INTRODUCTION INTO QUANTUM CHROMODYNAMICS

– THE THEORY OF STRONG INTERACTIONS

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DESY

GRADUIERTENKOLLEG: HUMBOLDT UNIVERSITÄT BERLIN, DESY-ZEUTHEN, FREIE UNIV. BERLIN, APRIL 1997

4 LECTURES

1) QUARKS AND GLUONS – THE FIELDS IN QCD
2) ASYMPTOTIC FREEDOM AND THE BEHAVIOUR OF THE STRONG COUPLING CONSTANT
3) THE PARTON MODEL, PARTON DISTRIBUTIONS & DIS STRUCTURE FUNCTIONS
4) SCALING VIOLATIONS, MASS SINGULARITIES AND EVOLUTION EQUATIONS

LECTURE 1

1) QUARKS AND GLUONS – THE FIELDS IN QCD

HADRONS : MESONS : $\pi^\pm, \pi^0, \rho^\pm, \rho^0, \omega$

M

$|q\bar{q}\rangle$

BARYONS : $n, p, \Lambda, \Omega^-, \ldots$

B

$|qqq\rangle$

FLAVOUR : $q = u, d, s \longrightarrow$ LIGHT FLAVOURS

$m_q \approx 0$

$$\left.\begin{array}{ll} \text{MESONS:} & 3 \otimes \bar{3} = 1 \oplus 8 \\ \text{BARYONS:} & 3 \otimes 3 \otimes = 1 \oplus 8 \oplus 8 \oplus 10 \end{array}\right\} \text{Multiplets.}$$

QUANTUM NUMBERS OF QUARKS:

• SPIN : $$\left.\begin{array}{l} S(B) = \frac{1}{2}, \frac{3}{2} \\ S(M) = 0, 1 \end{array}\right\} \curvearrowright \quad S(q) = S(\bar{q}) = \frac{1}{2}$$

Fermions

$\longrightarrow \Delta^{++} = |uuu\rangle \qquad Q(\Delta^{++}) = 2$

• CHARGE : $Q(u, c, t) = +\frac{2}{3}$

$Q(d, s, b) = -\frac{1}{3}$.

HOW CAN A FERMIONIC $|qqq\rangle$ STATE EXIST?

PAULI PRINCIPLE !

NEW QUANTUM NUMBER : COLOUR

$\boxed{\text{QUARKS ARE COLOUR TRIPLETS}}$ $\quad 3$

ANTIQUARKS ARE COULOR ANTITRIPLETS $\bar{3}$

$\longrightarrow$ COMPLEX RERESENTATION REQUESTED.

WHAT IS THE RIGHT GAUGE GROUP ?

REQUESTS: • M & B ARE COLOUR SINGLETS.

$$\left.\begin{array}{l} M: \; 3 \otimes \bar{3} = \underline{1} \oplus 8 \\ B: \; 3 \otimes 3 \otimes 3 = \underline{1} \oplus 8 \oplus 10. \end{array}\right\} \text{EXAMPLE: } SU(3)_c$$

• $|qq\rangle, |\bar{q}\bar{q}\rangle, |qqqq\rangle$ ARE NO STABLE GROUNDSTATES

$\longleftrightarrow$ NO COLOUR SINGLETS

$|q\bar{q}\rangle$:

$3 \otimes 3 = 3^* \oplus 6$ ✓

$|qqqq\rangle$:

$3 \otimes 3 \otimes 3 \otimes 3 = 3 \oplus 3 \oplus 3 \oplus 6^* \oplus 6^* \oplus 15 \oplus 15 \oplus 15 \oplus 15'$ ✓

WHY $SU(3)_c$?

CONSIDER THE CLASS OF COMPACT SIMPLE LIE GROUPS

• $SU(N+1)$, $SU(2N+1)_{N \geq 1}$, $Sp(N)$, $SO(2N)_{N>2}$, E_6, E_7, E_8, F_4, G_2

- THOSE WITH 3 DIMENSIONAL IRRED. REPRESENTAT

SU(2), SU(3), SO(3), Sp(1)

$\hookrightarrow$ $SO(3) \simeq SU(2) \simeq Sp(1)$ ARE ISOMORPH.
THEIR TRIPLET REPR.
IS <u>REAL</u>!

$\curvearrowright$ ONLY $\underline{SU(3)_c}$ HAS A COMPLEX 3-REPRESENTATION.

- QUARKS WERE NOT OBSERVED AS ASYMPTOTIC STATES! $\leftrightarrow$ CONFINEMENT

- MAY ONLY BE STUDIED AT <u>SHORT</u> DISTANCES.
$\leftrightarrow$ HIGH $\underline{Q^2}$!

WHY DO WE BELIEVE IN QUARKS ?

- <u>FRACTIONAL CHARGE</u>

$M_{V^0}^2 \Gamma(V^0 \to f\bar{f}) \sim e_q^2 |\psi(0)|^2$ – STUDY RATIOS OF THESE OBSERVABLES.

$$|\rho_0\rangle = \frac{1}{\sqrt{2}} |u\bar{u} - d\bar{d}\rangle \quad : \quad e_q^2 := \frac{1}{2}\left(\frac{2}{3}+\frac{1}{3}\right)^2 = \frac{1}{2}$$

$$|\omega_0\rangle = \frac{1}{\sqrt{2}} |u\bar{u} + d\bar{d}\rangle \quad : \quad e_q^2 := \frac{1}{2}\left(\frac{2}{3}-\frac{1}{3}\right)^2 = \frac{1}{18}$$

$$|\phi\rangle = s\bar{s} \quad : \quad e_q^2 := \frac{1}{9}$$

$$M_i^2 \Gamma(V_i \to f\bar{f}) \quad : \quad \rho_0 : \omega_0 : \phi = \frac{1}{2} : \frac{1}{18} : \frac{1}{9}$$

$|\psi(0)|^2 \sim$ same value.

o STRUCTURE FUNCTION RATIO AND FRACTIONAL CHARGE:

$$\frac{F_2^{ed}}{W_2^{\nu d} + W_2^{\bar{\nu} d}} = \frac{\frac{5}{18}\Sigma + \overbrace{\frac{1}{6}(c-s)}^{\approx 0}}{\Sigma} \approx \frac{5}{18}$$

$$\Sigma = \sum_{i=1}^{N_f} x\,(q_i + \bar{q}_i)$$

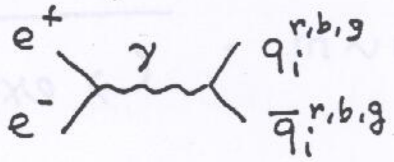
$\overset{(-)}{q}_i$: quark (antiquark) densit

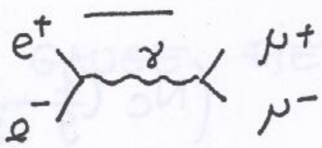
x – Bjorken x.

o FRACTIONAL CHARGE & COLOUR:

$$R = \frac{\sigma(e^+e^- \to \text{HADRONS})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx \sum_f e_f^2\, N_c\, \theta(s - 4m_{q_f}^2)$$

N_c number of colours





$Q(e) = Q(\mu) = 1$

$R_i = R$ (i quark flavours)

$$R_3 = 3 \cdot \left(\frac{4}{9} + 2\,\frac{1}{9}\right) = 2$$

$$R_4 = 3 \cdot \left(\frac{8}{9} + \frac{2}{9}\right) = \frac{10}{3}$$

$$R_5 = 3 \cdot \left(\frac{8}{9} + \frac{3}{9}\right) = \frac{11}{3}$$

$$R_6 = 3 \cdot \left(\frac{12}{9} + \frac{3}{9}\right) = 5$$

: u, d ; c, s ; t, b .

o $\mathrm{Br}\,(\tau^- \to e^- \bar{\nu}_e \nu_\tau) = \dfrac{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}{\Gamma(\tau^- \to \mathrm{all})}$

$$= \frac{1}{1_e + 1_\mu + N_c\, 1_{\bar{u}d}} \sim \frac{1}{5}$$

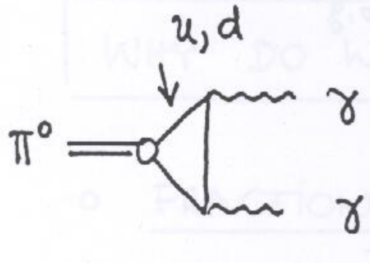
$$= (17.44 \pm 0.85)\,\% \leftarrow \text{incl. corrections}$$

o <u>The width of π^0:</u>

$$\Gamma(\pi^0 \to \gamma\gamma) = \left(\frac{\alpha}{2\pi}\right)^2 \left[N_c (e_u^2 - e_d^2) \right]^2 \frac{m_\pi^2}{8\pi f_\pi}$$

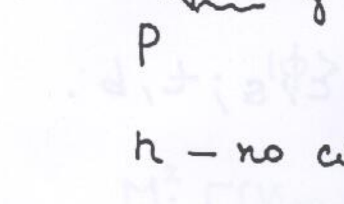
$$= \begin{cases} 0.86 \text{ eV}, & N_c = 1 \\ \underline{7.75 \text{ eV}}, & N_c = 3. \end{cases}$$

$\hookrightarrow$ exp: 7.86 ± 0.54 eV.



$$\left(N_c \left(\tfrac{4}{9} - \tfrac{1}{9}\right)\right)^2 = \left(3 \cdot \tfrac{1}{3}\right)^2 = \underline{\underline{1}}$$

NOTE, HOWEVER, J. STEINBERGER: PHYS. REV. 76 (1949) 118[illegible]



$$\Gamma(\pi^0 \to \gamma\gamma) = \left(\frac{\alpha}{2\pi}\right)^2 \cdot 1 \cdot \frac{m_\pi^2}{8\pi f_\pi}.$$

n – no contribution!

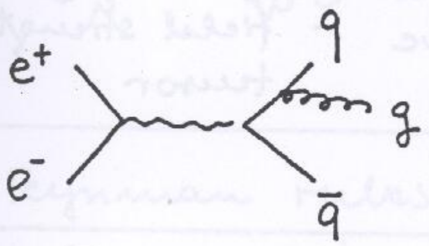
- ARE THERE OTHER PARTICLES THAN QUARKS CONTAINED IN HADRONS (THE NUCLEON e.g.) ?

$$P_q = \int_0^1 dx \; F_2^{\nu N}(x) \simeq 0.47 \qquad Q^2 \sim 20\,\mathrm{GeV}^2$$

PERCENTAGE OF THE NUCLEON MOMENTUM CARRIED BY QUARKS.

WHO CARRIES THE REMAINING PART ?

GLUONS – CHARGE NEUTRAL $(SU_2 \times U_1)$ BOSONS.



$$\frac{d\sigma^{e^+e^-}}{d\theta_g} \longrightarrow$$

- GLUONS ARE VECTOR PARTICLES
- $m \sim 0$ (< few MeV experimenta

GLUONS ARE THE COLOR GAUGE FIELDS.

$N_c = 3 \longrightarrow \underline{SU(3)_c}$.

THE LAGRANGE DENSITY OF QCD :

$$\boxed{\mathcal{L} = \sum_{k=1}^{N_f} \bar{\psi}^k (i \gamma_\mu D_\mu - m_k) \psi^k - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a}$$

(classical Lagrangian).

$D_\mu = \partial_\mu - i g T^a A_{\mu a}$ covariant derivative

ψ^k - quark field: flavour a

m^k —"— mass —"—

$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_{\mu b} A_{\nu c}$ - field strength tensor

A_μ^a - gluon field

g - strong coupling constant.

f^{abc} SU(3) STRUCTURE CONSTANTS.

$\dim [\mathcal{L}] = 4$

$\dim [m] = \dim [\partial_\mu] = 1$

$\dim [A_\mu] = \frac{D-2}{2} = 1$

$\dim [\psi] = \frac{D-1}{2} = \frac{3}{2}$.

$$\delta \mathcal{L} = 0 \Big|_{SU(3)_c}$$

QUANTUM LEVEL:

$$\boxed{\mathcal{L} = \mathcal{L}_{class.} + \mathcal{L}_{GF} + \mathcal{L}_{FP}}$$

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2$$ GAUGE FIXING TERM

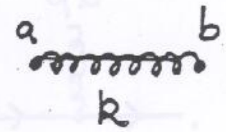
$$\mathcal{L}_{FP} = \left(\partial^\mu \chi^a\right)^* D^{ab}_\mu \chi^b$$ FADDEEV-POPOV GHOST TERM

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - \delta^{ab}\, i g\, T^c A_{\mu c}$$

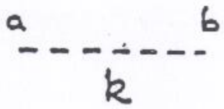
$\boxed{\text{Feynman rules}}$:

A: PROPAGATORS

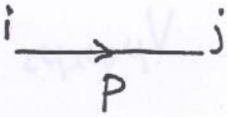
gluon: $$D^{ab}_{\mu\nu}(x) = \delta^{ab}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\varepsilon}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)$$



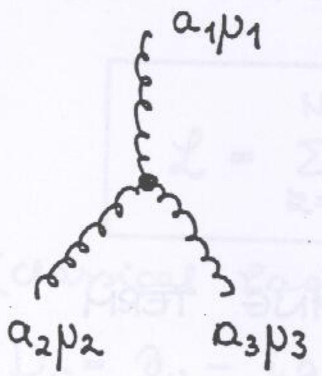
ghost: $$D^{ab}(x) = \delta^{ab}\int\frac{d^4k}{(2\pi)^4}\frac{-1}{k^2+i\varepsilon}\, e^{-ikx}$$



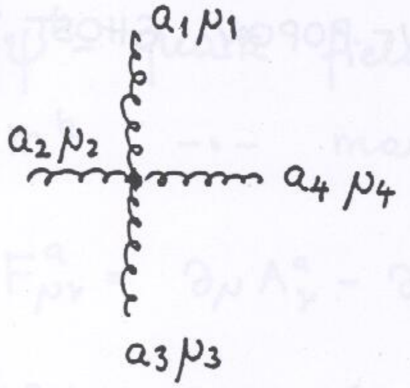
quarks: $$S(x) = \int\frac{d^4p}{(2\pi)^4}\frac{1}{m-\not{p}}\, e^{-ipx}\,\delta^{ij}$$



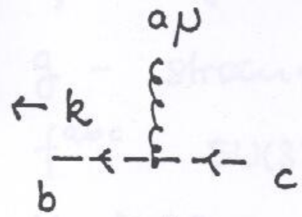
B: VERTICES



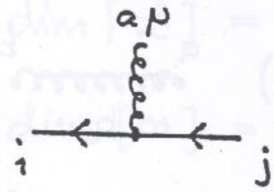
$$-ig f^{a_1a_2a_3} V_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$$



$$-g^2 W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}$$



$$-ig f^{abc} k_\mu$$



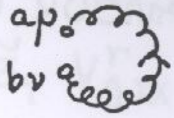
$$g\gamma_\mu T^a_{ij}$$

$$V_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = (k_1-k_2)_{\mu_3} g_{\mu_1\mu_2} + (k_2-k_3)_{\mu_1} g_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2} g_{\mu_3\mu_1}$$

$$W^{a_1\dots a_4}_{\mu_1\dots\mu_4} = (f^{13,24} - f^{14,32})\, g_{\mu_1\mu_2} g_{\mu_3\mu_4} + (f^{12,34} - f^{14,23})\, g_{\mu_1\mu_3} g_{\mu_2\mu_4} + (f^{13,42} - f^{12,34})\, g_{\mu_1\mu_4} g_{\mu_3\mu_2}$$

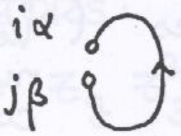
$$f^{ij,kl} = f^{a_ia_ja} f^{a_ka_la}$$

C: LOOPS:



$\uparrow k$ $\qquad$ $\displaystyle \int \frac{d^4k}{(2\pi)^4 i}\, \delta^{ab} g^{\mu\nu}$

$\uparrow k$ $\qquad$ $\displaystyle -\int \frac{d^4k}{(2\pi)^4 i}\, \delta^{ab}$



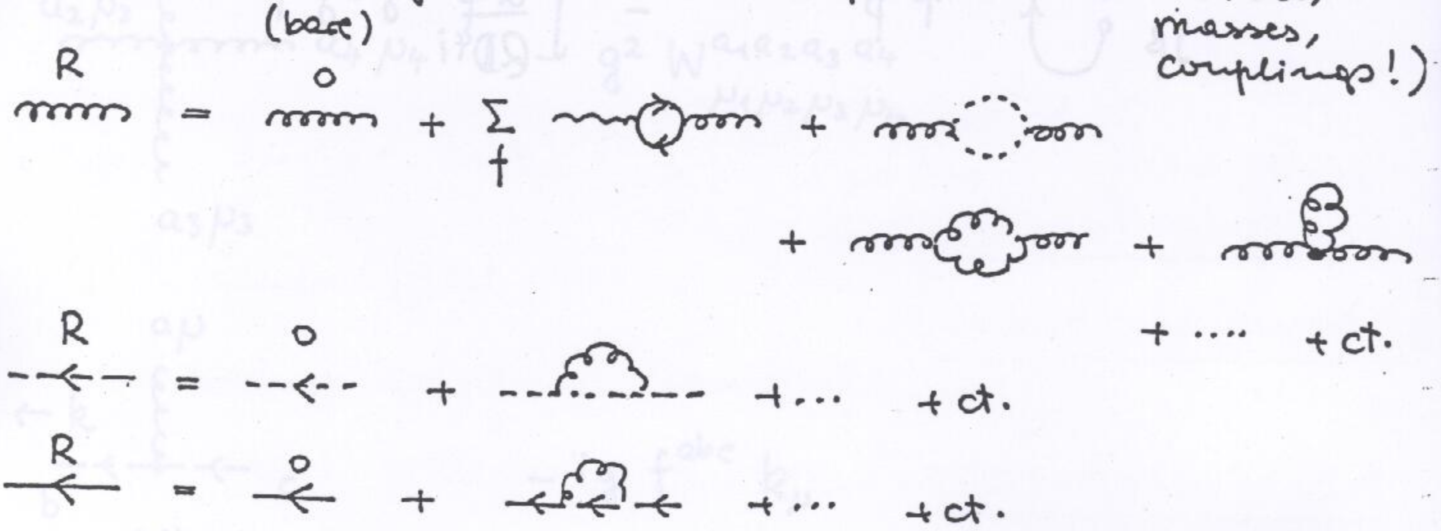
$\uparrow p$ $\qquad$ $\displaystyle -\int \frac{d^4p}{(2\pi)^4 i}\, \delta^{ij} \delta^{\alpha\beta}$

LECTURE 2

ASYMPTOTIC FREEDOM & THE BEHAVIOUR OF THE STRONG COUPLING CONSTANT

A) Renormalization:

Ultraviolet divergencies $\longleftrightarrow$ bare quantities (propagators, vertices, masses, couplings!)



