From Moments to Functions in Higher Order QCD

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- Introduction
- Single Scale Feynman Integrals as Recurrent Quantities
- Establishing and Solving Recurrences
- Application to 3-Loop Anomalous Dimensions and Wilson coefficients
- Conclusions

1. Introduction

- Higher order calculations in Quantum Field Theories easily become tedious due to the larger number of terms and the sophistication of the Feynman parameter integrals.
- This even applies to Zero Scale and Single Scale Quantities.
- Even more this is the case for higher scale problems.
- While in the latter case the mathematical structure of the solution for the Feynman Integrals is widely unknown, it is explored to a certain extent for Zero Scale and Single Scale quantities.
- Zero Scale quantities are the expansion coefficients of the running couplings and masses, fixed moments of splitting functions etc.
- They can be expressed by rational numbers and certain special numbers as multiple ζ-values and related quantities.

Introduction

- Single Scale quantities depend on a scale $z \in [0,1]$, with z a ratio of Lorentz invariants. One may perform a Mellin Transform over z

$$\int_0^1 dz z^{N-1} f(z) = M[f](N)$$

- Here one assumes $N \in \mathbf{N}, N > 0$. Due to this the problem on hand becomes discrete.
- One may seek a description in terms of difference equations.
- Zero Scale problems are obtained from Single Scale problems treating N as a fixed integer or considering the limit $N \to \infty$.

Some Remarks about MZV's

- General question on the bases of MZV's: length in the non-alternating and alternating cases.
- Do Zero Scale Feynman integrals always lead to MZV's ?
- No! e.g. Y. Andre, 2008.
- At lower orders in perturbation theory one has just MZV's even in single-mass problems.
- J.B., Broadhurst, Vermaseren, 2008: explicit calculation of bases and all relations of alternating MZV's to w=12 and non-alternating MZV's to w=22. [World Record.]; Verification to w=26.
- Broadhurst 1996 conjecture is proven. shuffles, stuffles, doubling, gen. doubling relations However, we did not find further reductions - which still may exist.

Introduction

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments ?
- This is possible for recurrent quantities.
- At least up to 3-loop order, presumably to higher orders, single scale quantities belong to this class.
- Goal : design a general formalism to solve the problem.

2. Single Scale Feynman Integrals as Recurrent Quantities

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments ?
- Polynomials and Nested Harmonic Sums obey recurrence relations, so do their polynomials.
- Example: Harmonic Sums or linear combinations thereof:

$$F(N+1) - F(N) = \frac{\mathrm{sign}(a)^{N+1}}{(N+1)^{|a|}}$$

is solved by $S_a(N)$; and similarly for deeper nested sums

$$S_{a,\vec{b}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(a))^k}{k^{|a|}} S_{\vec{b}}(k)$$

.

Single Scale Feynman Integrals as Recurrent Quantities

- Feynman integrals have often a form like

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z} H_{\vec{a}}(z), \qquad \int_0^1 dz \frac{(-z)^{N-1}-1}{1+z} H_{\vec{a}}(z)$$

- This structure leads to recurrences.
- It is very likely that single scale Feynman diagrams do always obey difference equations.

3. Establishing and Solving Recurrences

- One seeks the relation

$$\sum_{k=0}^{l} \left[\sum_{i=0}^{d} c_{i,k} N^i \right] F(N+k) = 0 \ .$$

- The corresponding linear system is dense.
- Rational number arithmetics is not applicable for the large systems to be determined; $C_{2,q,C_F^3}^{(3)}$ would require 11 Tb of memory.
- Use arithmetic in finite fields together with Chinese remaindering $\Longrightarrow$ few Gb of memory
- The linear system approximately minimizes for $l \approx d$.
- Join different recurrences found to reduce l to a minimal value.

