} - 192 S_{2,1} \right\} 
\]

\[
+ \frac{96(N-1)(N^2 + 3N - 2)}{N(N+1)^2(N+2)} S_{-2} - 2N + N^2 \right\} S_{-2} + \frac{8P_4}{(N-1)(N+1)(N+2)} S_{11} 
\]

\[
+ \frac{8P_4}{(N-1)(N+1)(N+2)} S_{11} - \frac{128(N-1)}{(N+1)(N+2)^2} S_{-2} \right\}
\]
Results: $A_{Qg}^{(3)}$

- 2010: $N_F$ terms
- 2017: calculation of all $\zeta$-color factors except of the pure rational and $\zeta_3$ terms
- 1000 even moments are available; 8000 moments for the $T_F^2$ terms
- Difference equations for the $T_F^2$ available:
  
  $d,o = (1407; 46)$ \([T_F^2 C_A]\), $d,o = (447; 24)$ \([T_F^2 C_A \zeta_3]\), \(d,o = (654; 27)\) \([T_F^2 C_F]\), \(d,o = (283; 15)\) \([T_F^2 C_F \zeta_3]\). 
- The 1st difference eq. is more voluminous than the biggest for the massless WCs
  
  [JB, Kauers, Klein, Schneider, (2008)]
- 3-loop massive form factor: 2 equal mass cases: \((d,o) = (1324; 55)\).
- The solution of the associated differential equations lead to an exponential singularity in $z \in [0, 1]$ both for the rational and $\zeta_3 T_F^2$ terms
- The unification of both difference eqs. is necessary to cancel this spurious singularity.
- New method to solve the associated differential eq.: several formal Laurent series to a given finite order around $z_0 \in [0, 1]$ to map out the whole region with overlapping convergence radii.
  
  [J. Ablinger, J. Blümlein, C. Schneider, 2021]
- In this way a full representation, tuneable to any precision is obtained.
  
  Here it is irrelevant, if elliptic, hyper-elliptic or whatsoever structures are present.
- One should notice that even any HPL solution finally needs an efficient numerical solution.
- The generation of high number of moments for the non $T_F^2$ terms is underway.
Single mass case: mathematical structures

- Before 1998:

- 1997: Shuffle algebras [Hoffman], 2003 algebraic relations [Blümlein]
- 1998: Harmonic sums [Vermaseren; Blümlein & Kurth]
- 1999: Harmonic polylogarithms [Remiddi, Vermaseren]
- 2004: infinite binomial and inverse binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2009: structural relations of harmonic sums [Blümlein]
- 2009: MZV data mine [Blümlein, Broadhurst, Vermaseren]
- 2011: cyclotomic harmonic sums and polylogarithms [Ablinger, Blümlein, Schneider]
- 2013: detailed theory generalized sums [Ablinger, Blümlein, Schneider]
- 2014: finite binomial and inverse binomial sums [Ablinger, Blümlein, Raab, Schneider]
- 2014: complete elliptic integrals in Feynman diagrams: [very many authors: Broadhurst et al., Remiddi et al., Weinzierl et al., Duhr et al., ........]
- 2017: method of arbitrary high moments [Blümlein, Schneider]
- 2021: Iterated integrals over letters induced by quadratic forms [Ablinger, Blümlein, Schneider]

List far from being complete.
2-mass contributions
## Two-mass contributions: mathematical functions

**Harmonic Sums**

\[ A^{(3),NS}_{qq,Q}, A^{(3)}_{gq,Q} \]

Harmonic Sums

\[ \sum_{i=1}^{N} \frac{1}{\beta} \sum_{j=1}^{i} \frac{1}{i} \]

HPLs

\[ \int_{0}^{x} d\tau_1 \int_{0}^{\tau_1} \frac{d\tau_2}{1-\tau_2} \]

**Generalized harmonic and binomial sums**

\[ A^{(3)}_{gg,Q} \]

Generalized harmonic and binomial sums

\[ \sum_{i=1}^{N} \frac{4^{i}(1-\eta)^{-i}}{i^{2}} \sum_{j=1}^{i} \frac{(1-\eta)^{j}}{j^{2}} \]

Iterated integrals over root and \( \eta \) valued letters

\[ \int_{0}^{x} d\tau_1 \sqrt{\frac{\tau_1(1-\tau_1)}{1-\tau_1(1-\eta)}} \int_{0}^{\tau_1} \frac{d\tau_2}{\tau_2} \]