— ABSORB THESE SINGULARITIES INTO THE "DRESSED" PROPAGATORS (FIELDS), VERTICES, COUPLINGS, MASSES

$\longrightarrow$ REFORMULATE THE THEORY INTRODUCING Z-FACTORS

$$A_\mu^{a,0} = Z_3^{1/2} A^a_{r\mu} \qquad g_0 = Z_g\, g_r$$

$$\chi^{a,0} = \tilde{Z}_3^{1/2} \chi^a_r \qquad \alpha_0 = Z_3\, \alpha_r$$

$$\psi^0 = Z_2^{1/2} \psi_r \qquad m_0 = Z_m\, m_r$$

$\uparrow$ bare $\qquad$ $\uparrow$ renormalized

quantities.

$$V^{o}_{ggg} = Z_1 \, V^{r}_{ggg}$$

$$V^{o}_{gqq} = Z_{1F} \, V^{r}_{gqq}$$

$$V^{o}_{gXX} = \tilde{Z}_1 \, V^{r}_{gXX}$$

$$V^{o}_{gggg} = Z_4 \, V^{r}_{gggg}$$

THE FOLLOWING RELATIONS ARE IMPLIED FOR THESE Z FACTORS:

$$Z_1 := Z_g \, Z_3^{3/2}$$

$$Z_{1F} := Z_g \, Z_3^{1/2} \, Z_2^{2/2}$$

$$\tilde{Z}_1 := Z_g \, Z_3^{1/2} \, \tilde{Z}_3^{2/2}$$

$$Z_4 := Z_g^2 \, Z_3^{4/2}$$

$$\boxed{\frac{Z_1}{Z_3} = \frac{Z_{1F}}{Z_2} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_4}{Z_1} = Z_g \, Z_3^{1/2}}$$

SLAVNOV-TAYLOR IDENDITIES

m, g - and the propagators, vertices have always the same Z-factor $\leftrightarrow$ relations.

$\overline{MS}$ SCHEME : $\varepsilon = (4-D)/2$; $D = 4-2\varepsilon$ DIMENSIONS

UV-SINGULARITY $-\ \frac{1}{\varepsilon}$ POLES.

<u>1 LOOP RESULTS:</u>

ren. coupling

ren. gauge parameter

$$Z_3 = 1 - \frac{g_r^2}{(4\pi)^2}\left[\frac{4}{3}T_R N_f - \frac{1}{2}C_G\left(\frac{13}{3}-\alpha_r\right)\right]\frac{1}{\varepsilon} + \ldots$$

$$\tilde{Z}_3 = 1 + \frac{g_r^2}{(4\pi)^2}\, C_G \left(\frac{3-\alpha_r}{4}\right)\frac{1}{\varepsilon} + \ldots$$

$$Z_2 = 1 - \frac{g_r^2}{(4\pi)^2}\, C_F\, \alpha_r \frac{1}{\varepsilon} + \ldots$$

$$Z_1 = 1 - \frac{g_r^2}{(4\pi)^2}\left[C_G\left(-\frac{17}{12} + \frac{3\alpha_r}{4}\right) + \frac{4}{3}T_R N_f\right]\frac{1}{\varepsilon} + \ldots$$

$$\tilde{Z}_1 = 1 - \frac{g_r^2}{(4\pi)^2}\, C_G \frac{\alpha_r}{2}\frac{1}{\varepsilon} + \ldots$$

$$Z_{1F} = 1 - \frac{g_r^2}{(4\pi)^2}\left(\frac{3+\alpha_r}{4}C_G + \alpha_r C_F\right)\frac{1}{\varepsilon} + \ldots$$

$$Z_4 = 1 - \frac{g_r^2}{(4\pi)^2}\left[\left(-\frac{2}{3}+\alpha_r\right) + \frac{4}{3}T_R N_f\right]\frac{1}{\varepsilon} + \ldots$$

- PROBLEM: CHECK ALL SLAVNOV-TAYLOR IDENDITIES AL 1-LOOP LEVEL.

EXAMPLES:

$$\frac{Z_1}{Z_3} = 1 - \frac{g_r^2}{(4\pi)^2}\frac{1}{\varepsilon}\left[C_G\left(-\frac{17}{12} + \frac{3\alpha_r}{4} + \frac{13}{6} - \frac{\alpha_r}{2}\right) + \frac{4}{3}T_R N_f - \frac{4}{3}T_R N_f\right] + \dots$$

$$= 1 - \frac{1}{\varepsilon}\frac{g_r^2}{(4\pi)^2} C_G\left(\frac{3+\alpha_r}{4}\right) + \dots \equiv Z_g Z_3^{1/2}.$$

$$\frac{Z_{1F}}{Z_2} = 1 - \frac{g_r^2}{(4\pi)^2}\frac{1}{\varepsilon}\left[\frac{3+\alpha_r}{4} C_G + \alpha_r C_F - \alpha_r C_F\right] + \dots$$

$$= 1 - \frac{1}{\varepsilon}\frac{g_r^2}{(4\pi)^2} C_G\left(\frac{3+\alpha_r}{4}\right) + \dots$$

$$\curvearrowright\ Z_g = \frac{Z_1}{Z_3}\frac{1}{Z_3^{1/2}} = 1 - \frac{g_r^2}{(4\pi)^2}\frac{1}{\varepsilon}\left[\left(\frac{3+\alpha_r}{4}\right)C_G - \frac{2}{3}T_R N_f + \frac{1}{4}C_G\left(\frac{13}{3} - \alpha_r\right)\right] + \dots$$

$$\underline{Z_g = 1 - \frac{g_r^2}{(4\pi)^2}\frac{1}{\varepsilon}\left\{\frac{1}{6}\left[11 C_G - 4 T_R N_f\right]\right\}}$$

$\rightarrow$ No dependence on the gauge parameter α_r!

EXAMPLE: QED @ 1 LOOP

$T_R = C_F = 1, \quad C_G = 0.$

$$Z_e = 1 - \frac{e_r^2}{(4\pi)^2}\frac{1}{\varepsilon}\cdot\left(-\frac{2}{3}\right) = 1 + \frac{\alpha_{QED}}{4\pi}\cdot\frac{2}{3}\frac{1}{\varepsilon}$$

↑ no asymptotic freedom (see below).

QCD:

$$Z_g = 1 - \frac{\alpha_{sr}}{4\pi}\frac{1}{\varepsilon}\left[\frac{1}{2}\left(\frac{11C_G - 4T_R N_f}{3}\right)\right]$$

$C_G = 3 = N_c$

$C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$

$T_R = 1/2$

> 0 FOR

$33 > 2N_f$ $\quad N_f = 6$ (so far

GAUGE PARAMETER INDEPENDENCE OF g_R IN MS$^{..}$-SCHEMES

$$g_0 = Z_g\, g_r, \quad Z_g = 1 + \sum_{k=1}^{\infty}\frac{a_k(g_R, \alpha_R)}{\varepsilon^k}$$

$$\frac{d}{d\alpha_R} g_0 = 0 \equiv \frac{dg_r}{d\alpha_r} + \frac{1}{\varepsilon}\left(a_1\frac{dg_r}{d\alpha_r} + \frac{da_1}{d\alpha_r} g_r\right) + \cdots$$

ε - general parameter.

$$\frac{dg_r}{d\alpha_r} = 0 \quad \curvearrowright \left(\frac{da_k}{d\alpha_r} = 0\right) \quad \forall k$$

This IS NOT THE CASE FOR:

$$Z_g = 1 + a_0(g_r, \alpha_r) + \cdots \qquad !$$

THE SCALE DEPENDENCE OF g_r:

$$g_R = \left(\frac{\mu_0}{\mu}\right)^{\varepsilon} Z_g^{-1} \, g_0$$

$$g_0 = g_R Z_g \left(\frac{\mu}{\mu_0}\right)^{\varepsilon}$$

$$\mu \frac{\partial}{\partial \mu} g_0 = 0 = \mu \frac{\partial g_R}{\partial \mu} Z_g \left(\frac{\mu}{\mu_0}\right)^{\varepsilon} + g_R \mu \frac{\partial Z_g}{\partial \mu} \left(\frac{\mu}{\mu_0}\right)^{\varepsilon}$$

$$+ g_R Z_g \, \varepsilon \left(\frac{\mu}{\mu_0}\right)^{\varepsilon} = 0$$

DEFINE: $\beta = \mu \left.\frac{\partial g_R}{\partial \mu}\right|_{g, \alpha}$ BETA FUNCTION.

$$0 = \beta + \varepsilon g_R + \frac{1}{Z_g} \mu \frac{\partial Z_g}{\partial \mu} g_R$$

$$\beta = - \varepsilon g_R - \frac{1}{Z_g} \mu \frac{\partial Z_g}{\partial \mu} g_R$$

1 LOOP:

$$Z_g = 1 - \frac{g_R^2}{(4\pi)^2} \frac{1}{\varepsilon} \frac{1}{6} (11 C_G - 4 T_R N_f)$$

$$-\mu \frac{\partial Z_g}{\partial \mu} g_R = \frac{2 g_R^2}{(4\pi)^2} \frac{1}{6\varepsilon} (11 C_G - 4 T_R N_f) \, \beta(g_R)$$

$$\beta(g_R) \to -\varepsilon g_R + \dots$$

$$\beta(g_R) = - \underset{\to 0}{\varepsilon g_R} - \frac{1}{(4\pi)^2} \frac{11 C_G - 4 T_R N_f}{3} g_R^3 + \dots$$

$$\beta_0 \equiv \frac{11 C_G - 4 T_R N_f}{}$$

$$\beta = \frac{\partial g_R}{\partial \log(\mu/\mu_0)} = -\beta_0 \frac{1}{(4\pi)^2} g_R^3$$

$$\frac{\partial g_R^2}{\partial \log \mu^2} = -\frac{\beta_0}{(4\pi)^2} g_R^4$$

$$g_R^2 = \alpha_s \cdot 4\pi$$

$$\frac{1}{4\pi} \frac{\partial \alpha_s}{\partial \log(\mu^2/\mu_0^2)} = -\sum_{k=0}^{\infty} \beta_k \left(\frac{\alpha_s}{4\pi}\right)^{2+k}$$

$\underline{\alpha_s / 4\pi := a_s}$

$$\frac{1}{a_s(\mu^2)} - \frac{1}{a_s(\mu_0^2)} = \beta_0 \log\left(\frac{\mu^2}{\mu_0^2}\right).$$

$$\boxed{a_s(\mu^2) \underset{\text{Df.}}{=:} \frac{1}{\beta_0 \log(\mu^2/\Lambda^2)}} \quad \text{LO.}$$

$$\Lambda^2 = \mu_0^2 \exp\left[-\frac{4\pi}{\beta_0 \alpha_s(\mu_0^2)}\right]$$

EXAMPLE: QED , 1-electron QED , $N_f = 1$

$$\beta_0 = -\frac{4}{3}$$

$$\beta_1 = -4$$

$$\beta_2 = \frac{62}{9} \qquad \text{VLADIMIROV, 1979}, \ \overline{MS}$$

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha(Q_0^2)} + \frac{\beta_0}{4\pi}\log\left(\frac{Q^2}{Q_0^2}\right) + \phi^{(n)}(\alpha(Q^2);\beta_i) - \phi^{(n)}(\alpha(Q_0^2);\beta_i)$$

$$\phi^{(2)}_{(x)} = -\frac{\beta_1}{8\pi\beta_0}\log\left|\frac{16\pi^2 x^2}{16\pi^2\beta_0 + 4\pi\beta_1 x + \beta_2 x^2}\right|$$

$$+ \frac{\beta_1^2 - 2\beta_0\beta_2}{8\pi\sqrt{\beta_1^2 - 4\beta_0\beta_2}}\log\left|\frac{\beta_2 x + 2\pi(\beta_1 - \sqrt{\beta_1^2 - 4\beta_2\beta_0})}{\beta_2 x + 2\pi(\beta_1 + \sqrt{\beta_1^2 - 4\beta_2\beta_0})}\right| , \quad x = \alpha_s$$

QED:

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha(Q_0^2)} - \frac{1}{3\pi}\log\left(\frac{Q^2}{Q_0^2}\right) - \frac{3}{8\pi}L_1(\dots) - \frac{16 + \frac{8}{3}\cdot\frac{62}{9}}{8\pi\sqrt{16 + \frac{16}{3}\frac{62}{9}}}L_2(\dots)$$

$L_1, L_2 > 0.$

$\curvearrowright$ $\alpha(Q^2)$ GROWS WITH Q^2.

$\longrightarrow$ NO ASYMPTOTIC FREEDOM.

QCD : ASYMPTOTIC FREEDOM, $\alpha(Q^2)$ FALLS WITH GROWING Q^2.

$\hookrightarrow$

$SU(3)_c$: MS

$$\beta_0 = 11 - \frac{2}{3} N_f$$ t' Hooft, Politzer, Gross, Wilczek 1972,

$$\beta_1 = 102 - \frac{38}{3} N_f$$ Caswell, Jones 1974

$$\beta_2 = -\frac{2857}{2} + \frac{5033}{18} N_f - \frac{325}{54} N_f^2$$ Tarasov, Vladimirov, Zakh... 1980

Larin, Vermaseren 1993

$$\begin{aligned}\beta_3 - &\left(\frac{149753}{6} + 3564\, \zeta_3\right) \\ &-\left(\frac{1078361}{162} + \frac{6508}{27} \zeta_3\right) n_f \\ &+\left(\frac{50065}{162} + \frac{6472}{81} \zeta_3\right) n_f^2 + \frac{1093}{729} n_f^3 .\end{aligned}$$

Larin, van Ritbergen, Vermaseren 1997

$$\frac{1}{\alpha_s(Q^2)} - \frac{1}{\alpha_s(Q_0^2)} + \frac{\beta_0}{4\pi} \log\left(\frac{Q^2}{Q_0^2}\right) + \dots$$

$\alpha_s(Q^2)$ falls with $Q^2 \to \infty$.

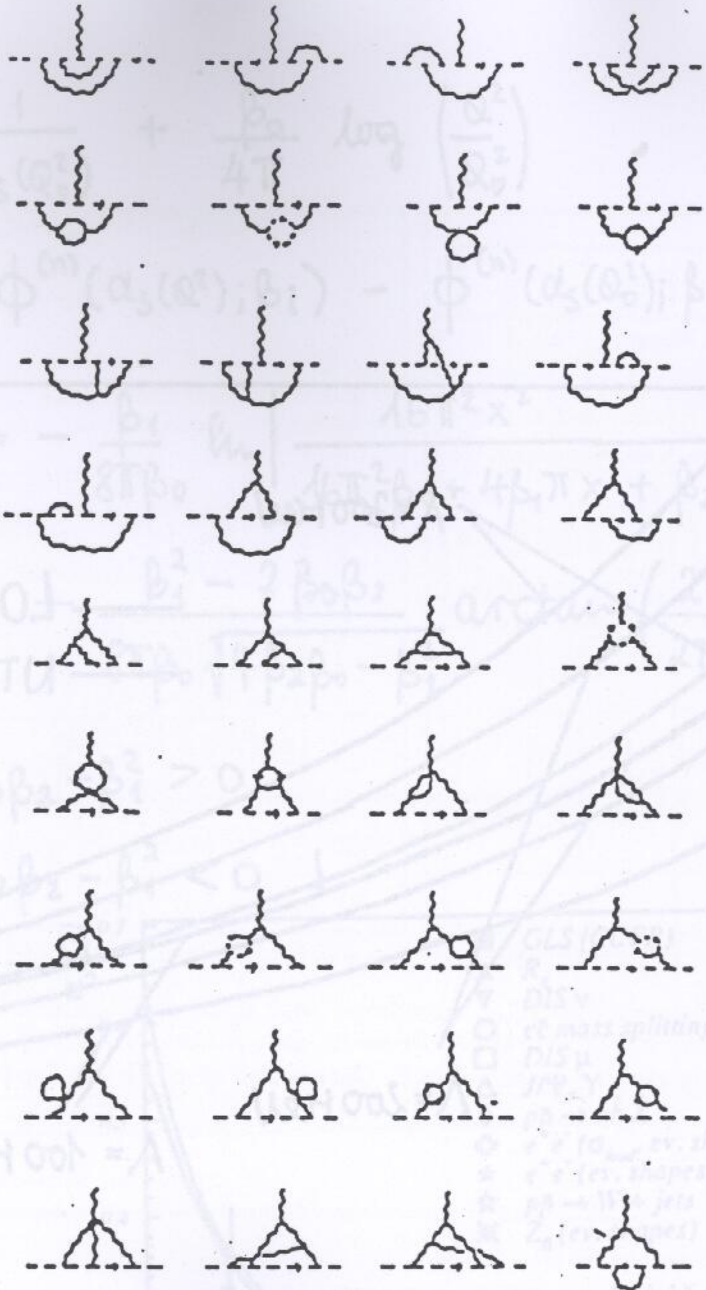
ASYMPTOTIC FREEDOM.