Establishing and Solving Recurrences

- For the solution of the recurrence low degrees are clearly preferred.
- The linear difference equation of order l with polynomial coefficients is equivalent to a linear system in l variables.
- It is solved in $\Pi - \Sigma$ fields.
- Apply advanced symbolic summation methods: telescoping, creative telescoping and its refinement. Code: `sigma`.
- The solutions are found as linear combinations of rational terms in N combined with functions, which cannot be further reduced in the $\Pi - \Sigma$ fields. In the present application they turn out all to be harmonic sums $S_{\vec{b}}(N)$.
- Other or higher order applications may consist of other sums too, which are uniquely found by the algorithm.

4. Application to 3-Loop Anomalous Dimensions and Wilson coefficients

- We apply the method for the unfolding of the unpolarized anomalous dimensions and Wilson coefficients up to 3-loop order.
- $\Longrightarrow$ analyze for individual color factors; 141 contributions from $1 - 3$ loops
- Input: Moch, Vermaseren, Vogt, 2004/05. The expressions are given in terms of harmonic sums.
- Calculate the moments (rational numbers) recursively through recursions for the harmonic sums; MAPLE code.
- Establish the corresponding difference equation by a recurrency finder; build a difference equation of minimal order possible; test the recurrency.
- Solve the difference equation order by order with the summation package `sigma` C. Schneider.; most complicated cases: 4 weeks @ $\leq$ 10Gb, 2 GHz Proc.

Input

```
C2qq3CF^3
N=3:
#11 digits / #10 digits

-98268084191 / 1166400000

N=500:
#1262 digits / #1256 digits

1641840770424196780953020619176376506284303544481262083057197600746507008493793994
4224110323441591630311482222058287688942209570859151121677307585313995100978363179
2518952817622034037186132846974627021672678012913675099511203807811938593043910803
5044345920218696052588332036355325089998361354226882367322149037631053761764348772
5403810874264968729520075619227285471802419403727207822473765999900236383740315299
2050533601633484348249454757555344664210814111140065475391136798689167410065076749
3578709478683390573977410013520894494463909291327425815766566386397276158317387748
5945471392646089700875157445075073192328542890965462004805711998748144414379386093
9937361798029044425789953726133675199790523770427298500510063464061985840066296071
3372543015648919155964069606994597363886301185067827291937065300754786947063672848
9382081926871078600328628131936766057475970450896556667622163365895808773428119721
5352792131089063577045069693962213061198894057033606068695607123271969726981060056
0115846094360239986233917872260722277322690450132376836253549152130116645670565045
9666945920164586023958060271746606798898861360772333088030741775605546518788793327
2264368297071217405654474375844238250889238538974548421298170425909521742559494728
720178770039473965622616598603668391544078534623381716482270131342667953202518 47
/
```

3057444614247225372882570514367358697278130741348282122206492932820352440850471902
7491046962105336645563654873675690796713906565688820365601907263710863954826386081
3227580037879361869941003802807590860358894142891046776447162895908787986423254678
5776778283337231702130612499429819559798501074020676282769289102955679421885795867
1982932998601320344971927374905889934059987271939760212836368619501189238215442366
3805773701929509268157747992859384837403751183019423692868569168206789710047557452
5131217382272060267681480496298975522467614707848639773185909858278799786637303834
1017166676276847525704755493166263297079720470719813623901545811953853986456533543
9994182050551827959988760121168490745476969259468454613431624179198860751513076481
0304734205926703138519418575731315944374873897873646706993620825697218523316375559
4068222004765962715924208526106008109740402380126260947524640509361283802755722132
4856690051525724685919792641506082307567956962328560073471086799287131287564668441
6256698083504233897436484702002471314330803421467773925541151273924985946178771189
2312437162213438137703896064734987157020801413153555435311326719739117599044341913
5922693587373856609594245948237469293148702516714038297077639382332251255360181047
49658623247509112659762997679737527882711111677459300352000000000000000000

N=5114:

