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Master Integrals for Massive 2-Loop Bhabha Scattering

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Contributed to the Workshop on Bhabha Scattering
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A project in collaboration with
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Janusz Gluza DESY (and Katowice)

See also:
- **hep-ph/0412164, to appear in PRD (April 2005)**
- **http://www-zeuthen.desy.de/theory/research/bhabha/**

- **What do we need?** $\rightarrow 10^{-4}$ **for** $d\sigma/d\cos\vartheta$ **at small** ϑ $\rightarrow$ **talk by Mönig**
- **What are 'Scalar Master Integrals'?**
- **Status**

The Needs

Need a cross-section prediction with 5 significant digits.

Perturbative orders:

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

Kinematics:

$$\begin{aligned} \sqrt{s} &= 90 \ldots 1000 \text{ GeV} \\ \vartheta &= 26 \ldots 82 \text{ mrad} \\ \cos\vartheta &\sim 0.999\,66 \ldots 0.996\,64 \\ T &= \frac{s}{2}(1-\beta^2\cos\vartheta) > 1.36 \text{ GeV}|_{GigaZ},\ 42.2 \text{ GeV}|_{ILC500} \end{aligned}$$

Conclude:

- t-**channel exchange of** γ **dominates everything else**
- $m_e^2/s \ < \ m_e^2/T \ \leq \ 10^{-5} \ldots 10^{-7}$
- **Calculate: 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung**

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The 1-loop electroweak corrections (plus some leading higher order terms) are well-known, with rising technical precision, since about 1988/91.
Böhm, Denner, Hollik; Bardin, TR $\rightarrow$ **Fig. 2004 Lorca, TR**

2-loop Bhabha scattering: What to be done?

- **Calculate:**

$$\sigma = (2 \to 2) + (2 \to 3) + (2 \to 4)$$

$$\begin{aligned}\sigma = & \;|\textbf{Born} + \textbf{1-loop} + \textbf{2-loop}|^2 \\ & + |(\textbf{Born} + \textbf{1-}\gamma) + (\textbf{1-loop} + \textbf{1-}\gamma)|^2 \\ & + |(\textbf{Born} + \textbf{2-}\gamma)|^2\end{aligned}$$

- **Do not include:** $|\textbf{2-loop}|^2$
  $|(\textbf{1-loop} + \textbf{1-}\gamma)|^2$

Results: Numerical comparison in all $f\bar{f}$

Bhabha $e^-e^+ \to e^-e^+\ (\gamma)$ **at LC:** $\sqrt{s} = 500\ \mathrm{GeV},\ E_{\max}(\gamma_{\mathrm{soft}}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right]_{\mathrm{Born}}$ (pb)	$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
-0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	aITALC
-0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	*FeynArts*
-0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	aITALC
-0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	*FeynArts*
-0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
$+0.0$	0.59814 23072 50330 3	0.54667 71794 69423 1	aITALC
$+0.0$	0.59814 23072 50329 4	0.54667 71794 69421 8	*FeynArts*
$+0.0$	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
$+0.9$	$0.18916\ 03223\ 32270\ 6 \cdot 10^3$	$0.17292\ 83490\ 66507\ 2 \cdot 10^3$	aITALC
$+0.9$	$0.18916\ 03223\ 32270\ 6 \cdot 10^3$	$0.17292\ 83490\ 66508\ 0 \cdot 10^3$	*FeynArts*
$+0.9$	$0.18916\ 03223\ 31848\ 5 \cdot 10^3$	$0.17292\ 83490\ 61347\ 4 \cdot 10^3$	$m_e = 0$
$+0.9999$	$0.20842\ 90676\ 46142\ 9 \cdot 10^9$	$0.19140\ 17861\ 11341\ 6 \cdot 10^9$	aITALC
$+0.9999$	$0.20842\ 90676\ 46436\ 4 \cdot 10^9$	$0.19140\ 17861\ 11979\ 0 \cdot 10^9$	*FeynArts*

Great independent agreement up to 14 digits! : limit in double precision

Previous agreement with *FeynArts*: 11 digits hep-ph/0307132**, SANC: 10 digits** hep-ph/0207156

Thanks to T. Hahn, numbers supplied with *FeynArts* + *FormCalc* + *LoopTools*

Zinnowitz, 28/04/2004 A. Lorca —Automatization and width effects with aITALC

Status by end of 2004:

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP", hep-ph/0306083 [1]

- BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]
- NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2]

Conclude: The pure, nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calulated and integrated into the MC programs.