3rd ORDER:

CASWELL, 1974
JONES 1974

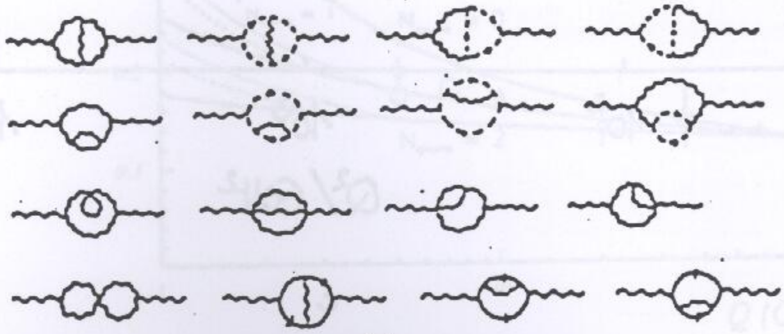
BELAVIN, MIGDAL
1974
(and diff. authors later)

$V_{G\tilde{g}\tilde{g}}$



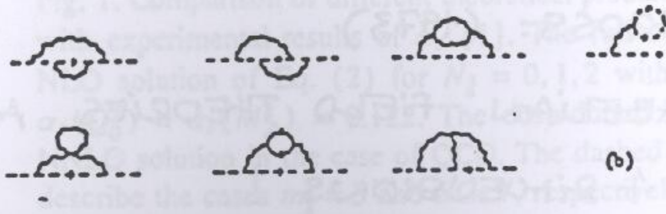
(c)

Σ_G



(a)

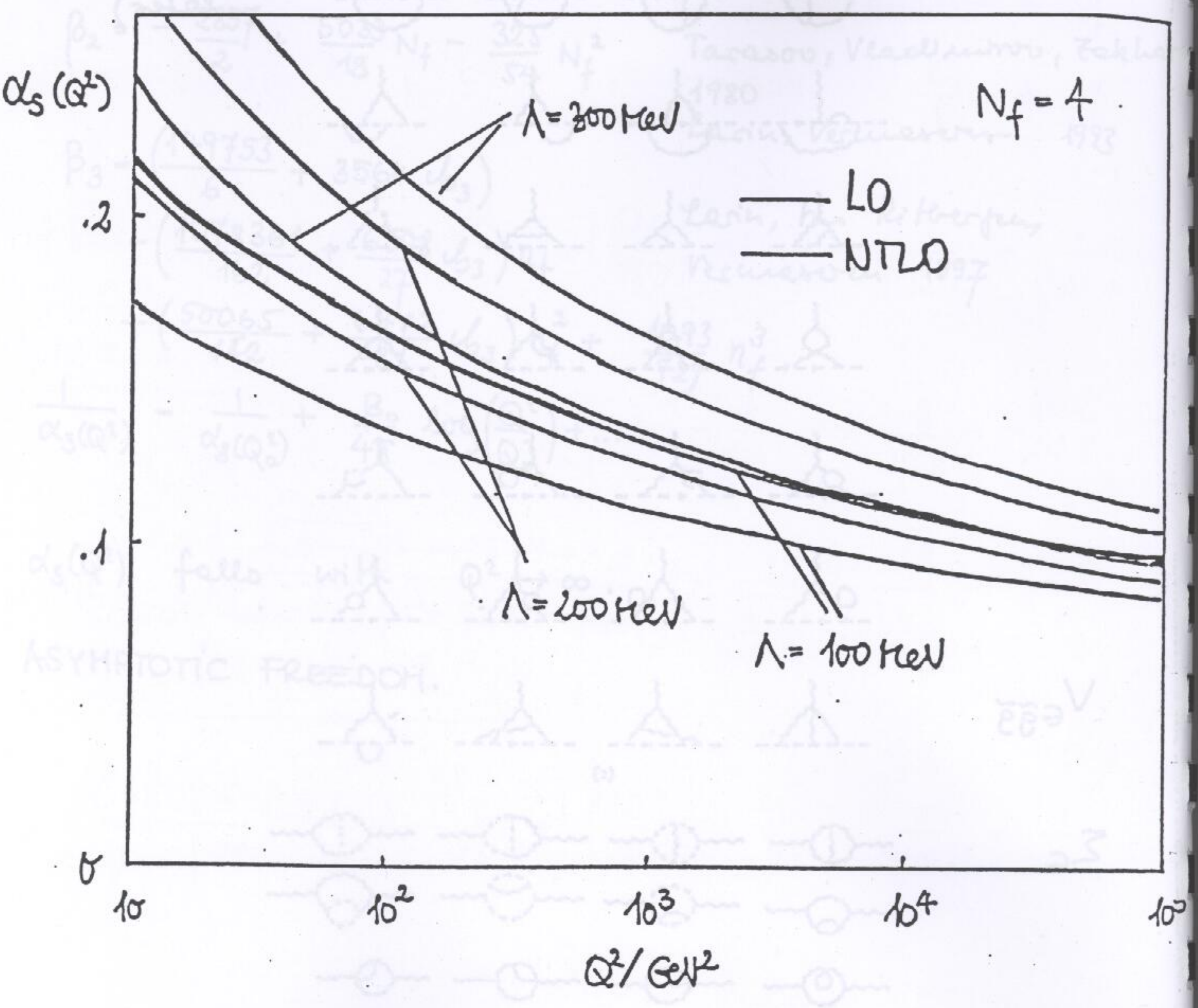
$\Sigma_{\tilde{g}}$



(b)

$$\beta_1 = -\frac{34}{3} C_2^2(G) + \frac{20}{3} C_2(G)\, T(R)\, N_f + 4\, C_2(R)\, T(R)\, N_f$$

$$= -102 + \frac{38}{3} N_f .$$



COLEMAN, GROSS: (1973)

ONLY NON-ABELIAN FIELD THEORIES ARE ASYMPTOTI FREE IN 4 DIMENSIONS !

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(Q_0^2)} + \frac{\beta_0}{4\pi}\log\left(\frac{Q^2}{Q_0^2}\right)$$

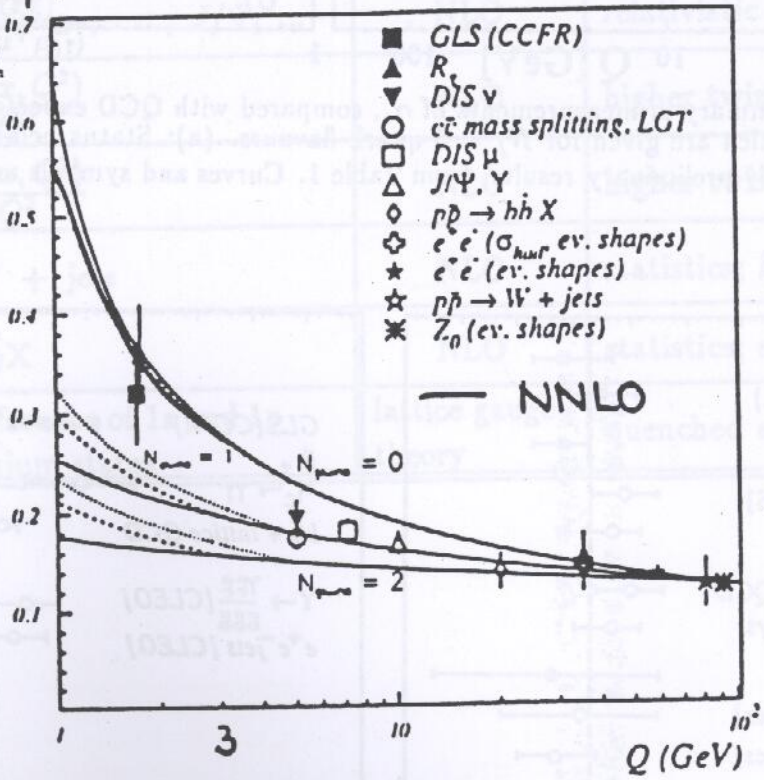
$$+ \phi^{(n)}(\alpha_s(Q^2);\beta_i) - \phi^{(n)}(\alpha_s(Q_0^2);\beta_i)$$

$$\phi_{(n)}(x;\beta_i) = -\frac{\beta_1}{8\pi\beta_0}\ln\left|\frac{16\pi^2 x^2}{16\pi^2\beta_0 + 4\beta_1\pi x + \beta_2 x^2}\right|$$

$$+ \frac{\beta_1^2 - 2\beta_0\beta_2}{8\pi\beta_0\sqrt{4\beta_2\beta_0 - \beta_1^2}}\arctan\left(\frac{2\pi\beta_1 + \beta_2 x}{2\pi\sqrt{4\beta_0\beta_2 - \beta_1^2}}\right)$$

$N_f \leq 5$: $4\beta_0\beta_2 - \beta_1^2 > 0$

$N_f = 6$: $4\beta_0\beta_2 - \beta_1^2 < 0$!



JB, J.BOTTS
1994

Fig. 1. Comparison of different theoretical predictions for $\alpha_s(Q^2)$ with experimental results of α_s [1]. The full curves denote the NLO solution of Eq. (2) for $N_{\tilde{g}} = 0, 1, 2$ with $m_{\tilde{g}} = 0$ taking $\alpha_s(Q_0^2) = \alpha_s(M_Z^2) = 0.122$. The dash-dotted line denotes the NNLO solution in the case of QCD. The dashed and dotted lines describe the cases $m_{\tilde{g}} = 3$ and 5 GeV, respectively.

S.BETHKE, 19'

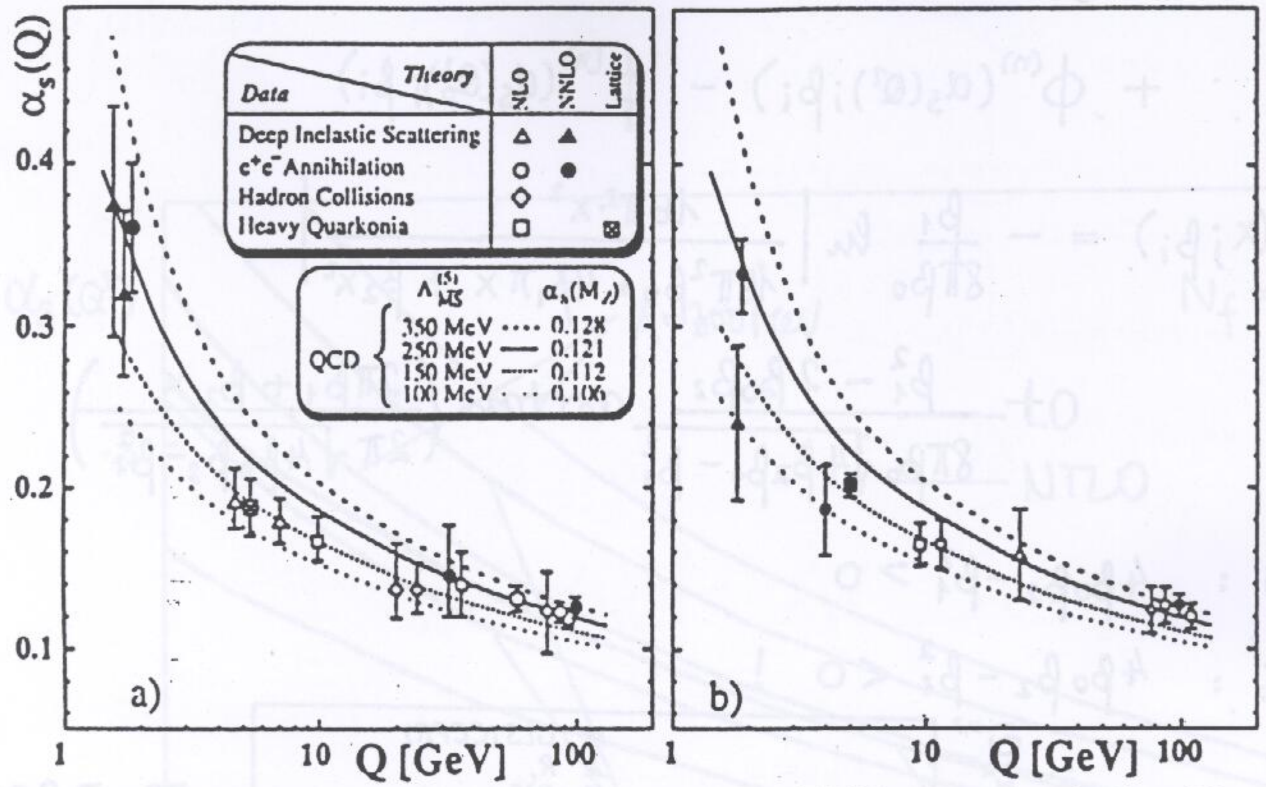


Figure 1. A Summary of measurements of α_s, compared with QCD expectations for four different values of $\Lambda_{\overline{MS}}$ which are given for $N_f = 5$ quark flavours. (a): Status before this conference. (b): Newest and mostly preliminary results, from Table 1. Curves and symbols are the same as in a).

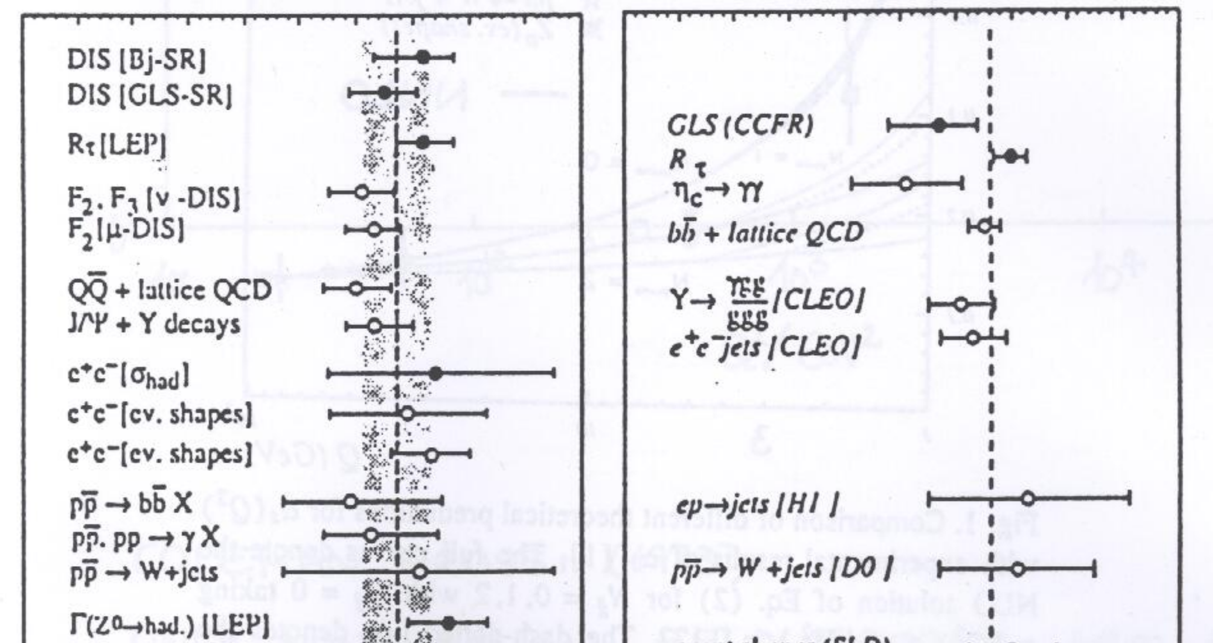


Figure 2. A Summary of measurements of $\alpha_s(M_{Z^0})$. Filled symbols are derived using $O(\alpha_s^3)$ QCD; open symbols are in $O(\alpha_s^2)$ or based on lattice calculations. (a): Status before this conference; vertical line and shaded area represent the world average of $\alpha_s(M_{Z^0}) = 0.117 \pm 0.006$. (b): Newest and mostly preliminary results, from Table 1; vertical line represents $\alpha_s(M_{Z^0}) = 0.116$.

OTHER OBSERVABLES

S·BETHKE

Table 4. Processes and Observables from which significant determinations of α_s are derived.

Process	Observable	Theory	Caveats
e^+e^-	hadronic event shapes, jet production rates, energy correlations	NLO and resummed NLO	hadronization corrections
	$R_Z = \frac{\Gamma(Z^0 \to \text{hadrons})}{\Gamma(Z^0 \to \text{leptons})}$	NNLO	small QCD corrections
	$R_\tau = \frac{Br(\tau \to \text{hadrons})}{Br(\tau \to e\nu)}$	NNLO	nonperturbative corrections
	scaling violations in $\frac{d\sigma}{dx}$ spectra	NLO	only through MC models
	$\frac{\Gamma(\Upsilon \to ggg)}{\Gamma(\Upsilon \to \mu^+\mu^-)}$; .... ; J/Ψ; ...	NLO	relativistic corrections
DIS	$\frac{d \ln F_2(x,Q^2)}{d \ln Q^2}$	NLO	higher twist; $g(x,Q^2)$
	$\frac{d \ln F_3(x,Q^2)}{d \ln Q^2}$	NLO	higher twist
$p\bar{p}$	$p\bar{p} \to W +$ jets	NLO	statistics; k-factors
	$p\bar{p} \to b\bar{b}X$	NLO	statistics; exp. systematics
$c\bar{c}$ states	mass difference of 1s and 1p charmonium states	lattice gauge theory	quenched approximation

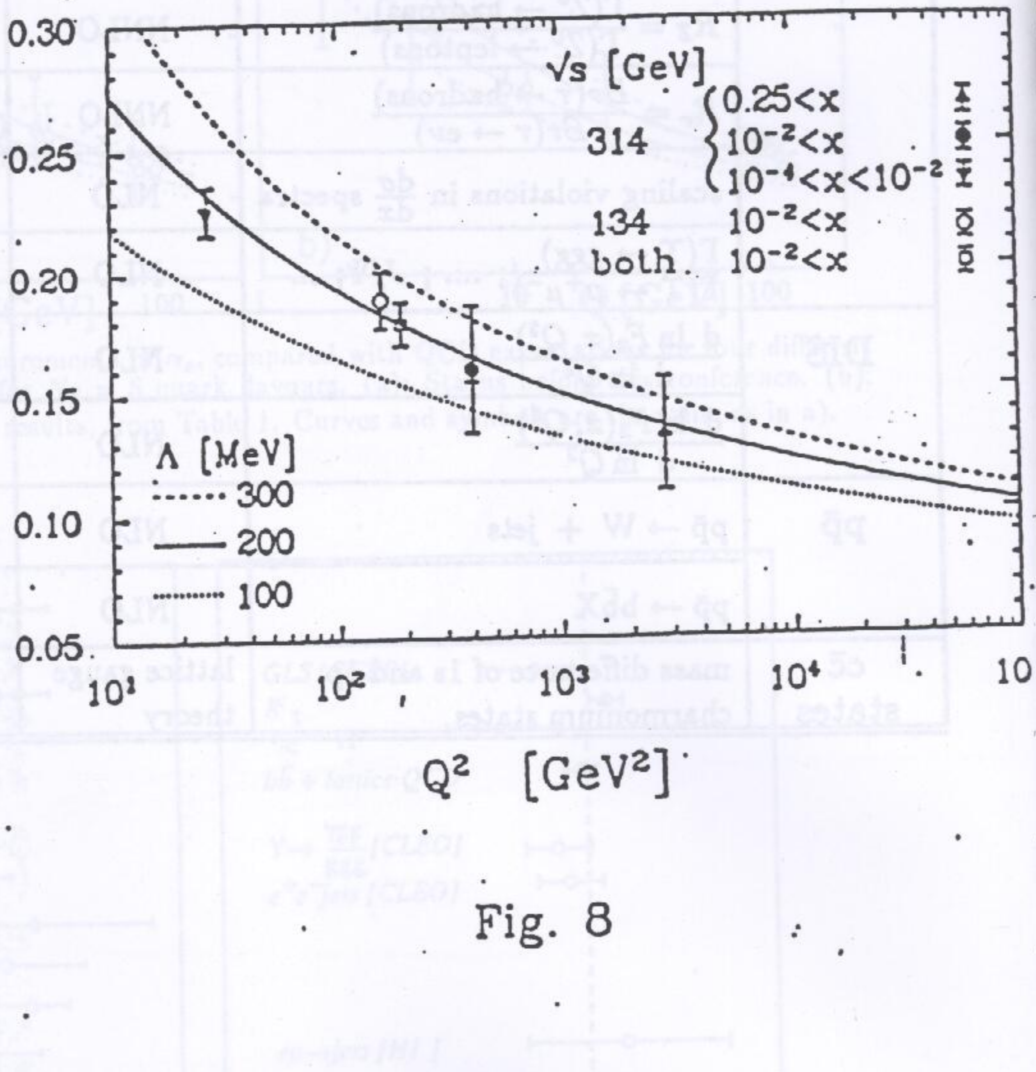


Fig. 8

Ag.-Nr. 521/583/23
Rg.-Nr. 1/12/6

- DIS $\nu F_2, F_3$ 5 $.193 {}^{+0.019}_{-0.018}$ $.111 \pm .006$ $.004$ $.004$ NLO
- DIS μF_2 7.1 $.180 \pm .014$ $.113 \pm .005$ $.003$ $.004$ NLO

Process	Ref.	$\langle Q \rangle$ [GeV]	$\alpha_s(Q)$	$\alpha_s(M_{Z^0})$	$\Delta\alpha_s(M_{Z^0})$ exp.	theor.	Theory
GLS (CCFR)	[15]	1.73	0.24 ± 0.047	$0.107 {}^{+0.007}_{-0.009}$	${}^{+0.006}_{-0.007}$	${}^{+0.004}_{-0.006}$	NNLO
R_τ (CLEO)	[16]	1.78	0.302 ± 0.024	0.116 ± 0.003	0.002	0.002	NNLO
R_τ (ALEPH)	[17]	1.78	0.355 ± 0.021	0.122 ± 0.003	0.002	0.002	NNLO
R_τ (OPAL)	[17]	1.78	$0.375 {}^{+0.032}_{-0.025}$	0.123 ± 0.003	0.002	0.002	NNLO
R_τ (Raczka)	[18]	1.78	0.333 ± 0.021	0.120 ± 0.003	0.002	0.002	NNLO
$\eta_c \to \gamma\gamma$ (CLEO)	[16]	2.98	0.187 ± 0.029	0.101 ± 0.010	0.008	0.006	NLO
$Q\overline{Q}$ states	[19]	5.0	0.188 ± 0.018	0.110 ± 0.006	0.000	0.006	q LGT
$b\overline{b}$ states	[19]	5.0	0.203 ± 0.007	0.115 ± 0.002	0.000	0.002	LGT
Υ(1S) (CLEO)	[16]	9.46	0.164 ± 0.013	0.111 ± 0.006	0.001	0.006	NLO
• $e^+e^- \to$ jets (CLEO)	[16]	10.53	0.164 ± 0.015	0.113 ± 0.006	0.002	0.006	NLO
$ep \to jets$ (H1)	[20]	5 - 60		0.123 ± 0.018	0.014	0.010	NLO
$p\bar{p} \to$ W jets (D0)	[21]	80.6	0.123 ± 0.015	0.121 ± 0.014	0.012	0.005	NLO
• $e^+e^- \to Z^0$:							
• scal. viol. (ALEPH)	[17]	91.2		0.127 ± 0.011	–	–	NLO
• ev. shapes (SLD)	[22]	91.2		0.120 ± 0.008	0.003	0.008	resum.
• $\Gamma(Z^0 \to$ had.) (LEP)	[23]	91.2		0.127 ± 0.006	0.005	${}^{+0.003}_{-0.004}$	NNLO

QUAR

Table 1. Summary of most recent measurements of α_s, presented at this conference. Abbreviations: GLS-SR = Gross-Llewellyn-Smith sum rules; (N)NLO = (next-)next-to-leading order perturbation theory; LGT = lattice gauge theory (q stands for quenched approximation); resum. = resummed next-to-leading order. Most results are still preliminary.