#13388 digits / #13381 digits

Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$P_{NS,0}$	14	2	3	0.05	0.087	1 [1]	0.55
$P^-_{NS,1,C_F^2}$	142	5	31	3.32	4.666	6 [10]	7.45
$P^-_{NS,1,C_AC_F}$	109	4	24	1.91	2.834	6 [7]	6.28
$P^-_{NS,1,C_FN_F}$	24	2	7	0.13	0.271	2 [2]	0.92
$P^+_{NS,1,C_F^2}$	142	5	31	3.35	4.707	6 [10]	7.45
$P^+_{NS,1,C_AC_F}$	109	4	23	1.88	2.703	6 [7]	6.23
$P^+_{NS,1,C_FN_F}$	24	2	7	0.09	0.271	2 [2]	0.89
$P^-_{NS,2,C_F^3}$	1079	16	192	3152.19	529.802	25 [68]	1194.41
$P^-_{NS,2,C_F^3\zeta_3}$	48	3	11	0.49	0.643	1 [1]	1.56
$P^-_{NS,2,C_AC_F^2}$	974	15	181	1736.08	450.919	25 [62]	1194.41
$P^-_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.53	0.643	1 [1]	1.53
$P^-_{NS,2,C_A^2C_F}$	749	12	147	1004.12	242.892	25 [62]	1100.88
$P^-_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.56	0.643	1 [1]	1.56
$P^-_{NS,2,C_FN_F^2}$	39	2	11	0.31	0.369	3 [3]	1.20
$P^-_{NS,2,C_F^2N_F}$	377	8	68	76.34	33.946	12 [24]	72.22
$P^-_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.53
$P^-_{NS,2,C_AC_FN_F}$	356	7	62	65.25	23.830	12 [20]	52.67
$P^-_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.55
$P^+_{NS,2,C_F^3}$	1079	16	192	4713.27	527.094	25[68]	1165.22
$P^+_{NS,2,C_F^3\zeta_3}$	48	3	11	0.55	0.643	1[1]	1.562
$P^+_{NS,2,C_AC_F^2}$	974	15	178	1715.03	442.031	25[62]	889.047
$P^+_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.531
$P^+_{NS,2,C_A^2C_F}$	749	12	146	991.22	240.325	25[50]	516.812
$P^+_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.593
$P^+_{NS,2,C_F^2N_F}$	377	8	69	111.38	33.872	12[24]	71.235
$P^+_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.531
$P^+_{NS,2,C_AC_FN_F}$	307	7	61	48.62	23.808	12[24]	71.235
$P^+_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.547
$P^+_{NS,2,C_FN_F^2}$	39	2	11	0.40	0.369	3[3]	1.172
$P^-_{NS,2,N_Fd_{abc}}$	39	2	11	0.55	0.369	3 [3]	1.19

Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function $F_2(x, Q^2)$.