Beware:

$$m_e, m_\gamma, (d-4), E_\gamma$$

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The massive 2-loop contributions

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \to e^+e^-)$$

We are using a scheme with

(1) $m_e \neq 0$ **(good with the MC's)**

(2) $m_\gamma = 0$ **(bad with the MC's; $\to$ Mastrolia, Remiddi 2003)**

(3) dim.reg. for UV and IR divergences

Also:

M. Argeri, R. Bonciani, A. Ferroglia, P. Mastrolia, Remiddi, v.d.Bij: Same approach

A. Penin, Z. Bern, L. Dixon, A. Ghinculov, N. Glover, B. Tausk, J. v.d.Bij: Small m_e approach

Heinrich, Smirnov: Calculation of complicated Feynman integrals

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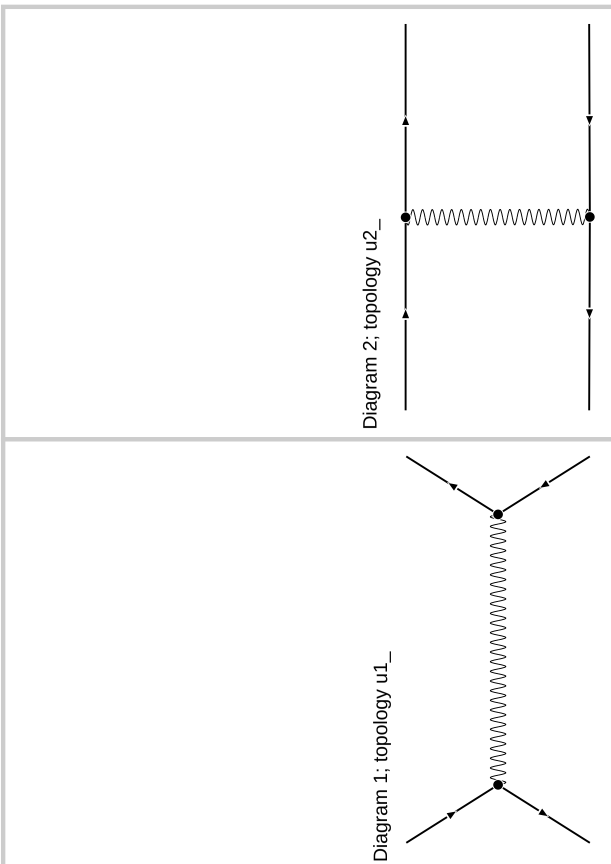
The massive diagrams (See also webpage)

Assume 3 leptonic flavors, do not look at loops in external legs.