S.BETHKE 199

DIS: $\overline{\alpha}_s(M_Z) = 0.112 \pm 0.004$

e^+e^-: $\overline{\alpha}_s(M_Z) = 0.121 \pm 0.004$

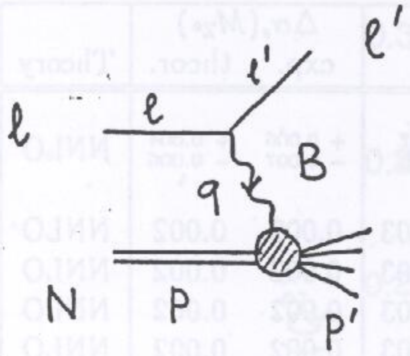
TWO CLUSTERS !

- LGT WITHIN BETWEEN
  MORE CALCULATIONS NEEDED
  $\to$ PROPER TREATMENT OF QUARKS

LECTURE 3

THE PARTON MODEL, PARTON DISTRIBUTIONS & DIS STRUCTUREFUNCTIONS

1) DEEP INELASTIC SCATTERING:



$\ell = e^{\pm}, \mu^{\pm}, \nu, \bar{\nu}$

$B = \gamma, W^{\pm}, Z$

$N = p, n, d, A$

KINEMATIC QUANTITIES: (BORN LEVEL)

$$\ell + p = \ell' + p'$$

$\ell - \ell' = p' - p = q$ – MOMENTUM TRANSFER

$Q^2 = -q^2 = (\ell - \ell')^2 \approx 2\ell\ell'$, $\quad m_\ell \sim 0$

$$y := \frac{P \cdot q}{P \cdot \ell} = \frac{2M(E-E')}{2ME} = 1 - \frac{E'}{E} \leq 1$$

$\uparrow$ Target – Rest – System. $(P = (M, \vec{0}))$

– Bjorken y $\qquad y \in [0,1]$.

$x := \dfrac{Q^2}{2Pq}$ – Bjorken x

$W^2 = (p+q)^2 = M^2 + 2pq - Q^2 \geq M^2$ (Hadronic mass)^2.

$2pq \geq Q^2 \qquad \curvearrowright \; 0 \leq x \leq 1 .$

$$Q^2 = -(E-E')^2 + (E - E'\cos\theta)^2 + E'^2 \sin^2\theta$$

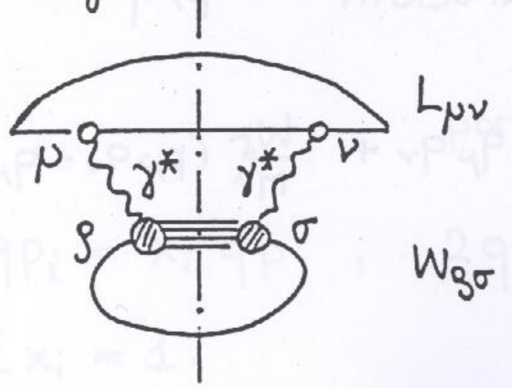
TS

$$= - E^2 - E'^2 + 2EE' + E^2 - 2EE'\cos\theta + E'^2$$

$$= 2EE'(1-\cos\theta) \geq 0.$$

LORENTZ STRUCTURE OF THE SCATTERING CROSS SECTION:

$$\frac{d^2\sigma}{dx\,dy} \propto \sum_{s'} \overline{|M|^2} = \overline{L_{\mu\nu}\, g^{\mu\rho}\, W_{\rho\sigma}\, g^{\nu\sigma}}$$



$$L_{\mu\nu} = \frac{1}{2} \mathrm{tr}\, [(\not{l}+m_e)\gamma_\mu(\not{l}'+m_e)\gamma_\nu]$$

$$\approx 2(l_\mu l'_\nu + l_\nu l'_\mu - l l' g_{\mu\nu})$$

$$\longrightarrow \qquad L_{\mu\nu} = L_{\nu\mu}$$

$$\underline{q_\mu L_{\mu\nu} = q_\nu L_{\mu\nu}} = 2\,[ql(l_\nu - q_\nu) + l_\nu(ql - q^2) + \tfrac{1}{2}q^2 q_\nu]$$

$$\left. ql \equiv q^2/2 \right) \hookrightarrow = 2\left[\frac{q^2}{2}(l_\nu - q_\nu) - l_\nu \frac{q^2}{2} + \frac{q^2}{2} q_\nu\right] \equiv 0.$$

CURRENT CONSERVATION

SINCE $L_{\mu\nu}$ IS SYMMETRIC ONLY THE SYMMETRIC PART OF $W_{\mu\nu}$ CONTRIBUTES.

THE PROPERTY $q_\mu L_{\mu\nu} = q_\nu L_{\mu\nu} = 0$ IMPLIES:

$$q_\mu W_{\mu\nu} = q_\nu W_{\mu\nu} = 0.$$

STRUCTURE OF $W_{\mu\nu}$:

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_n \langle P | J_\mu^{em\dagger}(0) | n \rangle \langle n | J_\nu^{em}(x) | P \rangle (2\pi)^4 \delta^{(4)}(P+q-p_n)$$

TWO INDEPENDENT OUTER 4 MOMENTA: p, q

$$W_{\mu\nu} = g_{\mu\nu} W_1 + \frac{W_2}{M^2} p_\mu p_\nu + \frac{W_4}{M^2} q_\mu q_\nu + \frac{W_5}{M^2}(p_\mu q_\nu + q_\mu p_\nu)$$

$q_\mu W_{\mu\nu} = 0$:

$$q_\nu \left[W_1 + \frac{W_4}{M^2}(p\cdot q) + \frac{W_5}{M^2}(p\cdot q)\right] = 0$$

$$p_\nu \left[\frac{W_2}{M^2}(p\cdot q) + \frac{W_5}{M^2} q^2\right] = 0$$

$$\curvearrowright \; W_5 = -W_2 \frac{q^2}{p\cdot q}$$

$$W_4 = -\left[W_1 - \frac{W_2 q^2}{M^2}\right]\frac{M^2}{p\cdot q}.$$

$$\boxed{W_{\mu\nu}^{\gamma^*} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \frac{1}{M^2}\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) W_2}$$

$q_\mu W_{\mu\nu} = q_\nu W_{\mu\nu} \equiv 0$ in explic way.

THE BJORKEN LIMIT:

$$Q^2 \to \infty \; ; \quad \nu \equiv \frac{sy}{2M} \to \infty \; ; \qquad x = \frac{Q^2}{2M\nu} = \frac{Q^2}{sy} = \text{const.}$$

$$\left.\begin{array}{l} \nu W_2(\nu, Q^2) \to F_2(x) \\ M W_1(\nu, Q^2) \to F_1(x) \end{array}\right\} \text{DEPEND ON } Q^2 \,\&\, \nu \text{ ONLY THROUGH } \frac{Q^2}{2M\nu} \; !$$

$\longrightarrow$ SCALING

local scattering of partons!

$$W_2(\nu, Q^2) = \sum_i \int_0^1 dx_i \underbrace{f(x_i)}_{\text{parton density}} x_i \underbrace{e_i^2 \, \delta\left(\frac{qp_i}{M} - \frac{Q^2}{2M}\right)}_{e_i^2 \, \delta\left(\nu - \frac{Q^2}{2Mx_i}\right)}$$

$$qp_i = x_i qP \; , \quad 2qP = Q^2/x \; , \quad M\nu = qP$$

$$\sum_i x_i = 1 .$$

$$\nu W_2(\nu, Q^2) = \sum_i \int_0^1 dx_i \, f(x_i) \, e_i^2 \, \nu \, \delta\left(\nu - \frac{Q^2}{2Mx_i}\right)$$

$$\rightarrow \int_{-\infty}^{+\infty} dx \, \delta(f(x)) = \left. \frac{1}{|\partial f/\partial x|} \right|_{f(x)=0}$$

$$f = \nu - \frac{Q^2}{2Mx} \; , \quad \left|\frac{\partial f}{\partial x}\right| = \frac{Q^2}{2Mx^2}$$

$$f(x) = 0 \;\curvearrowright\; x = \frac{Q^2}{2M\nu} \; ; \quad \left.\frac{1}{|\partial f/\partial x|}\right|_{f(x)=0} = \left.\frac{2Mx^2}{Q^2}\right|_{x = \frac{Q^2}{2M\nu}}$$

$$\boxed{\nu W_2 \longrightarrow \sum_i e_i^2\, x_i\, f(x_i) = F_2(x)} \qquad x_i = x = \frac{Q^2}{Sy}.$$

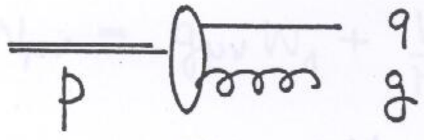
charge weighted momentum density of quarks.

$\boxed{\text{PARTONS}}$

Infinite momentum frame: $(E; P, 0, 0)$

$$|P| \gg M \quad , \; E = \sqrt{M^2 + P^2}.$$

→ non-covariant approach.



$$E = P\sqrt{1 + \frac{M^2}{P^2}} \sim P + \frac{1}{2}\frac{M^2}{P}$$

$$q = (q_0\,;\, q_3, \vec{q}_\perp)$$

$$q^2 \approx \vec{q}_\perp^{\,2}$$

<u>ONE ANSATZ:</u>

$$q_0 = \frac{2M\nu + q^2}{4P} \quad , \quad q_3 = -\frac{2M\nu - q^2}{4P}$$

$$= \frac{Q^2}{4P}\left(\frac{1-x}{x}\right) \qquad = -\frac{Q^2}{4P}\left(\frac{1+x}{x}\right).$$

$$-q^2 = Q^2 = (q_3 - q_0)(q_3 + q_0) + q_\perp^2$$

$$= \frac{Q^4}{(4P)^2}\left[-\frac{2}{x}\right][-2] + q_\perp^2 = \frac{Q^4}{4xP^2} + q_\perp^2 \approx q_\perp^2$$

$$\approx 0.$$

SINCE: $Q^2 = 2M\nu x$

$$\frac{Q^4}{4xP^2} = \frac{(M\nu)^2 x}{P^2}.$$

CHARACTERISTIC TIMES IN THE INFINITE MOMENTUM FRAME:

$$\tau_{int} \sim \frac{1}{q_0} = \frac{4P}{2M\nu + q^2} = \frac{4Px}{Q^2(1-x)}$$ INTERACTION TIME.

$$\tau_{life} \sim \frac{1}{\sum_i E_i - E}$$

PARTON LIFE TIME (LIFE TIME OF THE SINGLE PARTON FLUCTUATION OFF BACKGROUND)

$$\sum_i E_i - E = \sum_i \sqrt{x_i^2 P^2 + M_i^2 + k_{\perp i}^2} - \sqrt{P^2 + M^2}$$

$$\approx \sum_i x_i P \sqrt{1 + \frac{M_i^2 + k_{\perp i}^2}{x_i^2 P^2}} - P - \frac{M^2}{2P}$$

$$= \underbrace{\sum_i x_i P}_{\equiv P} + \sum_i \frac{M^2 + k_{\perp i}^2}{2 x_i P} - P - \frac{M^2}{2P}.$$

$$\tau_{life} = \frac{2P}{\sum_i \frac{k_{\perp i}^2 + M_i^2}{x_i} - M^2}$$

$$\frac{\tau_{life}}{\tau_{int}} = \frac{\left(\sum_i \frac{k_{\perp i}^2 + M_i^2}{x_i} - M^2\right)^{-1}}{\left[2x/Q^2(1-x)\right]} \Big|_{\text{indep of } \underline{P}}^{\approx} \ !$$

<u>Example:</u> $k_{\perp 1}^2 = k_{\perp 2}^2 = k_\perp^2 \ , \quad M_i^2 \approx 0$

$x_1, x_2 \quad ; \ x_1 + x_2 = 1$

$$\curvearrowright \quad \frac{\tau_{life}}{\tau_{int}} = \left[\frac{2k_\perp^2}{Q^2}\frac{1}{(1-x)^2}\right]^{-1} \left\{ \begin{array}{l} \tau_{life} \sim \dfrac{2Px(1-x)}{k_\perp^2} \\ \tau_{int} \sim \dfrac{4Px}{Q^2(1-x)} \end{array} \right.$$

$k_\perp^2 = 1\,\text{GeV}^2 , \quad Q^2 = 20\,\text{GeV}^2$

$$\frac{\tau_{life}}{\tau_{int}} \sim 5.$$

$x \to 0, 1 \quad ; \quad \tau_{life} \to 0$

WHEN DOES DIS PROBE SINGLE PARTONS?

$Q^2 \gg k_{\perp i}^2 \quad , \qquad x \text{ not } \sim 0, 1$

THE SPIN OF QUARKS & σ_{DIS}:

$$\frac{d^2\sigma}{dQ^2 dy} = \frac{s}{2M}\,\frac{4\pi\alpha^2}{Q^4}\,\frac{E'}{E}\left[2W_1 \sin^2(\theta/2) + W_2 \cos^2(\theta/2)\right]$$

TS

$$1-\frac{E'}{E} = y, \qquad 4EE'\sin^2\frac{\theta}{2} = Q^2$$

$$MW_1 = F_1 \qquad\qquad s = (M+E)^2 - E^2 = 2ME, \quad M \ll E.$$

$$\nu W_2 = F_2$$

$$\frac{d^2\sigma}{dQ^2 dy} = \frac{2\pi\alpha^2}{y\,Q^4}\left[2xy\,F_1 + 2(1-y)F_2 - \underbrace{\frac{2M^2}{s}\,xy\,F_2}_{\substack{\sim 0 \\ M^2 \ll s}}\right].$$

spin 0 partons: $\quad F_1 = 0$

spin 1/2 partons: $\quad F_1 = \frac{1}{2x} F_2$.

$$R = \frac{F_2 - 2xF_1}{2xF_1} = \begin{cases} \to \infty & \text{spin } 0 \\ \to 0 & \text{spin } \frac{1}{2} \end{cases}$$

y-dependence of the cross section:

$$\left.\frac{d\sigma}{dQ^2 dy}\right|_{\text{spin } \frac{1}{2}} \sim \frac{2\pi\alpha^2}{y\,Q^4}\left[1+(1-y)^2\right] F_2 \quad \nwarrow \text{ observed}$$

$$\left.\frac{d\sigma}{dQ^2 dy}\right|_{\text{spin } 0} \sim \frac{2\pi\alpha^2}{y\,Q^4}\left[2(1-y)\right] F_2 \quad \swarrow \text{ not observed!}$$

THE BORN CROSS SECTIONS

CHARGED LEPTONS :

NC :

$$\frac{d^2\sigma}{dx\,dQ^2} = 2\pi\alpha^2 \frac{M_N s}{(s-M^2)^2}\frac{1}{Q^4} L^{\mu\nu}\tilde{W}_{\mu\nu}$$

$e^\pm N \to e^\pm X \quad (\mu^\pm N \to \mu^\pm X)$

pure photon exchange:

$$L_{\mu\nu} = 2\left[l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu}\, l.l'\right]$$

$$W_{\mu\nu} = \frac{1}{4\pi}\sum_n \langle P | J_\mu^{em\,\dagger}(0) | n\rangle\langle n | J_\nu^{em}(0) | P\rangle (2\pi)^4 \delta^{(4)}(P+q-p_n)$$

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x,Q^2) + \frac{1}{M^2}\left[\left(P_\mu - \frac{Pq}{q^2}q_\mu\right)\left(P_\nu - \frac{Pq}{q^2}q_\nu\right)\right] \cdot W_2(x,Q^2)$$

$$\boxed{\begin{aligned} F_2(x,Q^2) &= x\left(-g_{\mu\nu} + \frac{12x^2}{Q^2}P_\mu P_\nu\right)W^{\mu\nu} \\ F_L(x,Q^2) &= \frac{8x^3}{Q^2}P_\mu P_\nu W^{\mu\nu} \end{aligned}}$$

$(\mathcal{O}(\alpha_s))$

2 STRUCTUREFCT. $\qquad F_L = F_2 - 2xF_1$

$\mathcal{O}(\alpha_s^0)$ 1 STRCT. FCT.

$$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{2\pi\alpha^2}{xQ^4} Y_+ F_2(x,Q^2)}$$

$(F_L = 0)$ lowest order

$$Y_\pm = 1 \pm (1-y)^2$$

INCLUSION OF ℓ-BEAM POLARIZATION

& Z EXCHANGE:

$$\boxed{\frac{d^2\sigma^\pm}{dx\,dQ^2} = \frac{2\pi\alpha^2}{x\,Q^4}\left\{ Y_+\,\mathbb{F}_2^\pm(x,Q^2) + Y_-\,x\mathbb{F}_3^\pm(x,Q^2)\right\}}$$

$$\mathbb{F}_2^\pm(x,Q^2) = F_2(x,Q^2) + K_Z(Q^2)\,(-v \mp \lambda a)\,G_2(x,Q^2) + K_Z^2(Q^2)\,(v^2+a^2 \pm 2\lambda v a)\,H_2(x,Q^2)$$

$$x\mathbb{F}_3^\pm(x,Q^2) = K_Z(Q^2)\,(\pm a + \lambda v)\,x G_3(x,Q^2) + K_Z^2(Q^2)\,(\mp 2va - \lambda(v^2+a^2))\,x H_3(x,Q^2)$$

5 Structurefct. (without longitud.)

+3 longitudinal Structfct.

$$K_Z(Q^2) = \frac{1}{4\sin^2\theta_W\cos^2\theta_W}\;\frac{Q^2}{Q^2+M_Z^2}$$

$$a \equiv a_e = -\frac{1}{2}$$

$$v \equiv v_e = -\frac{1}{2} + 2\sin^2\theta_W$$

$$\boxed{\text{CC :}}$$

$e^{\pm}(\mu^{\pm})N \to \overset{(-)}{\nu}_{e(\mu)} X$

$$\boxed{\frac{d^2\sigma^{\pm}}{dx\,dQ^2} = \frac{2\pi\alpha^2}{x\,Q^4}\; \kappa_W^2(Q^2)\left(\frac{1\pm\lambda}{2}\right)\cdot\left\{ Y_+ W_2^{\pm}(x,Q^2) \pm Y_- x W_3^{\pm}(x,Q^2)\right.}$$

$$\kappa_W(Q^2) = \frac{Q^2}{Q^2+M_W^2}\cdot\frac{1}{4\sin^2\theta_W}$$

4 structure fct.