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$C^{(1)}_{2,q,C_F}$	35	3	7	0.26	0.429	2[3]	1.13
$C^{(2)}_{2,q,C_F^2}$	689	11	137	1134.10	177.806	13[39]	258.24
$C^{(2)}_{2,q,C_AC_F}$	545	10	121	413.33	127.893	12[35]	178.73
$C^{(2)}_{2,q,C_F^2\zeta_3}$	15	2	3	0.27	0.100	1[1]	0.54
$C_{2,q,C_AC_F\zeta_3}$	15	2	3	0.27	0.112	1[1]	0.55
C_{2,q,N_FC_F}	71	4	16	2.68	1.655	4[10]	3.95
$C^{(3)}_{2,q,C_F^3}$	5114	35	938	1.78886×10^6	30394.173	58[289]	0.50924×10^6
$C^{(3)}_{2,q,C_F^3\zeta_3}$	284	8	64	31.02	32.363	7 [18]	27.60
$C^{(3)}_{2,q,C_F^3\zeta_4}$	19	2	5	0.08	0.163	1 [1]	0.47
$C^{(3)}_{2,q,C_F^3\zeta_5}$	19	2	5	0.08	0.163	1 [1]	0.47
$C^{(3)}_{2,q,C_F^2C_A}$	5059	35	930	1.69267×10^6	30122.380	60 [290]	0.47780×10^6
$C^{(3)}_{2,q,C_F^2C_A\zeta_3}$	284	8	64	34.00	33.400	7 [18]	28.53
$C^{(3)}_{2,q,C_F^2C_A\zeta_4}$	48	3	11	0.32	0.643	1[1]	1.01
$C^{(3)}_{2,q,C_F^2C_A\zeta_5}$	19	2	5	0.08	0.167	1 [1]	0.42
$C^{(3)}_{2,q,C_FC_A^2}$	4564	33	863	1.38918×10^6	24567.518	60 [258]	0.34941×10^6
$C^{(3)}_{2,q,C_FC_A^2\zeta_3}$	284	8	63	26.83	29.918	7 [17]	30.46
$C^{(3)}_{2,q,C_FC_A^2\zeta_4}$	48	3	11	0.32	0.643	1 [1]	1.01
$C^{(3)}_{2,q,C_FC_A^2\zeta_5}$	19	2	5	0.08	0.175	1 [1]	0.42
$C^{(3)}_{2,q,C_F^2N_F}$	1762	20	348	40237.45	2339.516	29 [107]	7548.56
$C^{(3)}_{2,q,C_F^2N_F\zeta_3}$	87	4	21	1.94	2.354	3 [5]	2.83
$C^{(3)}_{2,q,C_F^2N_F\zeta_4}$	15	2	3	0.07	0.101	1 [1]	0.34
$C^{(3)}_{2,q,C_FC_AN_F}$	1847	20	360	47661.64	2507.362	29 [111]	7525.89
$C^{(3)}_{2,q,C_FC_AN_F\zeta_3}$	89	4	24	2.47	2.935	3 [8]	3.19
$C^{(3)}_{2,q,C_FC_AN_F\zeta_4}$	15	2	3	0.06	0.101	1 [1]	0.34
$C^{(3)}_{2,q,C_FN_F^2}$	131	5	30	58.00	5.347	7 [22]	8.97
$C^{(3)}_{2,q,C_FN_F^2\zeta_3}$	15	2	3	0.06	0.101	1 [1]	0.38
$C^{(3)}_{2,q,dabc}$	1199	14	242	6583.27	738.498	14 [62]	841.24
$C^{(3)}_{2,q,dabc\zeta_3}$	109	4	25	2.33	3.164	2[7]	2.40
$C^{(3)}_{2,q,dabc\zeta_5}$	8	1	2	0.03	0.041	0[0]	0.10

A complicated example

$C_{2,q} \propto C_F^3$:

- 5114 moments needed. Use a clever way to calculate the input.
- Largest moment: fraction: numerator 13388 digits; denominator 13381 digits.
- CPU time to determine the recurrence: 20.7 days.

- modular prediction of the dimension: 4 h; modular LEQ's: 5.8 days; modular operator GCDs: 11 days; Chinese Remainder + Rat. Reconstruction: 3.8 days. 140 large primes needed.

31 MB recurrence is established; largest integer: 1227 digits; order: 35; degree: 938

- Solved by `sigma` within about one week.
- 3 loop anomalous dimensions: much smaller recurrences & shorter computation times.

$\Longrightarrow$ In practice no method does yet exists to calculate such a high number of moments.

$\Longrightarrow$ Existence proof of a quite general and powerful automatic difference-equation solver, standing rather demanding tests.