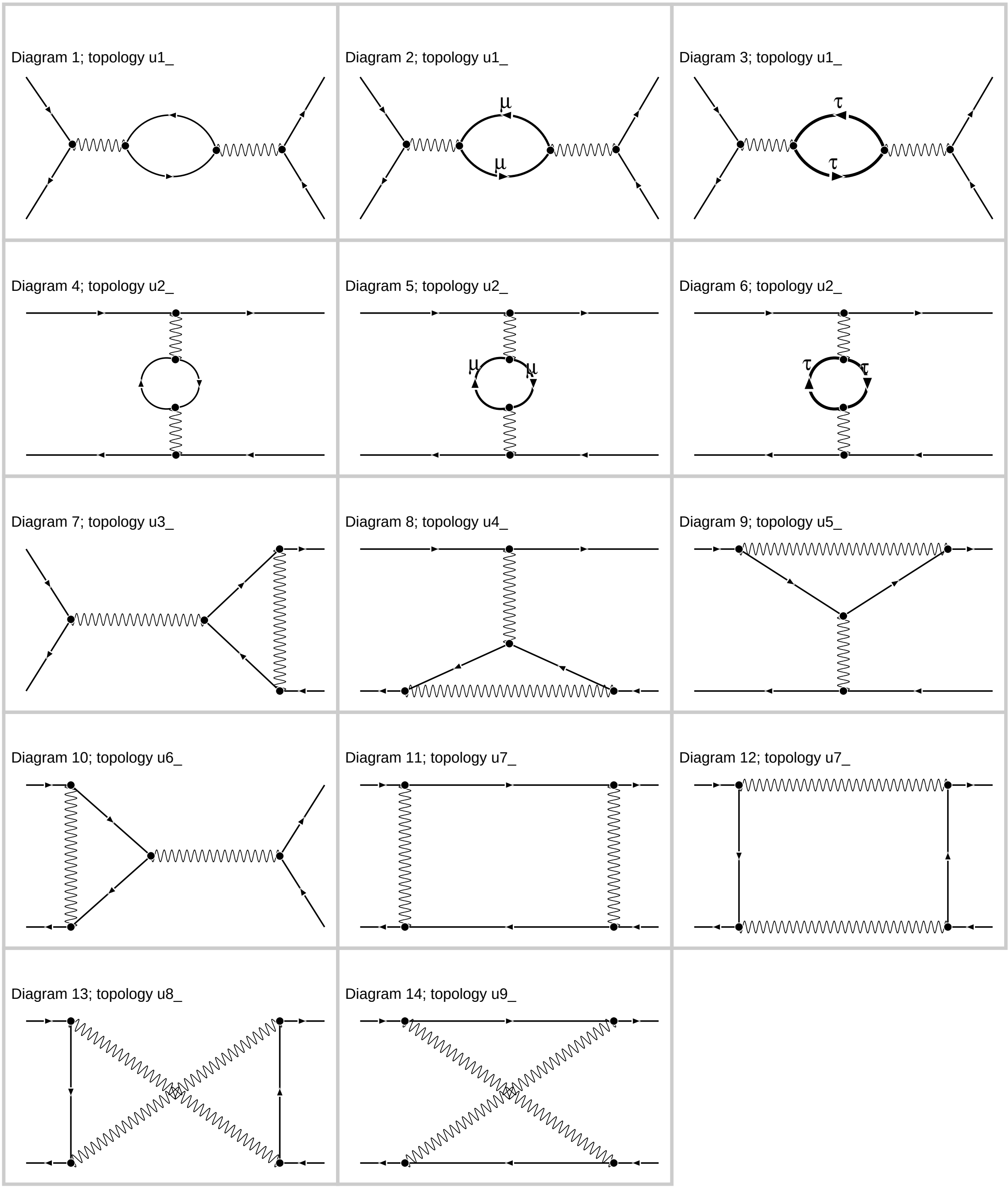
Not too many QED diagrams:

- **Born diagrams: 2**
- **1-loop diagrams: 14** — **interfere with Born and 1-loop**
- **2-loop diagrams: 154 (with 68 double-boxes)** — **interfere with Born only**