+ 2 long. structurefct.

BORN: $e^{\pm}p$ 14 structure functions !

$e^{\pm}d$ + ——— · ———

(composed out of : u, d, s, c, b & g

$\bar{u}, \bar{d}, \bar{s}, c=\bar{c}, b=\bar{b}$

$\leqq \underline{\underline{10}}$ parton densities)

PARTON MODEL AND FLAVOUR CONTENTS OF STRUCTURE FUNCTIONS

P

CHARGED LEPTON (BORN) STRUCTURE FCT.:

$$F_2(x,Q^2) = x\sum_q e_q^2\left[q(x,Q^2)+\bar{q}(x,Q^2)\right]$$

$$G_2(x,Q^2) = x\sum_q 2e_q v_q\left[q(x,Q^2)+\bar{q}(x,Q^2)\right]$$

$$H_2(x,Q^2) = x\sum_q (v_q^2+a_q^2)\left[q(x,Q^2)+\bar{q}(x,Q^2)\right]$$

$$xG_3(x,Q^2) = 2x\sum_q e_q a_q\left[q(x,Q^2)-\bar{q}(x,Q^2)\right]$$

$$xH_3(x,Q^2) = 2x\sum_q v_q a_q\left[q(x,Q^2)-\bar{q}(x,Q^2)\right]$$

$$W_2^+(x,Q^2) = 2x\sum_i\left[d_i(x,Q^2)+\bar{u}_i(x,Q^2)\right]$$

$$W_2^-(x,Q^2) = 2x\sum_i\left[u_i(x,Q^2)+\bar{d}_i(x,Q^2)\right]$$

$$xW_3^+(x,Q^2) = 2x\sum_i\left[u_i(x,Q^2)-\bar{d}_i(x,Q^2)\right]$$

$$xW_3^-(x,Q^2) = 2x\sum_i\left[d_i(x,Q^2)-\bar{u}_i(x,Q^2)\right]$$

$$\overset{(-)}{u}_i = (\overset{(-)}{u},\overset{(-)}{c},\overset{(-)}{t})$$

$$\overset{(-)}{d}_i = (\overset{(-)}{d},\overset{(-)}{s},\overset{(-)}{b})$$

NEUTRINO (BORN) STRUCTURE FCT.: **P**

$$F_2^{\nu}(x,Q^2) = 2x\left[a_{21}\sum_i (u_i+\bar{u}_i) + a_{22}\sum_i (d_i+\bar{d}_i)\right]$$
$$\equiv F_2^{\bar{\nu}}(x,Q^2)$$
$$xF_3^{\nu}(x,Q^2) = 2x\left[a_{31}\sum_i (u_i-\bar{u}_i) + a_{32}\sum_i (d_i-\bar{d}_i)\right]$$
$$\equiv -xF_3^{\bar{\nu}}(x,Q^2).$$

$$a_{21} = \frac{1}{4} - e_u \sin^2\theta_W + 2e_u^2 \sin^4\theta_W$$
$$a_{22} = \frac{1}{4} + e_d \sin^2\theta_W + 2e_d^2 \sin^4\theta_W$$
$$a_{31} = \frac{1}{4} - e_u \sin^2\theta_W$$
$$a_{32} = \frac{1}{4} + e_d \sin^2\theta_W$$

$$W_2^{\nu}(x,Q^2) = 2x\sum_i (d_i+\bar{u}_i)$$
$$xW_3^{\nu}(x,Q^2) = 2x\sum_i (d_i-\bar{u}_i)$$
$$W_2^{\bar{\nu}}(x,Q^2) = 2x\sum_i (u_i+\bar{d}_i)$$
$$xW_3^{\nu}(x,Q^2) = 2x\sum_i (u_i-\bar{d}_i)$$

DEUTERONS & ISOSCALAR NUCLEI

d's at colliders: $s \to s/2$!

quark contents:

$$\boxed{\overset{(-)}{u}_1, \overset{(-)}{d}_1 \equiv \overset{(-)}{u}, \overset{(-)}{d} \to \tfrac{1}{2}\overset{(\longrightarrow)}{(u+d)}}$$

$\longrightarrow$ previous formulae modify accordingly.

EXAMPLES:

$$F_2^{ed} = \frac{5}{18}x(u_v+d_v) + \frac{10}{9}xu_s + \frac{2}{9}xs + \frac{8}{9}xc + \frac{2}{9}xb$$

$$xG_3^{ed} = \frac{1}{2}x(u_v+d_v) = \frac{1}{2}V$$

$e^{\pm}d$

$$W_2^{e^{\pm}d} = x(u_v+d_v) + 4xu_s + 2xs + 2xc + 2xb \equiv \Sigma$$

$$xW_3^{e^{\pm}d} = x(u_v+d_v) \pm 2x\underline{(s-c)}$$

$$W_2^{\overset{(-)}{\nu} d} = \sum_i x\left[q_i(x,Q^2) + \bar{q}_i(x,Q^2)\right] \equiv \Sigma$$

$$\frac{1}{2}\left[x W_3^{\nu d} + x W_3^{\bar{\nu} d}\right] \underset{Df}{=} x W_3^{d} - x\,(u_v + d_v) \equiv V$$

WAYS TO UNFOLD PARTON DENSITIES

$\boxed{e^{\pm}p}$ 4 CROSS SECTIONS

$\sigma_{NC}^{\pm}\,,\ \sigma_{CC}^{\pm}$ $\hookrightarrow$ 4 COMBINATIONS OF PARTON DENSITIES

LINEAR MAPPING: $\overset{(-)}{U} = \sum_i x\overset{(-)}{u}_i \quad ; \quad \overset{(-)}{D} = \sum_i x\overset{(-)}{d}_i$

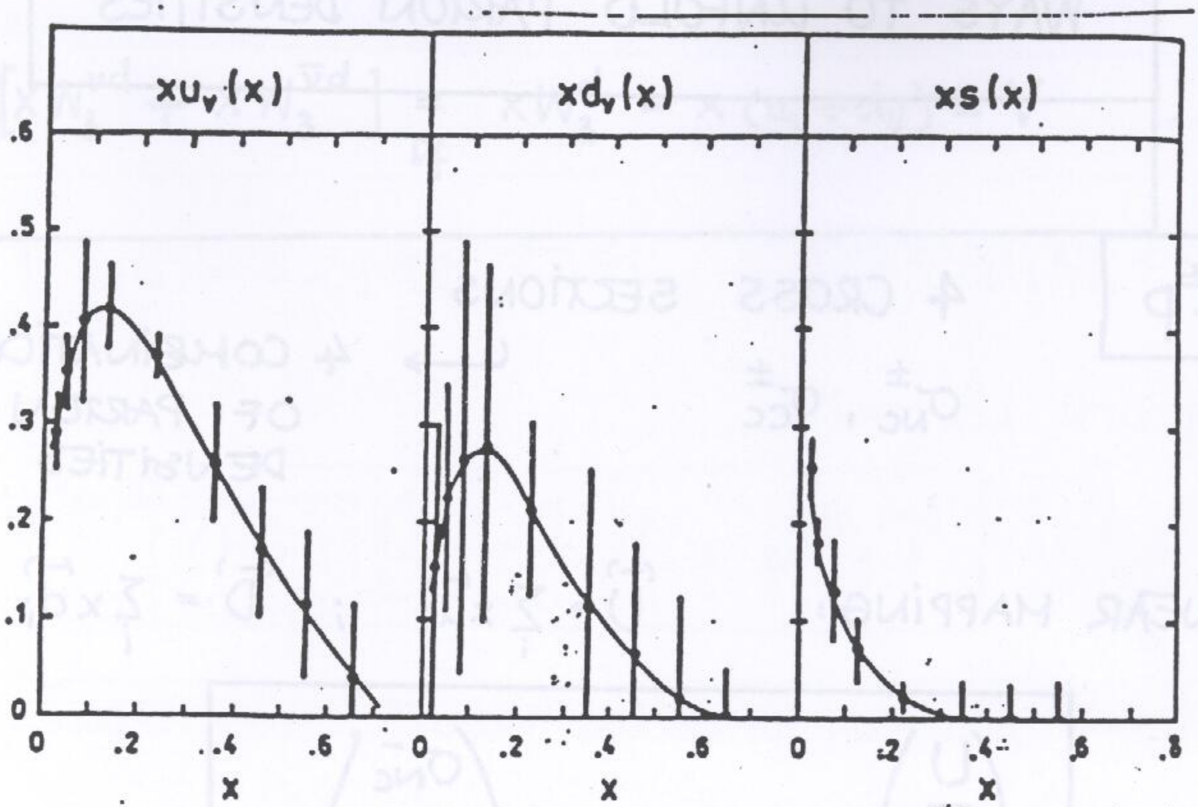
$$\boxed{\begin{pmatrix} U \\ \overline{U} \\ D \\ \overline{D} \end{pmatrix} = (A_{ij}) \begin{pmatrix} \sigma_{NC}^{-} \\ \sigma_{NC}^{+} \\ \sigma_{CC}^{-} \\ \sigma_{CC}^{+} \end{pmatrix}}$$

$$\det_4 A_{ij} \sim \left\{ \kappa_Z(Q^2)\,[1-(1-y)^4] \right\}^{-1}$$

(A_{ij}) becomes singular (degenerate) both for: $Q^2 \ll M_Z^2$, $y \ll 1$

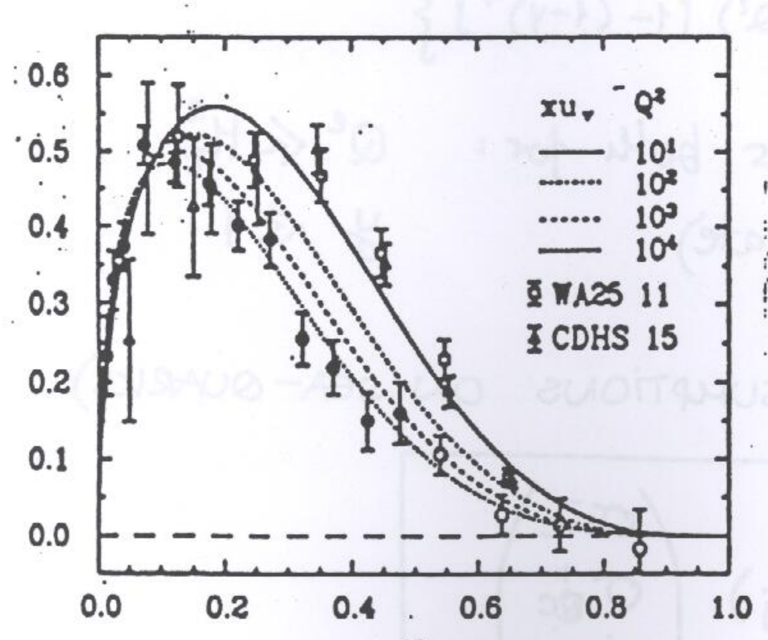
CONSIDER e.g. (WITH ASSUMPTIONS ON SEA-QUARKS)

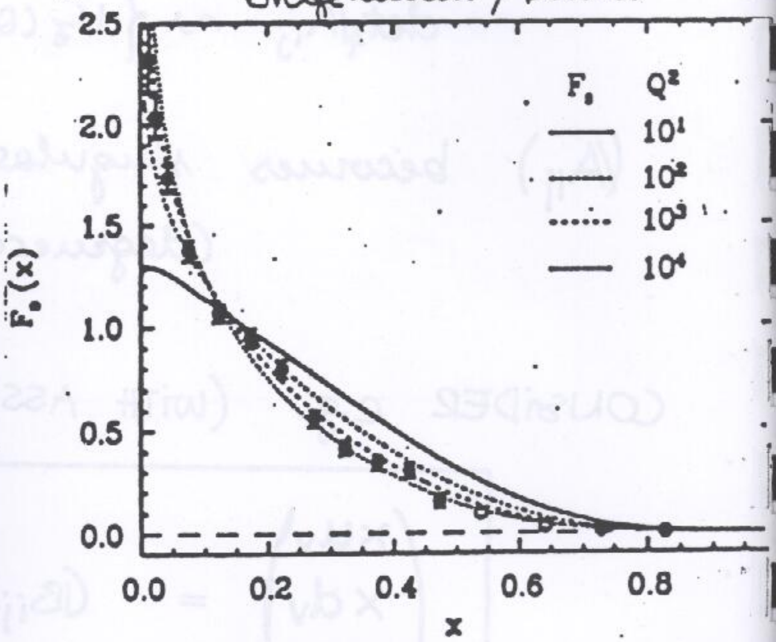
$$\boxed{\begin{pmatrix} xu_v \\ xd_v \\ xs \end{pmatrix} = (B_{ij}) \begin{pmatrix} \sigma_{NC}^{-} \\ \sigma_{NC}^{+} \\ \sigma_{CC}^{-} \end{pmatrix}}$$