Structure of the Results

- We carry out all algebraic reductions, J.B. 2003.
- Different color factor contributions lead to the same or nearly the same amount of sums at a given quantity.
- This points to the fact that the amount of harmonic sums is governed by topology rather than the fields and color.
- The linear harmonic sum representations by Vermaseren et al. 2004/05 require many more sums than our representation.
- There are reductions in the number of sums as $264 \longrightarrow 29$.
- Further use of structural relations will lead to maximally 35 sums for the 3-loop Wilson coefficients; J.B. 2008.

$$
\begin{aligned}
\gamma_{qq,(0)}(N) &= S_1(N) - \frac{3N^2+3N+2}{N(N+1)} && (1)\\
\gamma^{(+)}_{qq,1}(N) &= C_F N_F \left[S_1(N) - \frac{3N^2+3N+2}{N(N+1)} \frac{3N^4+6N^3+47N^2+20N-12}{9N^2(N+1)^2} - \frac{40}{9} S_1(N) + \frac{8}{3} S_2(N) \right] \\
&+ C_A C_F \left[\frac{-51N^5 - 153N^4 - 757N^3 - 144(-1)^N N^2 - 851N^2 - 208N + 132}{18N^2(N+1)^3} \right.\\
&\quad +8S_{-3}(N) + \frac{268}{9} S_1(N) + S_{-2}(N) \left(16S_1(N) - \frac{8}{N(N+1)} \right) - \frac{44}{3} S_2(N) + 8S_3(N) \\
&\quad \left. -16S_{-2,1}(N) \right] \\
&+ C_F^2 \left[\frac{-3N^6 - 9N^5 - 9N^4 + 32(-1)^N N^3 - 59N^3 - 40N^2 - 32N - 8}{2N^3(N+1)^3} - 16S_{-3}(N) \right.\\
&\quad + S_{-2}(N) \left(\frac{16}{N(N+1)} - 32S_1(N) \right) + S_1(N) \left(\frac{8(2N+1)}{N^2(N+1)^2} - 16S_2(N) \right) \\
&\quad \left. + \frac{4\left(3N^2+3N+2\right) S_2(N)}{N(N+1)} - 16S_3(N) + 32S_{-2,1}(N) \right] && (2)\\
\gamma^{(-)}_{qq,1}(N) &= C_F N_F \left[\frac{3N^4+6N^3+47N^2+20N-12}{9N^2(N+1)^2} - \frac{40}{9} S_1(N) + \frac{8}{3} S_2(N) \right] \\
&+ C_A N_F \left[\frac{-144(-1)^N N^3 - (N+1)\left(51N^5+102N^4+655N^3+484N^2+12N+144\right)}{18N^3(N+1)^3} \right.\\
&\quad +8S_{-3}(N) + \frac{268}{9} S_1(N) + S_{-2}(N) \left(16S_1(N) - \frac{8}{N(N+1)} \right) - \frac{44}{3} S_2(N) + 8S_3(N) \\
&\quad \left. -16S_{-2,1}(N) \right] \\
&+ C_F^2 \left[\frac{-3N^6 - 9N^5 - 9N^4 + 32(-1)^N N^3 + 5N^3 + 24N^2 + 32N + 24}{2N^3(N+1)^3} - 16S_{-3}(N) \right.\\
&\quad + S_{-2}(N) \left(\frac{16}{N(N+1)} - 32S_1(N) \right) + S_1(N) \left(\frac{8(2N+1)}{N^2(N+1)^2} - 16S_2(N) \right) \\
&\quad \left. + \frac{4\left(3N^2+3N+2\right) S_2(N)}{N(N+1)} - 16S_3(N) + 32S_{-2,1}(N) \right] && (3)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{qq,2}^{(-)}(N) &= C_F^2 N_F \zeta_3 \left[32 S_1(N) - \frac{8\,(3N^2+3N+2)}{N(N+1)} \right] \\
&+ C_A C_F N_F \zeta_3 \left[\frac{8\,(3N^2+3N+2)}{N(N+1)} - 32 S_1(N) \right] \\
&+ C_F N_F^2 \left[\frac{51N^6+153N^5+57N^4+35N^3+96N^2+16\ N-24}{27N^3(N+1)^3} - \frac{16}{27} S_1(N) - \frac{80}{27} S_2(N) \right. \\
&\left. + \frac{16}{9} S_3(N) \right] \\
&+ C_F^3 \zeta_3 \left[-\frac{24\,(5N^4+10N^3+9N^2+4N+4)}{N^2(N+1)^2} - 192 S_{-2}(N) \right] \\
&+ C_A C_F^2 \zeta_3 \left[\frac{36\,(5N^4+10N^3+9N^2+4N+4)}{N^2(N+1)^2} + 288 S_{-2}(N) \right] \\
&+ C_A^2 C_F \zeta_3 \left[-\frac{12\,(5N^4+10N^3+9N^2+4N+4)}{N^2(N+1)^2} - 96 S_{-2}(N) \right] \\
&+ C_A C_F N_F \\
&\left[-\frac{2(N+1)\,(270N^7+810N^6-463\ N^5-1392N^4-211N^3-206N^2-156N+144)}{27N^4(N+1)^4} \right. \\
&-2\frac{96N^4(4N+1)}{27N^4(N+1)^4} + \frac{64}{3} S_4(N) + S_{-3}(N) \left(\frac{32}{3} S_1(N) - \frac{16\,(10N^2+10N+3)}{9N(N+1)} \right) \\
&+ \frac{1336}{27} S_2(N) + S_{-2}(N) \left(\frac{16\,(16N^2+10N-3)}{9N^2(N+1)^2} - \frac{320}{9} S_1(N) + \frac{64}{3} S_2(N) \right) \\
&- \frac{8\,(14N^2+14N+3)\,S_3(N)}{3N(N+1)} + \frac{80}{3} S_4(N) + \frac{32\,(10N^2+10N-3)\,S_{-2,1}(N)}{9N(N+1)} \\
&+ S_1(N) \left(-\frac{4\,(209N^6+627N^5+627N^4+72N^3+281N^3+36N^2+36N+18)}{27N^3\ (N+1)^3} \right. \\
&\left. \left. + 16 S_3(N) + \frac{64}{3} S_{-2,1}(N) \right) - \frac{32}{3} S_{2,-2}(N) - \frac{64}{3} S_{3,1}(N) - \frac{128}{3} S_{-2,1,1}(N) \right] \\
&+ \text{various pages more.} \qquad (1) \\
\gamma_{qq,1}^{(v)}(N) &= \frac{d_{abc}}{N_c} N_F \left[\frac{51N^6+153N^5+57N^4+35N^3+96N^2+16N-24}{27N^3(N+1)^3} - \frac{16}{27} S_1(N) - \frac{80}{27} S_2(N) \right. \\
&\left. + \frac{16}{9} S_3(N) \right] \qquad (2)
\end{aligned}
$$