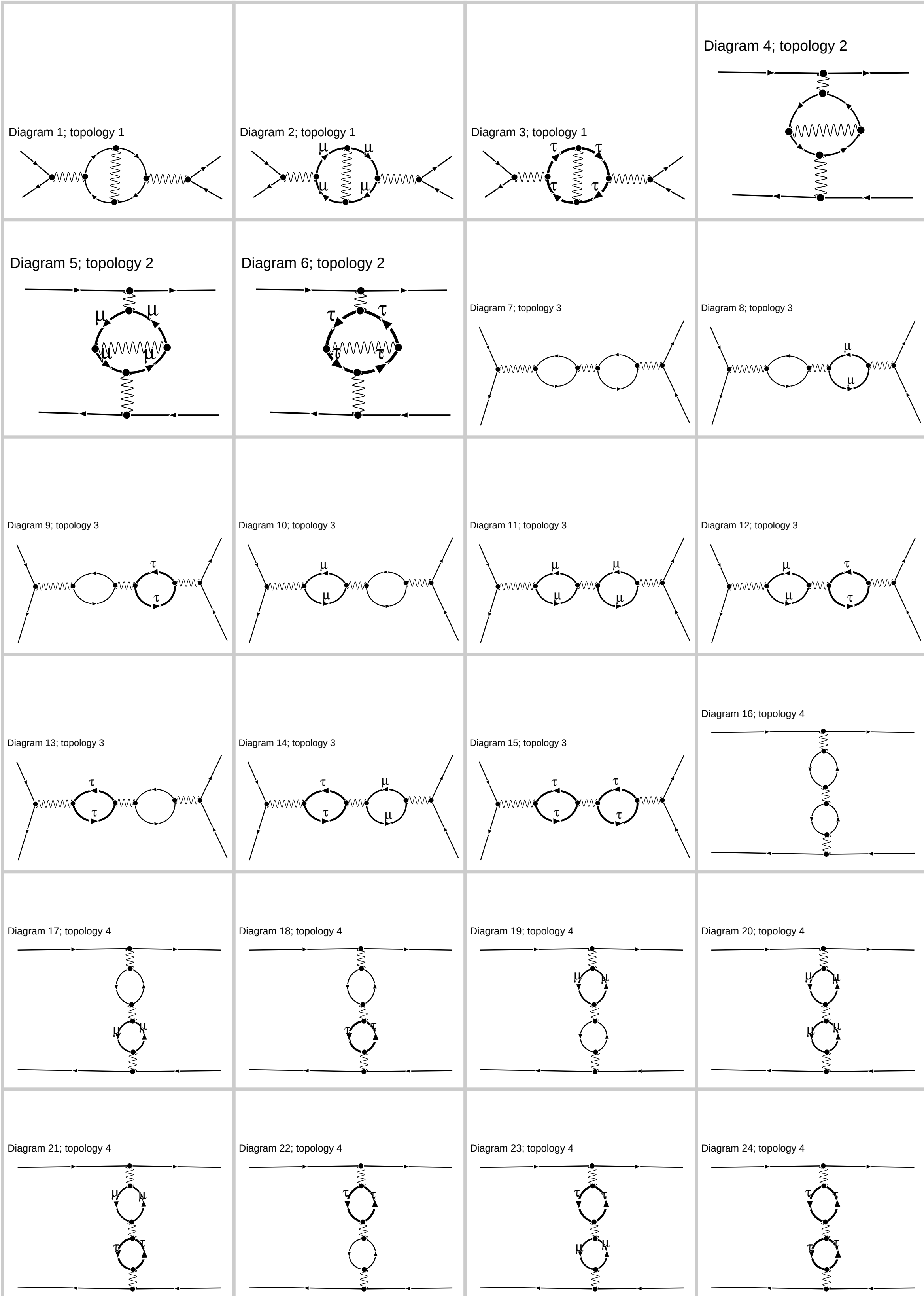


Born diagrams

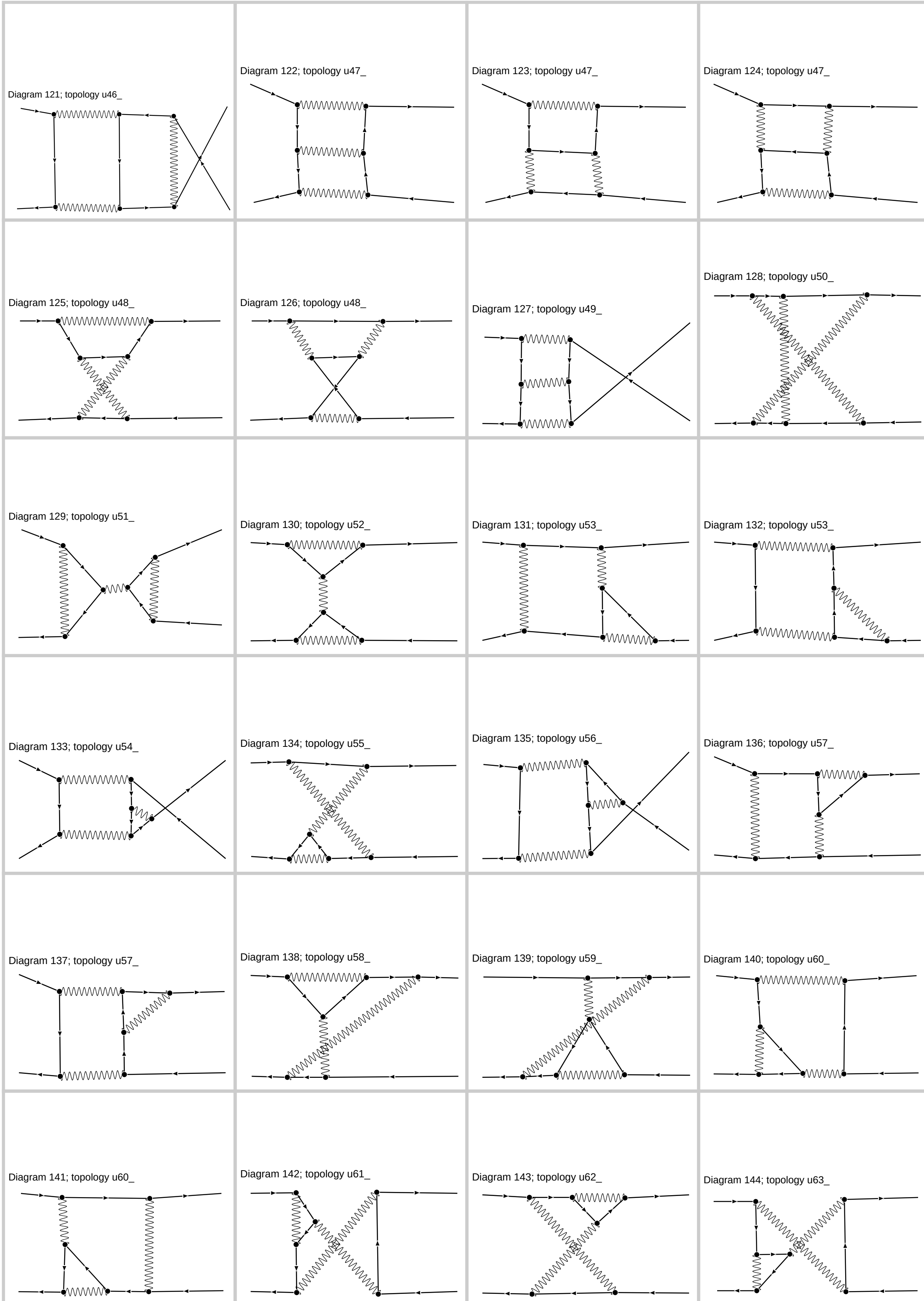
The one-loop diagrams



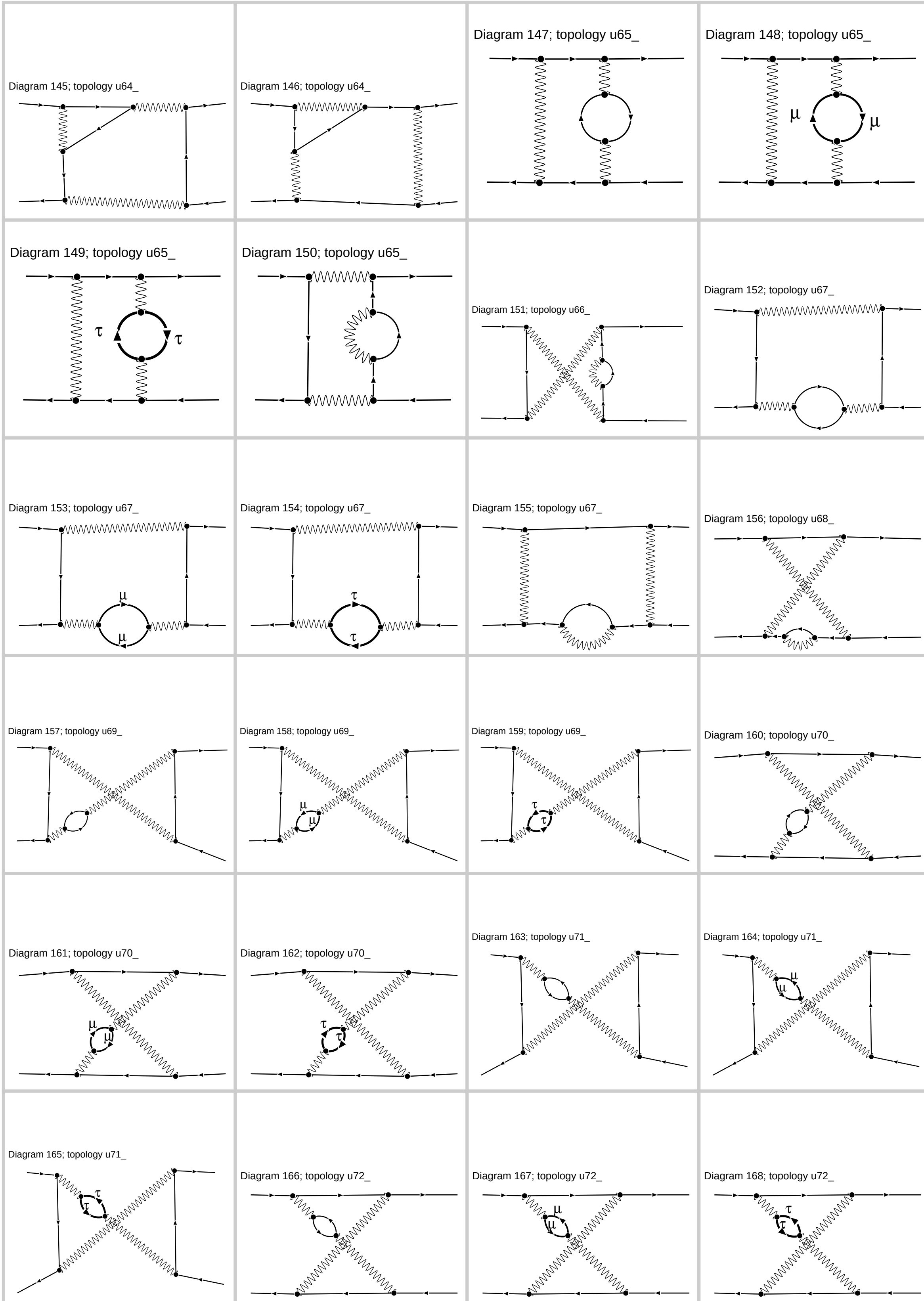