$\mathcal{L} = 100\,\mathrm{pb}^{-1}$ / per beam

Ingelman / Rückl

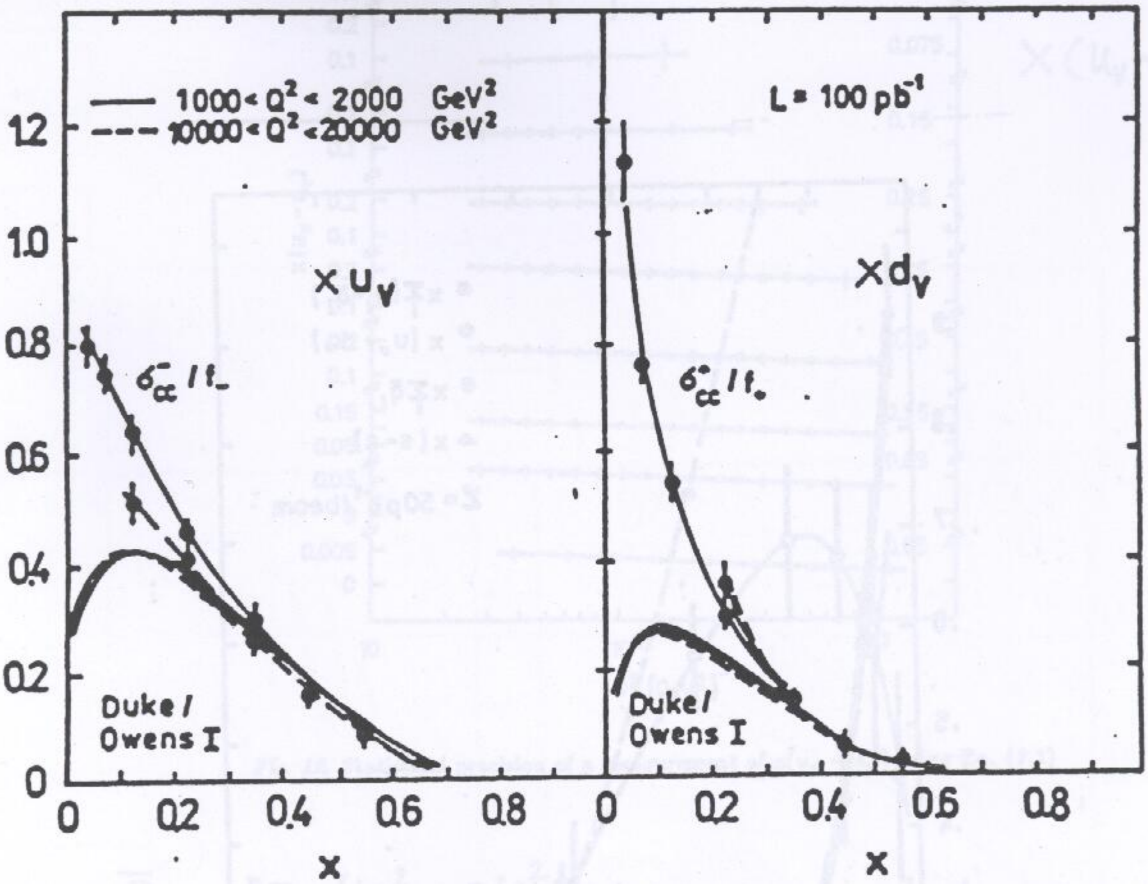




$\sum \mathcal{L} = 400\,\mathrm{pb}^{-1}$

APPROXIMATE REPRESENTATIONS

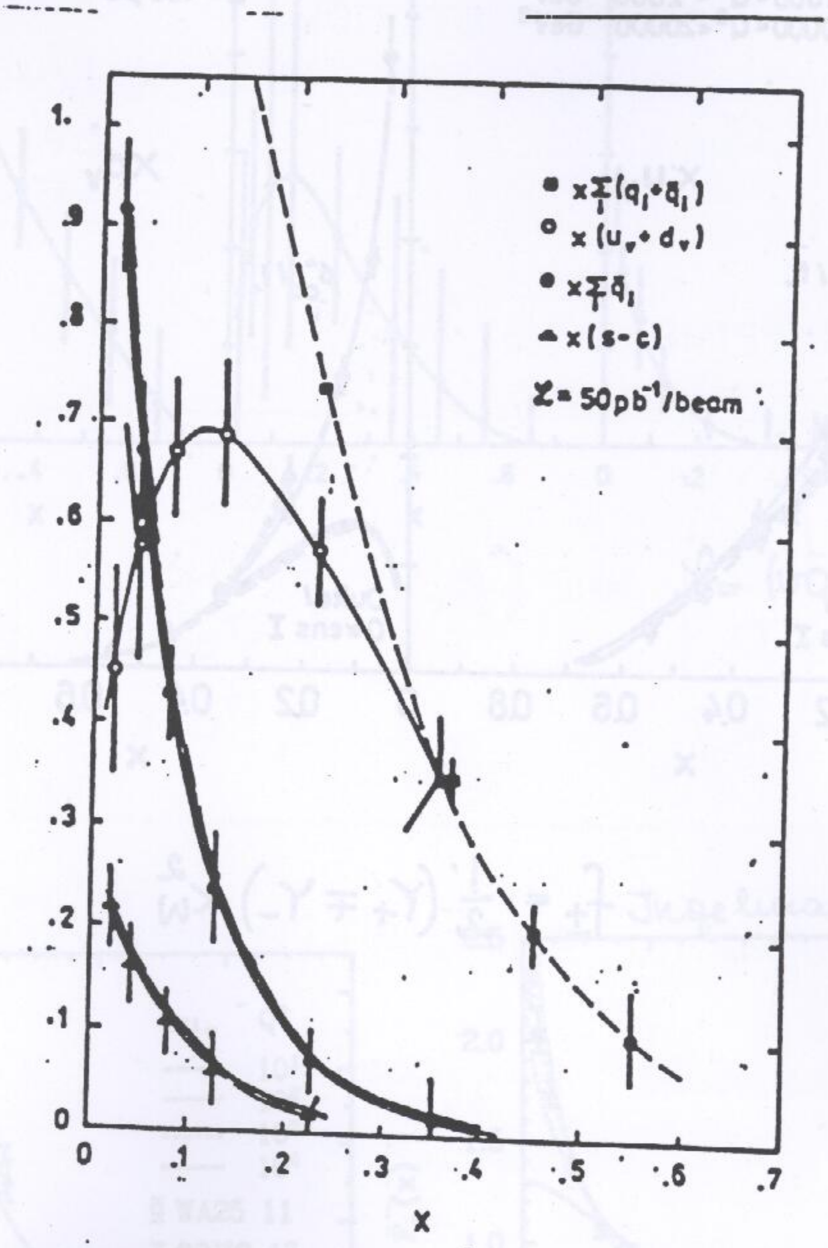
VALENCE RANGE



$$f_{\pm} = \frac{1}{2}\left(Y_+ \mp Y_-\right) K_W^2$$

$\mathcal{L} = 100\,pb^{-1}$

$e^{\pm}p$ & $e^{\pm}d$

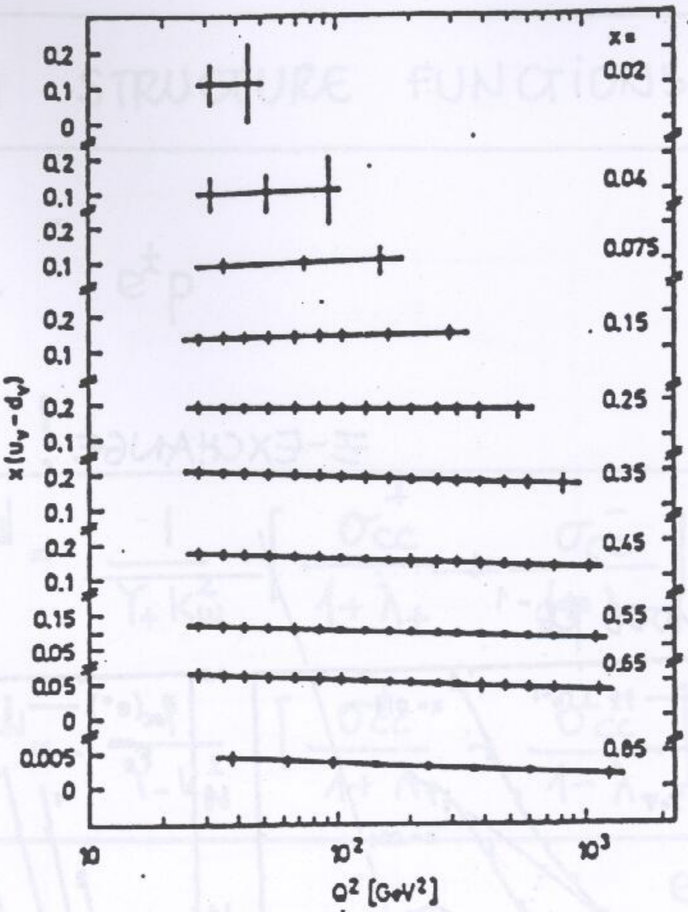


$$\bar{Q} = \frac{1}{2}\left(W_2^{eN} - x W_3^{eN}\right)$$

$$x(s-c) = \frac{5}{18} W_2^{eN} - F_2^{eN} \quad , \quad b \approx 0$$

$\mathcal{L} = 50\,pb^{-1}/beam$

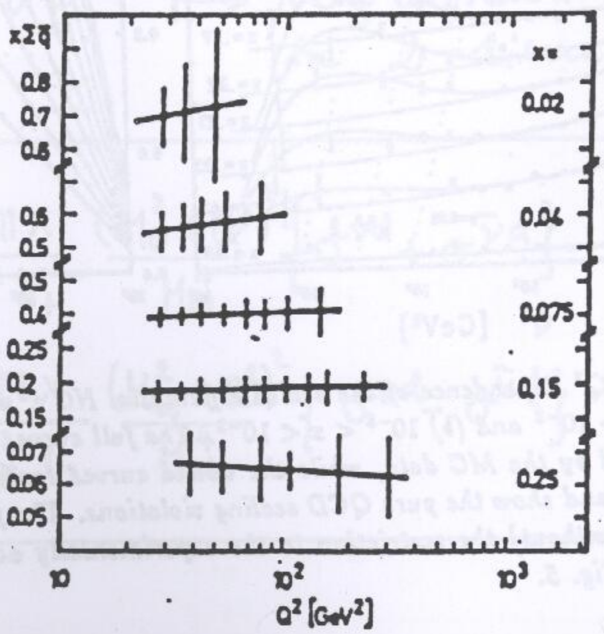
ν's: $$x(u_v - d_v) = \frac{4\pi x}{G_F^2}\,\frac{(M_W^2+Q^2)^2}{M_W^4}\,\frac{1}{Y_+ + Y_-}\left[\frac{1}{2}\sigma^{\nu d} - \sigma^{\nu p}\right]$$



$x(u_v - d_v)$

Fig. 12. Statistical precision of a measurement of $x(u_v - d_v)$ using Eq. (7.1)

$$\sum_i x\bar{q}_i = \frac{\bar{Q}}{2} = \frac{2\pi x}{G_F^2}\,\frac{(M_W^2+Q^2)^2}{M_W^4}\left[\sigma^{\bar{\nu}d} - \sigma^{\nu d}(1-y)^2\right]\frac{1}{Y_+Y_-} - \frac{x(s+b-c)}{Y_+}$$



$\sum_i x\bar{q}_i$

Fig. 13. Statistical precision of a measurement of the antiquark distribution Eq. (7.3)

S

Z-EXCHANGE !

LEP x LHC

$\mathcal{L}$ = 1000 pb^{-1} $\mathcal{L}$ = 100 pb^{-1}

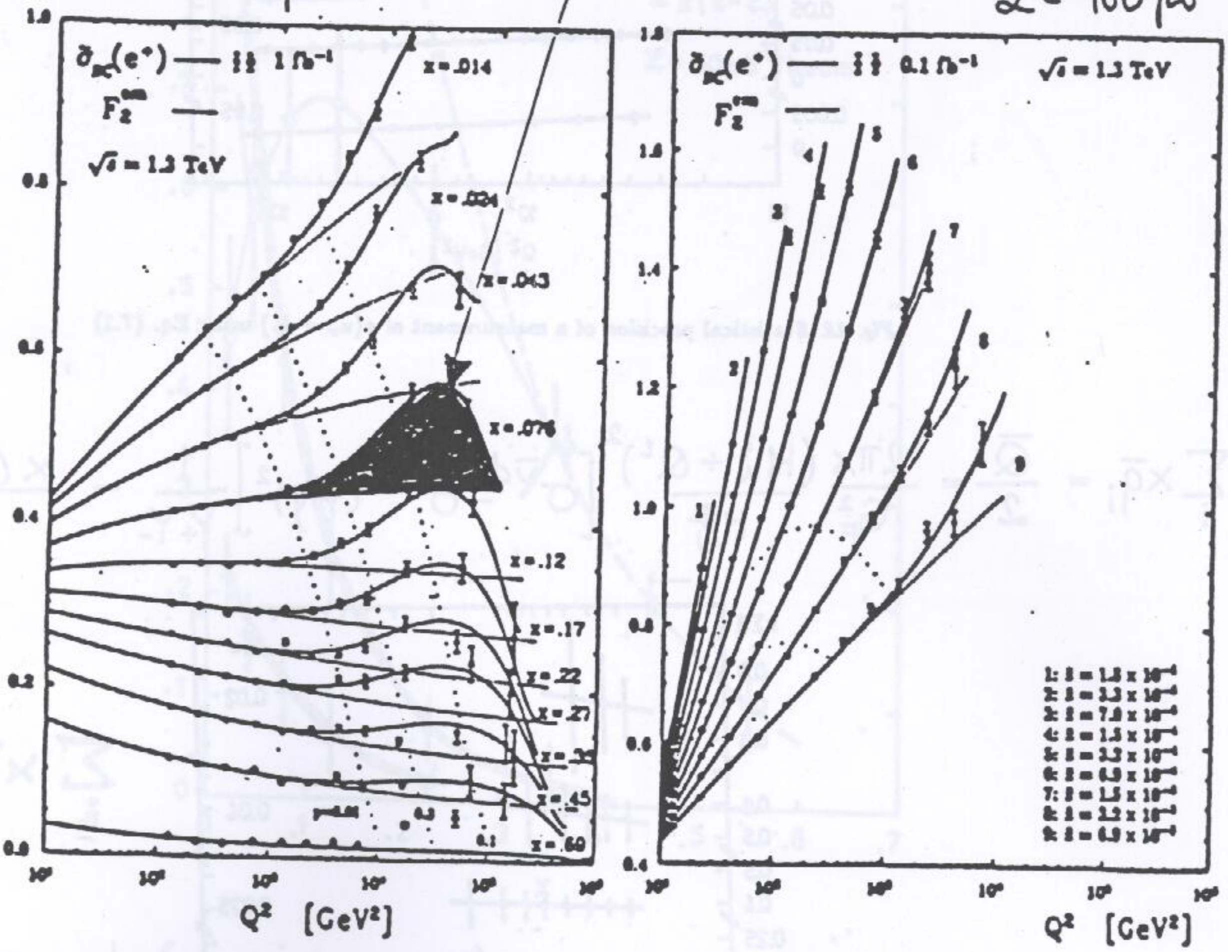


Figure 6: *Q^2 dependence of the scaled differential NC e^+p cross-section at LEP+LHC for (a) $x > 10^{-2}$ and (b) $10^{-5} < x < 10^{-2}$. The full curves correspond to $\tilde{\sigma}_{NC}(e^+)$, also represented by the MC data, while the dotted curves represent F_2^{em}, i.e. pure photon exchange, and show the pure QCD scaling violations. The full (open) MC data symbols are with (without) the restriction to the experimentally acceptable phase space region shown in Fig. 5.*

DEUTERON STRUCTURE FUNCTIONS

$e^\pm (\mu^\pm) d$

NC : cf. $e^\pm p$

CC :

$$W_2^{eN} = \frac{1}{Y_+ \kappa_W^2} \left[\frac{\sigma_{cc}^+}{1+\lambda_+} + \frac{\sigma_{cc}^-}{1-\lambda_-} \right]$$

$$x W_3^{eN} = \frac{1}{Y_- \kappa_W^2} \left[\frac{\sigma_{cc}^+}{1+\lambda_+} - \frac{\sigma_{cc}^-}{1-\lambda_-} \right]$$

e^+ & e^- requir. $\mathcal{L}$-splitting.

$\overset{(-)}{\nu}_\mu (\overset{(-)}{\nu}_e) d$

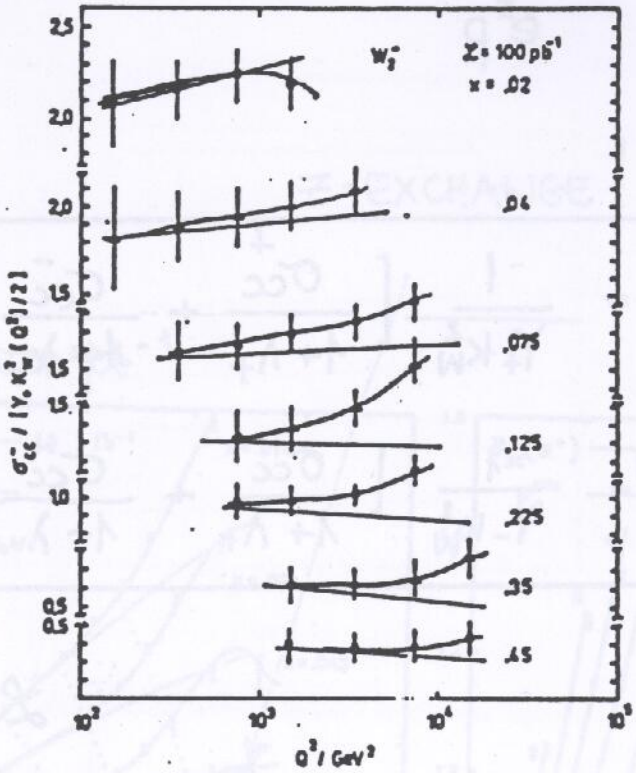
NC : cf. $\overset{(-)}{\nu} p$ $\longmapsto$ very difficult to measure in 2-dimensions (x, Q^2).

CC :

$$W_2^d = \frac{2\pi x}{G_F^2 Y_+} \frac{(M_W^2+Q^2)^2}{M_W^4} \left\{ \sigma^{\nu d} + \sigma^{\bar{\nu} d} \right\} - \underline{\frac{2x Y_-}{Y_+} (s+b-c)}$$

$$x W_3^d = \frac{2\pi x}{G_F^2 Y_-} \frac{(M_W^2+Q^2)^2}{M_W^4} \left\{ \sigma^{\nu d} - \sigma^{\bar{\nu} d} \right\}$$

SI:



W_2^-, CC, ep
$100 pb^{-1}$

$\sigma_{CC}^{e^-p} / [Y_+ \kappa_W^2/2]$

$Y_- \times W_3 \lneqq W_2$

treat as correctio

$W_2^d = \frac{1}{2}[W_2^{\nu d} + W_2^{\bar{\nu} d}]$ $\overset{(-)}{\nu} d$ $xW_3^d = \frac{1}{2}[xW_3^{\nu d} + xW_3^{\bar{\nu} d}]$

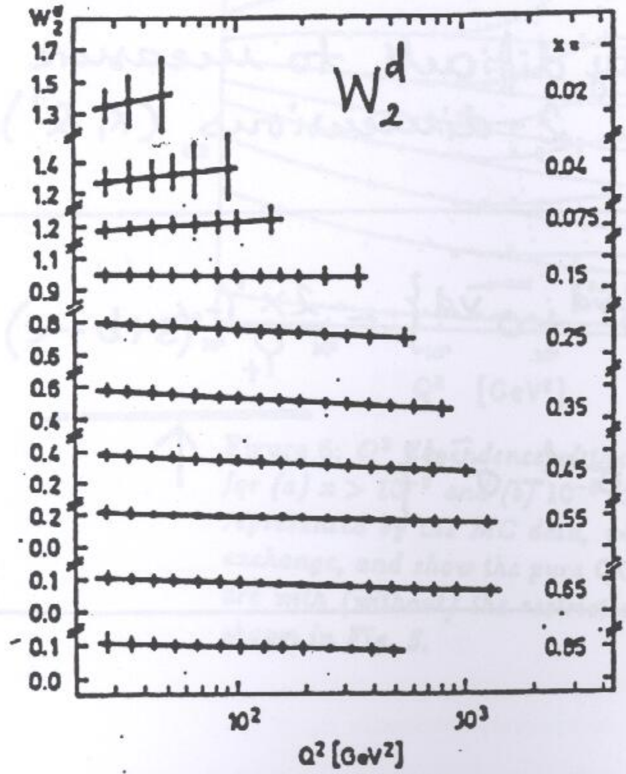


Fig. 9. **Statistical preparation of W_2 in $\overset{(-)}{\nu}$-WBB's, Eq. (5.2)**