Other Processes

- The present method can be applied irrespectively of the loop order to all single scale processes.

- As has been found before J.B. & Ravindran 2004/05, J.B. & Moch 2005, J.B. & S. Klein 2007 representing a large number of 2- and 3-loop processes in terms of harmonic sums, the basis elements emerging are always the same.

  {anomalous dimensions, Wilson coefficients, space- and time-like, polarized/unpolarized, Drell-Yan process, hadronic Higgs Boson production in the heavy mass limit, HO QED corrections in e^+e^- annihilation, soft+virtual corrections to Bhabha scattering}.

- The formalism also applies to Heavy Flavor Wilson Coefficients at $Q^2 \gg m^2$, c.f. Bierenbaum, J.B., Klein 2007/08.

- Basis to $w = 6$, c.f. J.B. 2008.

5. Conclusions

- We established a general algorithm to calculate the exact expression for single scale quantities from a finite (suitably large) number of moments (zero scale quantities).
- The latter ones are much more easily calculable.
- We applied the method to the anomalous dimensions and Wilson coefficients up to 3-loop order.
- To solve 3-loop problems this way is not possible at present, since the number of required moments is too large for the methods available.
- We attempted to solve the quantities for all color projections at once. This problem is too voluminous.
- Yet we showed that giant difference equations [order 35; degree $\sim$ 1000] can be reliably and fast established and solved unconditionally for advanced problems in Quantum Field Theory.