**Iterated integrals over root valued letters with restricted support**

\[ \theta(x - \eta) \int_{0}^{x} \frac{(1-x)/\eta}{d\tau} \sqrt{1-4\tau} \]

\[ A^{(3),PS}_{Qq} \]
\[ A_{gg, Q}^{(3)}(N) = \frac{1}{2} (1 + (-1)^N) \left\{ T_F^2 \left\{ \frac{32}{3} (L_1^3 + L_2^3) + \frac{64}{3} L_1 L_2 (L_1 + L_2) + 32 \zeta_2 (L_1 + L_2) + \frac{128}{9} \zeta_3 \right\} \\
+ C_F T_F^2 \left\{ \cdots + 32 \left( H_0^2(\eta) - \frac{1}{3} S_2 \right) S_1 + \frac{128}{3} S_{2,1} - \frac{64}{3} S_{1,1,1} \left( \frac{1}{1 - \eta}, 1 - \eta, 1, N \right) \right\} \\
- \frac{4 P_{41}}{3(N - 1)N^3(N + 1)^2(N + 2)(2N - 3)(2N - 1) \left( \frac{\eta}{1 - \eta} \right)^N} \left[ H_0^2(\eta) \right] \\
- 2 H_0(\eta) S_1 \left( \frac{\eta - 1}{\eta}, N \right) - 2 S_2 \left( \frac{\eta - 1}{\eta}, N \right) + 2 S_{1,1} \left( \frac{\eta - 1}{\eta}, 1, N \right) + \ldots \right\} \\
+ C_AT_F^2 \left\{ \cdots + \left[ \frac{8 P_{65}}{3645 \eta (N - 1)N^3(N + 1)^3(N + 2)(2N - 3)(2N - 1)} \right] \\
+ \frac{8 P_{37} H_0(\eta)}{45 \eta (N - 1)N^2(N + 1)^2(N + 2)} + \frac{2 P_{23} H_0^2(\eta)}{9 \eta (N - 1)N(N + 1)^2} + \frac{32}{27} H_0^3(\eta) + \frac{128}{9} H_{0,0,1}(\eta) \\
+ \frac{64}{9} H_0^2(\eta) H_1(\eta) - \frac{128}{9} H_0(\eta) H_{0,1}(\eta) \right\} S_1 \\
+ \frac{2^{-1 - 2N} P_{47}}{45 \eta^2(N - 1)N(N + 1)^2(N + 2)(2N - 3)(2N - 1)} \left( \frac{2N}{N} \right) \sum_{i=1}^{N} \frac{4^i \left( \frac{\eta}{1 - \eta} \right)^i}{i^{2i}} \left\{ \frac{1}{2} H_0^2(\eta) \right\} \\
S_{1,1} \left( \frac{\eta - 1}{\eta}, 1, i \right) + \ldots \right\}, \quad \eta = \frac{m_c^2}{m_b^2}. \]
Results: $A_{gg,Q}^{(3)}$

The two mass contributions over the whole $T_F^2$- contributions to the OME $A_{gg,Q}^{(3)}$:
The 2-mass variable flavor number scheme at NLO

\[ \Sigma(x, Q^2)^{2\text{-mass}} / \Sigma(x, Q^2) \]

\[ f_{b+b}(x, Q^2)^{2\text{-mass}} / f_{b+b}(x, Q^2) \]

- The ratio of the 2-mass contributions to the singlet parton distribution \( \Sigma(x, Q^2) \) (left) and the heavy flavor parton distribution \( f_{b+b}(x, Q^2) \) (right) over their full form in percent for \( m_c = 1.59 \text{ GeV}, m_b = 4.78 \text{ GeV} \) in the on-shell scheme. Dash-dotted line: \( Q^2 = 30 \text{ GeV}^2 \); Dotted line: \( Q^2 = 100 \text{ GeV}^2 \); Dashed line: \( Q^2 = 1000 \text{ GeV}^2 \); Full line: \( Q^2 = 10000 \text{ GeV}^2 \).

- For the PDFs the \( \text{NNLO} \) variant of ABMP16 with \( N_f = 3 \) flavors was used.

Alekhin et al., Phys. Rev. D 96 (2017) 1
HQ contributions in N$^3$LO QCD analyses

[Blümelin & Saragnese, 2021]

\[ F_2(x, Q^2)^{\text{NS}}(N) = E_{\text{NS}}(Q^2, Q_0^2; N) F_2(x, Q_0^2)^{\text{NS}}(N) \]

evolution operator: \( E_{\text{NS}}(Q^2, Q_0^2; N) \)

\( F_2(x, Q_0^2)^{\text{NS}}(z) \) is measured. \( \alpha_s(M_Z) \) from a one-parameter fit (knowing \( m_c \) and \( m_b \)).

Future high luminosity analyses at the EIC or LHeC.
Heavy Flavor contribution to $F_2$

$$Q^2 = 100 \text{ GeV}^2$$
Conclusions and Outlook

- Most of the massive 3–loop OMEs (VFNS) and asymptotic Wilson coefficients have been calculated in the unpolarized and polarized case (for single- and two-mass).
- \( A_{gg,Q}^{(3)} \) nearly completed; \( A_{Qg}^{(3)} \) going to come.
- Hugh difference equations can be solved either fully analytically or in representations which can be tuned to high precision.
- Various new computing technologies were developed for massive Feynman diagram calculations.
- The properties of the contributing mathematical function spaces were worked out.
- In the 2-loop case progress has been made in the analytic calculation of power corrections, relevant for the region of smaller virtualities.
- (\( T_F \) contributions to the) 3–loop anomalous dimensions appear as by-product, [see also the talks by P. Marquard and S. Moch.]
- Applications also to QED corrections in \( e^+ e^- \rightarrow Z^* / \gamma^* \) [K. Schönwald’s talk.]