UNK – WBB

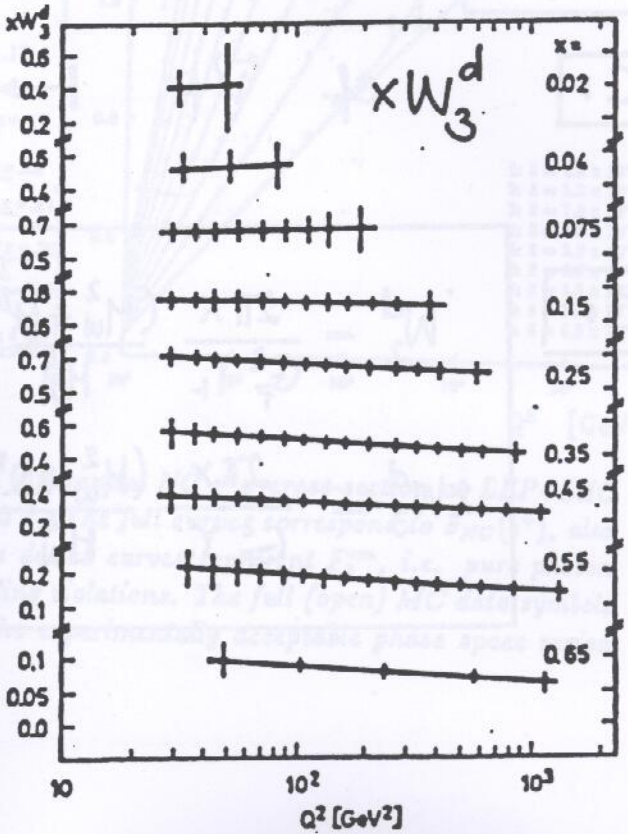
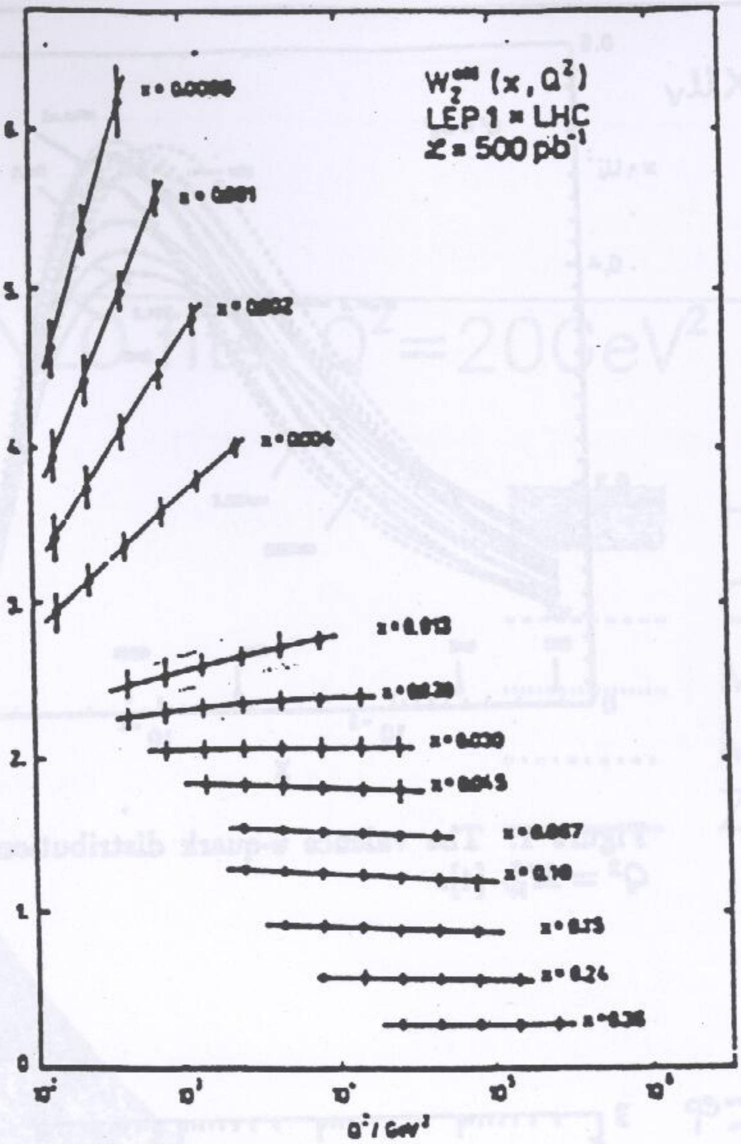
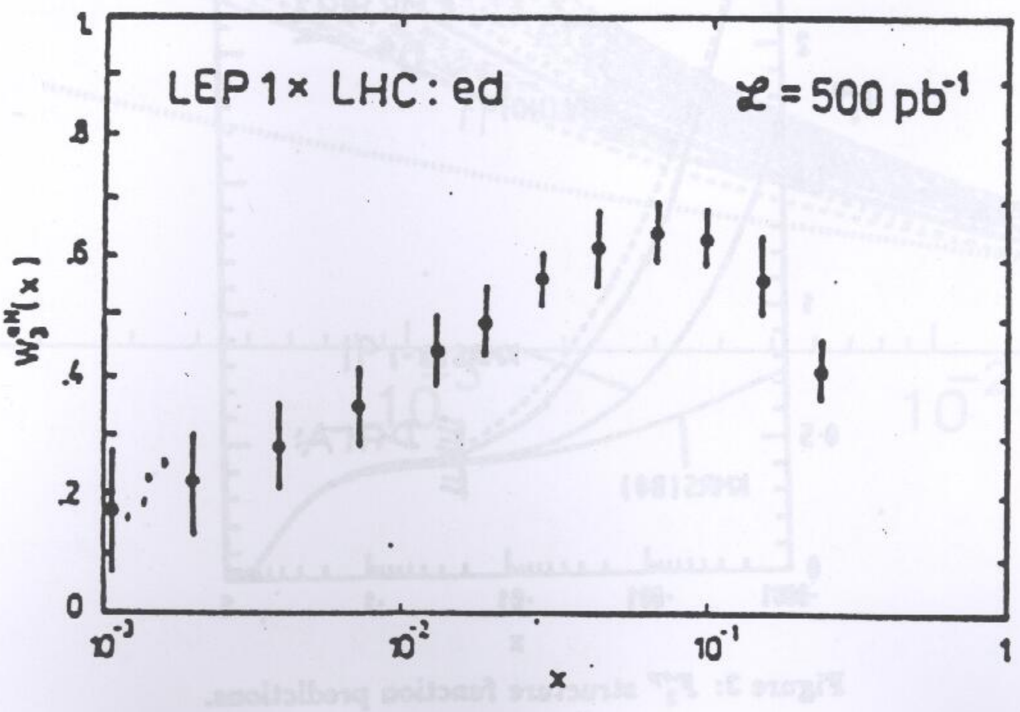


Fig. 10. **Statistical preparation of xW_3 in $\overset{(-)}{\nu}$-WBB's, Eq. (5.4)**

LEP1×LHC

$e^{\pm}d$





PARAMETRIZATIONS OF PARTON DISTRIBUTIONS

STIRLING

xu_v

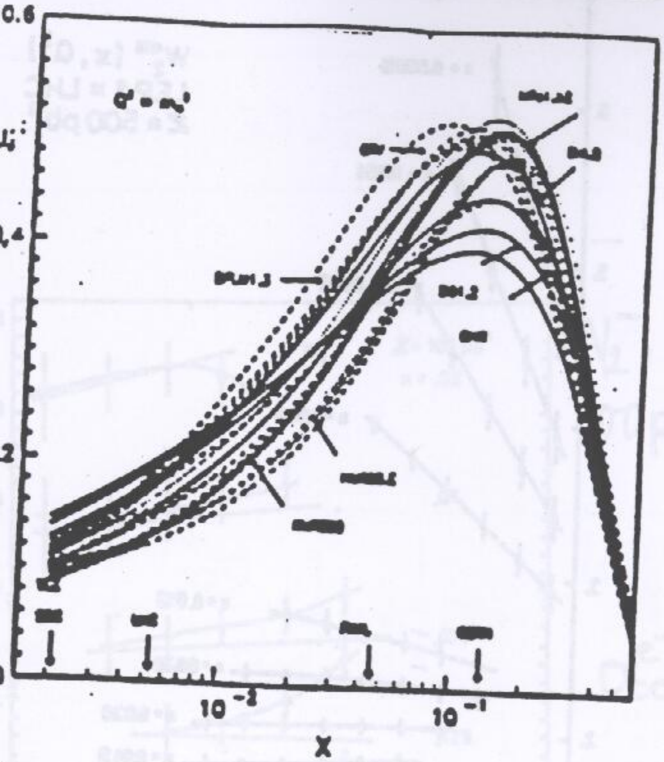


Figure 1: The valence u-quark distribution at $Q^2 = M_W^2$ [4].

F_2^{ep}

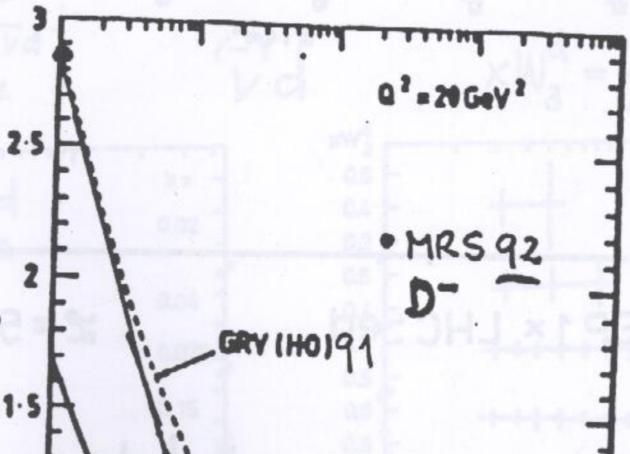
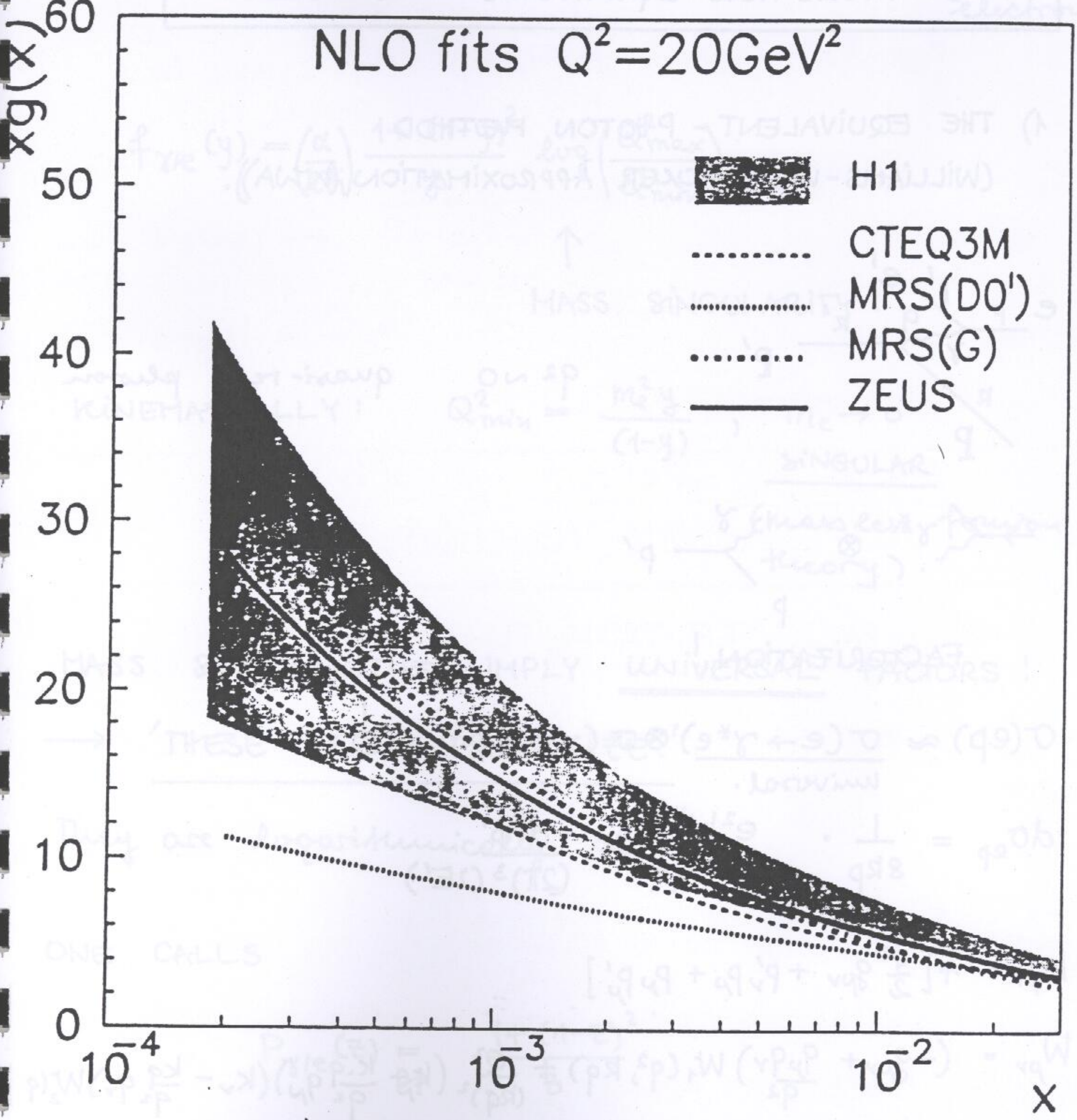


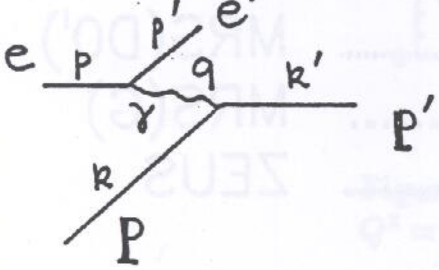
Figure 3: F_2^{ep} structure function predictions.


NLO fits Q^2=20GeV^2
xg(x)
H1
CTEQ3M
MRS(D0')
MRS(G)
ZEUS
0
10
20
30
40
50
60
10^{-4}
10^{-3}
10^{-2}
x

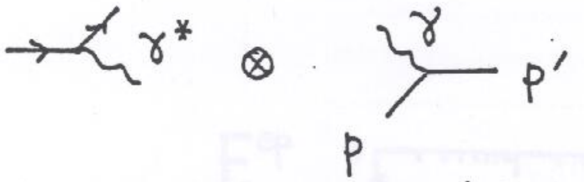
LECTURE 4

SCALING VIOLATIONS, MASS SINGULARITIES AND EVOLUTION EQUATIONS

1) THE EQUIVALENT - PHOTON METHOD (WILLIAMS - WEIZSÄCKER APPROXIMATION (WWA)).



$q^2 \sim 0$ quasi-real photon



FACTORIZATION !

$$\sigma(ep) \approx \underbrace{\sigma(e\to\gamma^* e)}_{\text{universal.}} \otimes \sigma(\gamma^* p \to p') \quad !$$

$$d\sigma_{ep} = \frac{1}{8kp} \cdot \frac{e^2 W^{\mu\nu} L_{\mu\nu}}{q^4} \frac{d^3p'}{(2\pi)^3 (2E')}$$

$$L_{\mu\nu} = 4\left[\frac{q^2}{2} g_{\mu\nu} + p'_\nu p_\mu + p_\nu p'_\mu\right]$$

$$W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(q^2, kq) + \frac{Q^2}{(kq)^2}\left(k_\mu - \frac{k.q}{q^2} q_\mu\right)\left(k_\nu - \frac{kq}{q^2} q_\nu\right) W_2(q$$

$$W_1(0, k.q) = W_2(0, k.q) + O(q^2)$$

$$d\sigma_{ep} = -W_1(0, k.q) \frac{\alpha}{4\pi} \frac{1}{s} \frac{1}{q^2} dq^2 dy \; \frac{1+(1-y)^2}{y^2}$$

$\sim$

$$d\sigma_{ep} = \sigma^{\gamma p} \cdot \underbrace{f_{\gamma/e}(y)\, dy}$$

<u>universal</u> photon (flux) contained in the electron.

$$f_{\gamma/e}(y) = \left(\frac{\alpha}{2\pi}\right) \frac{1+(1-y)^2}{y} \log\left(\frac{Q^2_{max}}{Q^2_{min}}\right)$$

$\uparrow$

MASS SINGULARITY!

KINEMATICALLY: $\quad Q^2_{min} = \dfrac{m_e^2\, y}{(1-y)}$, $\quad m_e \to 0$

<u>SINGULAR</u>

(massless fermion theory).

MASS SINGULARITIES IMPLY <u>UNIVERSAL</u> FACTORS!

$\longrightarrow$ <u>'THESE TERMS FACTORIZE.'</u>

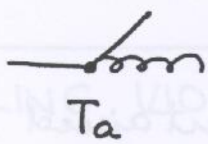
They are logarithmically large.

ONE CALLS

$$P_{\gamma/e}(z) \underset{Df}{=} \frac{1+(1-z)^2}{z}$$

THE SPLITTING FUNCTION OF AN ELECTRON INTO A PHOTON (IN LO QED).

QCD:



$$P_{g/q}(z) = C_F \, \frac{1+(1-z)^2}{z}$$

SPLITTING FCT OF A QUARK IN A GLUON (LO).

C_F : colour factor: $C_F = \dfrac{\mathrm{tr}\,[T_a T_a]}{N_C}$ ← (weight incom. quark)

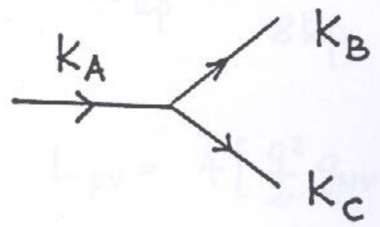
$$= \frac{N_c^2-1}{2} \cdot \frac{1}{N_c} = \frac{4}{3} \Big|_{N_c=3}.$$

---

2) LO QCD SPLITTING FUNCTIONS

(ALTARELLI-PARISI SPLITTING FUNCTIONS)

CONSIDER MASSLESS PARTICLE (FERMIONS, VECTORS) RADIATION.



INFINITE MOMENTUM FRAME : P

$$K_A = (P;\, P,\, \vec{0})$$

$$K_B = (zP + p_\perp^2/(2zP)\,;\, zP,\, \vec{p}_\perp)$$

$$K_C = ((1-z)P + p_\perp^2/(2(1-z)P)\,;\, (1-z)P,\, -\vec{p}_\perp).$$

CALCULATE :

$$d\hat{P}_{BA}(z)\, dz .$$

$$d\hat{P}_{BA}\, dz = \frac{E_B}{E_A}\, g^2 \frac{|M_{A\to B+C}|^2}{(2E_B)^2 (E_B+E_C-E_A)^2} \frac{d^3k_c}{(2\pi)^3(2E_c)}$$

ONE HAS :

$$\frac{E_B}{E_A} = z$$

$$(2E_B)^2 (E_B+E_C-E_A)^2 \approx 4\,(zP+\dots)^2 \left(P + \frac{p_\perp^2}{2zP} + \frac{p_\perp^2}{2(1-z)P} - \right.$$

$$= \frac{p_\perp^4}{(1-z)^2} .$$

$$\frac{d^3k_c}{(2\pi)^3(2E_c)} = \frac{P\, dz\, d^2\vec{p}_\perp}{2\cdot 8\pi^3 (1-z) P} = \frac{dz\, dp_\perp^2}{16\pi^2(1-z)}$$

$$\boxed{d\hat{P}_{BA}(z) = \frac{\alpha_s}{2\pi}\, \frac{z(1-z)}{2}\, \frac{1}{p_\perp^2}\, \overline{|M_{A\to B+C}|^2}\, d\log p_\perp^2}$$

EXAMPLE:

$$\sum_s \overline{|M_{G\to q\bar{q}}|^2} = \underbrace{\frac{1}{2}}_{\text{gluon spin}}\; \frac{tr[T_a T_a]}{8}\, tr[\not{k}_C \gamma_\mu \not{k}_B \gamma_\nu] \sum_{pol} \varepsilon_\mu^* \varepsilon_\nu$$

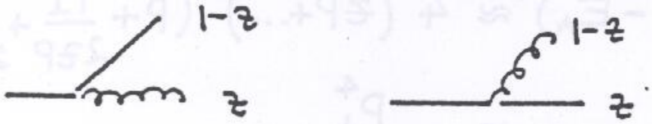
$$\frac{tr[T_a T_a]}{8} \equiv T_R \quad \text{initial gluons.}$$

$$= 2T_R\, p_\perp^2 \left(\frac{1-z}{z} + \frac{z}{1-z}\right)$$

$$d\hat{P}_{qg}(z) = \frac{\alpha_s}{2\pi} T_R \frac{z(1-z)}{2} \; \frac{(1-z)^2+z^2}{z(1-z)} \; d\ln p_\perp^2$$

$$P_{qg}(z) = T_R\left[z^2+(1-z)^2\right]$$

$P_{qq}(z)$:



$P_{gq}(z)$ $\qquad$ $P_{qq}(z)$

$z<1$: $\qquad P_{qq}(z) = P_{gq}(1-z) = C_F \frac{1+z^2}{1-z}$

$$P_{gg}(z) \underset{z<1}{=} 2C_G\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]$$

$$\equiv 2C_G\left[\frac{1}{z} + \frac{1}{1-z} - 2 + z(1-z)\right]$$

$P_{gg}(z) = P_{gg}(1-z)$

$P_{qg}(z) = P_{qg}(1-z)$

BEHAVIOUR OF $P_{qq}(z)$ AND $P_{gg}(z)$ FOR $z \to 1$:

A) FERMION NUMBER CONSERVATION.

$$q \to q \quad : \quad P^{-}_{qq}(z, \alpha_s) = \delta(1-z) + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^k P^{-(k)}_{qq}(z)$$

$k = 1 \qquad P_{qq} \equiv P^{-,(1)}_{qq}$.

WEIERSTRASS THEOREM $\rightsquigarrow$ $\int_0^1 dx\, P^{-,(k)}(x) = 0, \ \forall k$.