Reducible two-loop diagrams



Irreducible two-loop diagrams



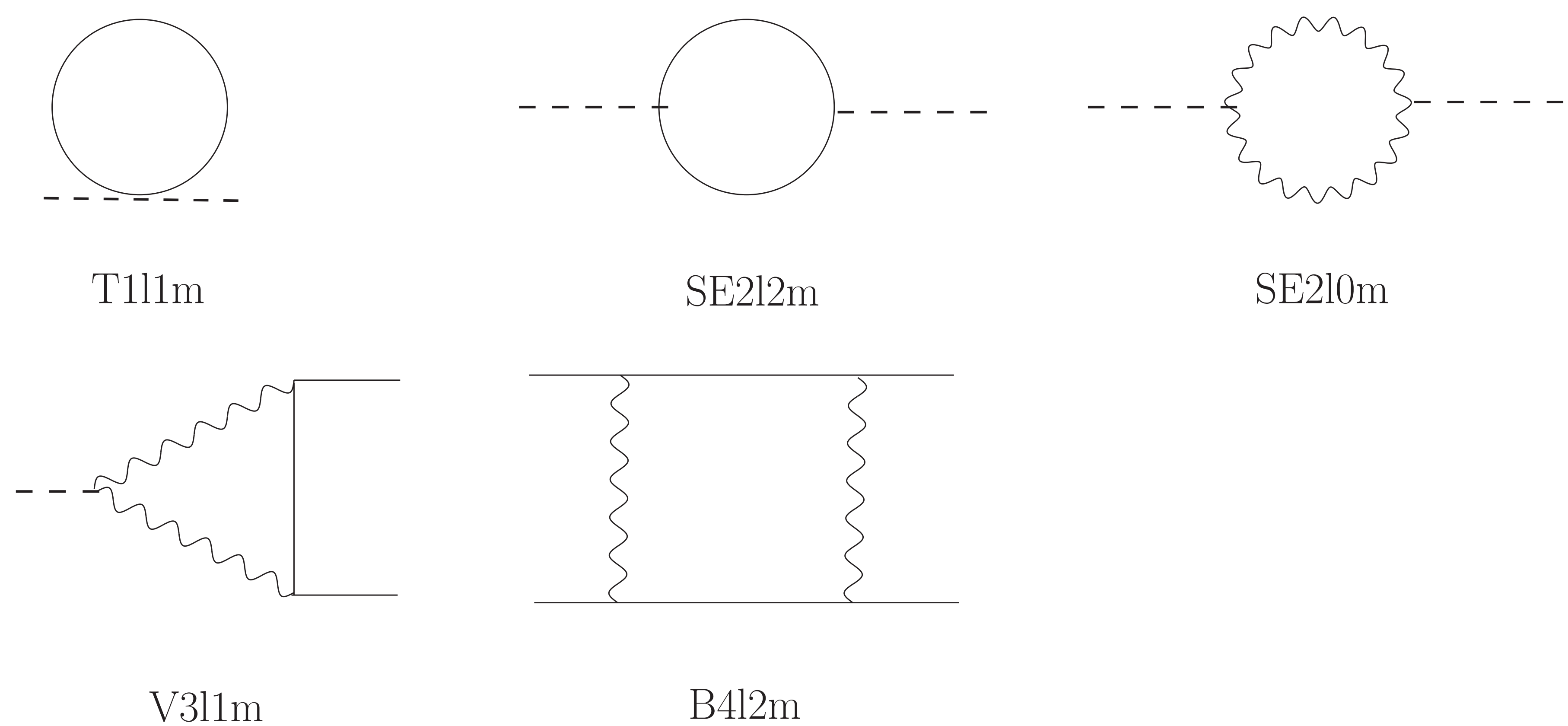
Irreducible two-loop diagrams



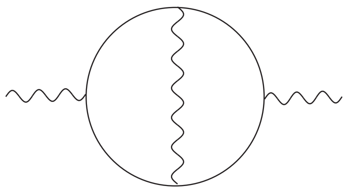
There are few topologies only:

- 1-loop: 5
- 2-loop self energies: 5 (3 for external legs)
- 2-loop vertices: 5
- 2-loop boxes: 6

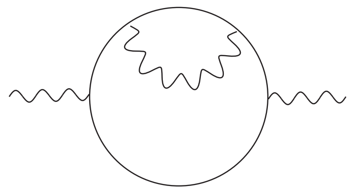
The many Feynman integrals may be reduced to 'few' master integrals by sophisticated methods (Remiddi-Laporta algorithm, 1996/2000).