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{1-z} - \left(\int_0^1 dx\, \frac{1+x^2}{1-x}\right)\delta(1-z)\right]$$

$$\equiv C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] \equiv C_F\left(\frac{1+z^2}{1-z}\right)_+ .$$

$+$ PRESCRIPTION:

$$\int_0^1 dx\, [f(x)]_+\, \varphi(x) = \int_0^1 dx\, f(x)\,[\varphi(x) - \varphi(1)]$$

B) MOMENTUM CONSERVATION.

$$\int_0^1 dz\, z\, [2N_f P_{gq}(z) + P_{gg}(z)] = 0 \qquad (M1)$$

$$\int_0^1 dz\, z\, [2N_f P_{qg}(z)] = 2N_f T_R \frac{1}{3} = -\frac{1}{2}\left[-\frac{4N_f T_R}{3}\right]$$

$$\int_0^1 dz\, z\left[P_{gg}(z) - \frac{1}{z(1-z)}\right] = -\frac{11 C_G}{2\cdot 3}$$

$$\Rightarrow \quad P_{gg}(z) = 2C_G\left[\frac{1}{z} + \frac{1}{(1-z)_+} - 2 + z(1-z)\right] + \frac{\beta_0}{2}\delta(1-z)$$

$$\beta_0 = \frac{11C_G - 4T_R N_f}{3}$$

FURTHERMORE, WE HAVE:

$$\int_0^1 dz\, z\,[P_{qq}(z) + P_{Gq}(z)] = 0 \qquad (M2)$$

$$C_F\int_0^1 dz\, z\left[\left(\frac{1+z^2}{1-z}\right)_+ + \frac{1+(1-z)^2}{z}\right] = C_F\int_0^1 dz\left\{[1+(1-z)^2] + (1+z^2)(-1)\right\} = 0$$

(M1) AND (M2) IMPLY TOTAL MOMENTUM CONSERVATION OF THE PROTON MOMENTUM

$$\frac{d}{dt}\int_0^1 dx\, x\left[\sum_{i=1}^{2N_f} q^i(x,t) + G(x,t)\right] = 0$$

$t = \log Q^2$.

4.1. Splitting Functions

$\underline{O(\alpha_s): \quad (LO)}$

$$P_{NS}^{(0)}(z) \equiv P_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right]$$

$$P_{qg}(z) = T_f \left((1-z)^2 + z^2 \right)$$

$$P_{gq}(z) = C_F \, \frac{1+(1-z)^2}{z}$$

$$P_{gg}(z) = 2 C_G \left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) \right] + \frac{1}{2}\beta_0 \, \delta(1-z)$$

GROSS, WILCZEK 1973
GEORGI, POLITZER 1973
GRIBOV, LIPATOV 1972
DOKSHITSER 1977
LIPATOV 1975

ALTARELLI, PARISI 1977
KIM, SCHILCHER 1977/78
et al.

$$\boxed{\int_0^1 dz \; z^{N-1} \, P_{ab}^{(0)}(z) = - \frac{\gamma_{ab}^{0,N}}{4}}$$

SPLITTING FUNCTION — ANOMALOUS DIMENSI

$\underline{O(\alpha_s^2)}$ CONTR. DUE TO:

FLORATOS, D ROSS, SACHRAJDA 1977-79, BARDEEN, BURAS, DUKE, MUTA 1978
CURCI, FURMANSKI, PETRONZIO 1980
FURMANSKI, PETRONZIO 1980
GONZALEZ-ARROYO, LOPEZ, YNDURAIN 1979/80
FLORATOS, KOUNNAS, LACAZE 1981 a,b

NON-SINGLET :

$$\boxed{P_{\pm}(x,\alpha) = \hat{P}_{qq}(x,\alpha) \pm \hat{P}_{q\bar{q}}(x,\alpha)}$$

$\overline{MS}$

$$\hat{P}_{qq}(x,\alpha) = \left(\frac{\alpha}{2\pi}\right) C_F \left(\frac{1+x^2}{1-x}\right) + \left(\frac{\alpha}{2\pi}\right)^2 \left[C_F^2 P_F(x) + \tfrac{1}{2} C_F C_G P_G(x) + C_F N_F T_F P_{N_F}(x)\right], \tag{4.50}$$

$$\hat{P}_{q\bar{q}}(x,\alpha) = \left(\frac{\alpha}{2\pi}\right)^2 \left(C_F^2 - \tfrac{1}{2} C_F C_G\right) P_A(x), \tag{4.51}$$

$$P_F(x) = -2\frac{1+x^2}{1-x}\ln x \ln(1-x) - \left(\frac{3}{1-x} + 2x\right)\ln x - \tfrac{1}{2}(1+x)\ln^2 x - 5(1-x), \tag{4.52}$$

$$P_G(x) = \frac{1+x^2}{1-x}\left[\ln^2 x + \tfrac{11}{3}\ln x + \tfrac{67}{9} - \tfrac{1}{3}\pi^2\right] + 2(1+x)\ln x + \tfrac{40}{3}(1-x), \tag{4.53}$$

$$P_{N_F}(x) = \tfrac{2}{3}\left[\frac{1+x^2}{1-x}\left(-\ln x - \tfrac{5}{3}\right) - 2(1-x)\right], \tag{4.54}$$

$$P_A(x) = 2\frac{1+x^2}{1+x}\int_{x/(1+x)}^{1/(1+x)} \frac{dz}{z} \ln\frac{1-z}{z} + 2(1+x)\ln x + 4(1-x). \tag{4.55}$$

Table I

Detailed contribution of various diagrams to $\Gamma_{qq}(x,\alpha,1/\epsilon)$

	$\Gamma_{qq}(x,\alpha,1/\epsilon)$	C_F^2						$\frac{1}{2}C_F C_G$					$\frac{1}{2}N_F C$
							SUM					SUM	
$1/\epsilon$ A	$\frac{1+x^2}{1-x}$	$7-\frac{2}{3}\pi^2$	$-7+\frac{2}{3}\pi^2$	0	0	0	0	$7-\frac{2}{3}\pi^2$	0	$-11+\pi^2$	$\frac{103}{9}-\frac{2}{3}\pi^2$	$\frac{67}{9}-\frac{1}{3}\pi^2$	[illegible]
A	$\frac{1+x^2}{1-x}\ln^2 x$	-2	1	-1	2	0	0	-1	1	1	0	1	
A	$(1+x)\ln x$	0	$-\frac{7}{2}$	2	-1	0	$-\frac{5}{2}$	$\frac{7}{2}$	-2	$\frac{1}{2}$	0	2	
A	$1-x$	3	-11	0	3	0	-5	11	0	-1	$\frac{10}{3}$	$\frac{40}{3}$	[illegible]
B	$(1+x)\ln^2 x$	0	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0	0	0	[illegible]
B	$\frac{1+x^2}{1-x}\ln^2(1-x)$	0	0	0	0	0	0	0	0	2	-2	0	[illegible]
B	$\frac{1+x^2}{1-x}\ln x\ln(1-x)$	-4	2	0	0	0	-2	-2	0	6	-4	0	[illegible]
B	$\frac{1+x^2}{1-x}\ln x$	0	$-\frac{3}{2}$	0	0	0	$-\frac{3}{2}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$\frac{11}{3}$	$\frac{11}{3}$	[illegible]
B	$\frac{1+x^2}{1-x}\ln(1-x)$	3	-3	4	-4	0	0	3	-4	5	-4	0	[illegible]
B	$(1-x)\ln x$	-4	2	0	3	0	1	-2	0	2	0	0	[illegible]
B	$(1-x)\ln(1-x)$	0	0	0	0	0	0	0	0	4	-4	0	[illegible]
B	$I_1\frac{1+x^2}{1-x}$	4	-4	0	0	0	0	4	0	-8	4	0	[illegible]
B	$I_0\frac{1+x^2}{1-x}$	0	0	4	-4	0	0	0	-4	8	-4	0	[illegible]
B	$I_0\frac{1+x^2}{1-x}(\ln x+\ln(1-x))$	-4	4	0	0	0	0	-4	0	8	-4	0	[illegible]
B	$I_0(1-x)$	-4	4	0	0	0	0	-4	0	8	-4	0	[illegible]

Appropriate colour factors are shown in the first line. Terms of type A satisfy the Gribov-Lipatov relation while those of type B break it.

SINGLET:

$P_{ij}^{(1)}(x)$:

$$\begin{aligned}\dot{P}_{FF}^{(1,S)} = {} & C_F^2[-1+x+(\tfrac{1}{2}-\tfrac{3}{2}x)\ln x-\tfrac{1}{2}(1+x)\ln^2 x-(\tfrac{3}{2}\ln x+2\ln x\ln(1-x))p_{FF}(x)+2p_{FF}(-x)S_2(x)] \\ & +C_FC_G[\tfrac{14}{3}(1-x)+(\tfrac{11}{6}\ln x+\tfrac{1}{2}\ln^2 x+\tfrac{67}{18}-\tfrac{1}{6}\pi^2)p_{FF}(x)-p_{FF}(-x)S_2(x)] \\ & +C_FT_RN_F[-\tfrac{16}{3}+\tfrac{40}{3}x+(10x+\tfrac{16}{3}x^2+2)\ln x-\tfrac{112}{9}x^2+\tfrac{40}{9}x^{-1}-2(1+x)\ln^2 x-(\tfrac{10}{9}+\tfrac{2}{3}\ln x)p_{FF}(x)]\,,\end{aligned}$$

$$\begin{aligned}\dot{P}_{FG}^{(1,S)} = {} & C_F^2[-\tfrac{5}{2}-\tfrac{7}{2}x+(2+\tfrac{7}{2}x)\ln x+(-1+\tfrac{1}{2}x)\ln^2 x-2x\ln(1-x)+(-3\ln(1-x)-\ln^2(1-x))p_{FG}(x)] \\ & +C_FC_G[\tfrac{28}{9}+\tfrac{65}{18}x+\tfrac{44}{9}x^2+(-12-5x-\tfrac{8}{3}x^2)\ln x+(4+x)\ln^2 x+2x\ln(1-x)+(-2\ln x\ln(1-x) \\ & +\tfrac{1}{2}\ln^2 x+\tfrac{11}{3}\ln(1-x)+\ln^2(1-x)-\tfrac{1}{6}\pi^2+\tfrac{1}{2})p_{FG}(x)+p_{FG}(-x)S_2(x)] \\ & +C_FT_RN_F[-\tfrac{4}{3}x-(\tfrac{20}{9}+\tfrac{4}{3}\ln(1-x))p_{FG}(x)]\,,\end{aligned}$$

$$\begin{aligned}\dot{P}_{GF}^{(1,S)} = {} & C_FT_RN_F[4-9x+(-1+4x)\ln x+(-1+2x)\ln^2 x+4\ln(1-x) \\ & +(-4\ln x\ln(1-x)+4\ln x+2\ln^2 x-4\ln(1-x)+2\ln^2(1-x)-\tfrac{2}{3}\pi^2+10)p_{GF}(x)] \\ & +C_GT_RN_F[\tfrac{182}{9}+\tfrac{14}{9}x+\tfrac{40}{9}x^{-1}+(\tfrac{136}{3}x-\tfrac{38}{3})\ln x-4\ln(1-x)-(2+8x)\ln^2 x+(-\ln^2 x \\ & +\tfrac{44}{3}\ln x-2\ln^2(1-x)+4\ln(1-x)+\tfrac{1}{3}\pi^2-\tfrac{218}{9})p_{GF}(x)+2p_{GF}(-x)S_2(x)]\,,\end{aligned}$$

$$\begin{aligned}\dot{P}_{GG}^{(1,S)} = {} & C_FT_RN_F[-16+8x+\tfrac{20}{3}x^2+\tfrac{4}{3}x^{-1}+(-6-10x)\ln x+(-2-2x)\ln^2 x] \\ & +C_GT_RN_F[2-2x+\tfrac{26}{9}x^2-\tfrac{26}{9}x^{-1}-\tfrac{4}{3}(1+x)\ln x-\tfrac{20}{9}p_{GG}(x)] \\ & +C_G^2[\tfrac{27}{2}(1-x)+\tfrac{67}{9}(x^2-x^{-1})+(-\tfrac{25}{3}+\tfrac{11}{3}x-\tfrac{44}{3}x^2)\ln x+4(1+x)\ln^2 x+(\tfrac{67}{9}-4\ln x\ln(1-x) \\ & +\ln^2 x-\tfrac{1}{3}\pi^2)p_{GG}(x)+2p_{GG}(-x)S_2(x)]\,.\end{aligned}$$

$$S_2(x)\equiv\int_{(1+x)/x}^{1/(1+x)}\frac{dz}{z}\ln\left(\frac{1-z}{z}\right);\quad S_1(x)\equiv\int_0^{1-x}\frac{dz}{z}\ln(1-z)\,.$$

3) EVOLUTION EQUATIONS

PURE QUARK : NON-SINGLET (FLAVOUR).

$$q^i(x,Q^2) - \bar{q}^j(x,Q^2) \underset{Df}{=:} q^{ij}_{NS}(x,Q^2)$$

$$dq_{NS}(x,Q^2) = \frac{\alpha_s}{2\pi} P_{qq}(x) \otimes q(x,Q^2)\, d\ln p_\perp^2$$

$$d\ln p_\perp^2 = d\ln Q^2$$

$\otimes$: Mellin convolution:

$$[A \otimes B](x) = \int_0^1 dx_1 \int_0^1 dx_2\, \delta(x - x_1 x_2)\, A(x_1)\, B(x_2)$$

Transform:

$$M[A](N) \underset{Df.}{=} \int_0^1 dz\, z^{N-1} A(z)$$

$$M[A \otimes B](N) = M[A](N) \cdot M[B](N)$$

$$\boxed{\frac{dq_{NS}(x,Q^2)}{d\log Q^2} = \frac{\alpha_s}{2\pi} P_{qq}(x) \otimes q_{NS}(x,Q^2)}$$

Integro differential equation.

$$M[q_{NS}^{(x)}] = \hat{q}_{NS}(N) \qquad M[P_{qq}(x)] = \hat{P}_{qq}(N)$$

$$\frac{d\hat{q}_{NS}(N)}{d\log Q^2} = \frac{\alpha_s}{2\pi} \hat{P}_{qq}(N) \cdot \hat{q}_{NS}(N)$$

Ordinary differential equation.

SINGLET CASE:

$$\Sigma = \sum_{i=1}^{N_f} (q_i + \bar{q}_i) \quad , \quad G$$

gluon density

$$\frac{d}{d\log Q^2} \begin{pmatrix} \Sigma \\ G \end{pmatrix} (x, Q^2) = \frac{\alpha_s}{2\pi} \mathbb{P}(x) \otimes \begin{pmatrix} \Sigma \\ G \end{pmatrix} (x, Q^2)$$

$$\mathbb{P}(x) = \begin{pmatrix} P_{qq}(x) & 2N_f P_{qg}(x) \\ P_{gq}(x) & P_{gg}(x) \end{pmatrix} .$$

NEXT-TO-LEADING ORDER EVOLUTION EQUATIONS

DEFINE BASIC DENSITIES:

NS:
$$\hat{q}_i^- = q_i - \bar{q}_i$$
$$\hat{q}_i^+ = q_i + \bar{q}_i \qquad q^+ = \sum_{i=1}^{N_f} q_i^+ \equiv \Sigma$$
$$q_i^- := q_i^-$$
$$q_i^+ := \hat{q}_i^+ - \frac{1}{N_f} q^+$$

$$P_{NS}^{\pm} = P_{NS}^{(0)} + \frac{\alpha_s}{2\pi} P_{NS}^{(1),\pm} + \dots$$

$$\mathbb{P} = \mathbb{P}^{(0)} + \frac{\alpha_s}{2\pi} \mathbb{P}^{(1)} + \dots$$

$$\left[\mathbb{P}^{(0)}, \mathbb{P}^{(1)}\right] \neq 0 .$$

$$\frac{d}{d\log Q^2}\, q_i^- = \frac{\alpha_s}{2\pi}\, P^-(x,\alpha_s) \otimes q_i^-$$

$$\frac{d}{d\log Q^2}\, q_i^+ = \frac{\alpha_s}{2\pi}\, P^+(x,\alpha_s) \otimes q_i^+$$

$$\frac{d}{d\log Q^2} \begin{pmatrix} \Sigma \\ G \end{pmatrix} = \frac{\alpha_s}{2\pi}\, \mathbb{P}(x,\alpha_s) \otimes \begin{pmatrix} \Sigma \\ G \end{pmatrix}.$$

FACTORIZING THE INPUT DENSITIES AT Q_0^2:

DEFINE:

$$q_i^-(x,t) := E^-(x,t) \otimes q_i^-(x)$$

$$\begin{aligned} q_i^+(x,t) := {} & E^+(x,t) \otimes q_i^+(x) \\ & + \frac{1}{N_f}\left(E_{11}(x,t) - E^+(x,t)\right) \Sigma(x) \\ & + \frac{1}{N_f}\, E_{12}(x,t) \otimes G(x) \end{aligned}$$

$$\begin{pmatrix} \Sigma(x,t) \\ G(x,t) \end{pmatrix} = \mathbb{E}(x,t) \otimes \begin{pmatrix} \Sigma(x) \\ G(x) \end{pmatrix}$$

t DENOTES THE EVOLUTION VARIABLE:

$$t = -\frac{2}{\beta_0} \log\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)$$

$E^\pm$, $\mathbb{E}$ are the EVOLUTION OPERATORS, WHICH MAP AN INPUT $D(x,t=0)$ TO THE EVOLVED DENSITY $D(x,t)$.

WITH

$$\lim_{t\to 0} E^{\pm}(x,t) = \delta(1-x)$$

$$\lim_{t\to 0} \mathbb{E}(x,t) = \mathbb{1}\cdot\delta(1-x)$$

TRANSFORMATION OF THE MEASURE:

$$\frac{\alpha_s(Q^2)\, d\log Q^2}{2\pi} \underset{\mathrm{NLO}}{=} \left[1 - \frac{\beta_1}{2\beta_0}\frac{\alpha_s(Q^2)}{2\pi} + \dots\right] dt.$$

SINCE THE INPUT DENSITIES ARE ARBITRARY ONE MAY DERIVE EVOLUTION EQUS. FOR THE OPERATORS THEMSELVES:

$$\frac{d}{dt} E^{\pm}(x,t) = \left\{ P_{NS}^{(0)}(x) + \frac{\alpha(t)}{2\pi} R^{\pm}(x) + \dots \right\} \otimes E^{\pm}(x,t)$$

$$\frac{d}{dt} \mathbb{E}(x,t) = \left\{ \mathbb{P}^{(0)}(x) + \frac{\alpha(t)}{2\pi} \mathbb{R}(x) + \dots \right\} \otimes \mathbb{E}(x,t)$$

$$R^{\pm}(x) = P^{\pm,(1)}(x) - \frac{\beta_1}{2\beta_0} P_{NS}^{(0)}(x)$$

$$\mathbb{R}(x) = \mathbb{P}^{(1)}(x) - \frac{\beta_1}{2\beta_0} \mathbb{P}^{(0)}(x)$$

DEPENDS ON Λ_{QCD} ONLY

IN A QCD FIT THE DETERMINATION OF THE INPUT PARAMETERS OF $q_{NS,i}$, Σ, G IS ORTHOGONAL (BUT CORRELATED) TO THE DETERMINATION OF Λ_{QCD} OR $\alpha_s(M_Z)$.