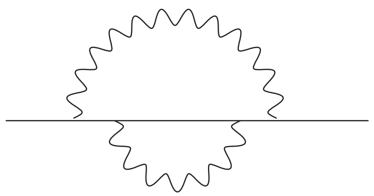
The one-loop prototypes



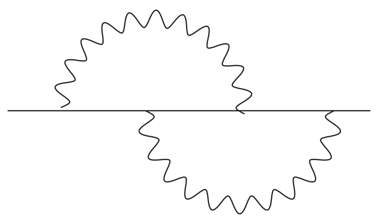
SE1



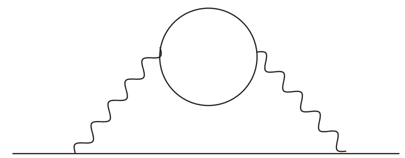
SE2



SE3



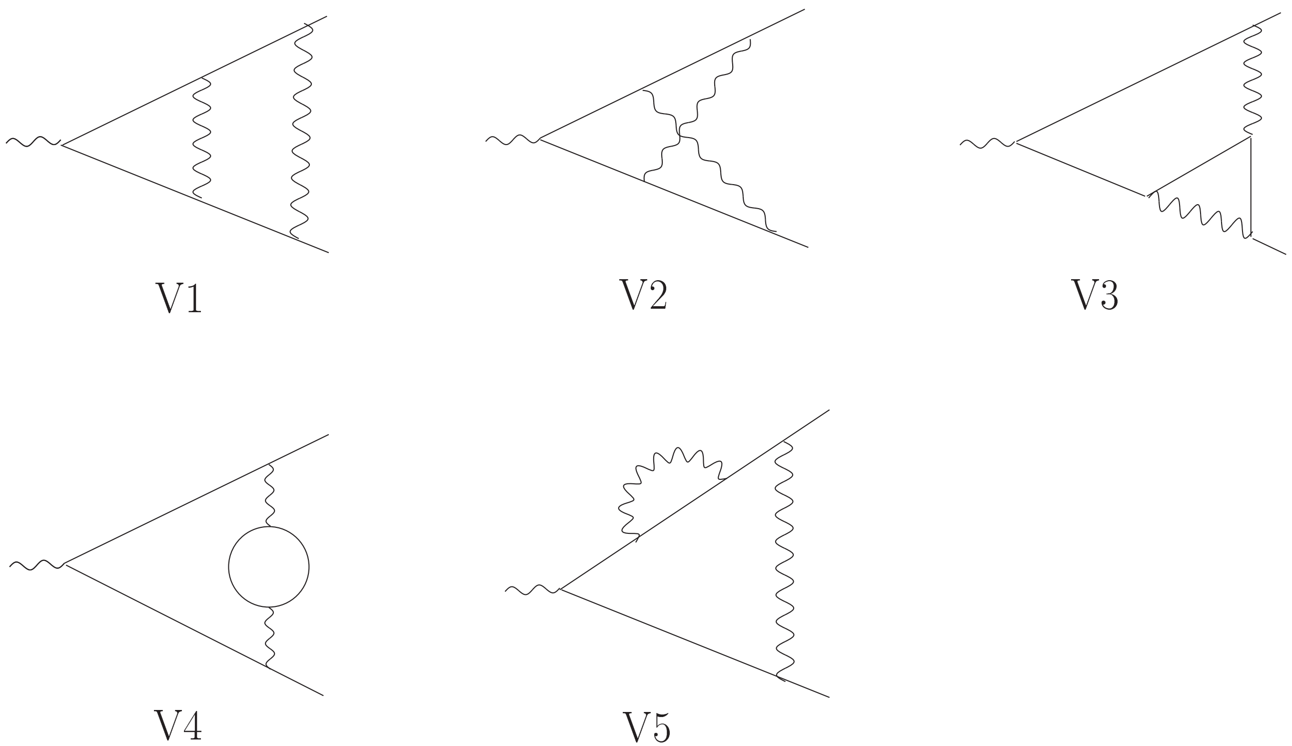
SE4



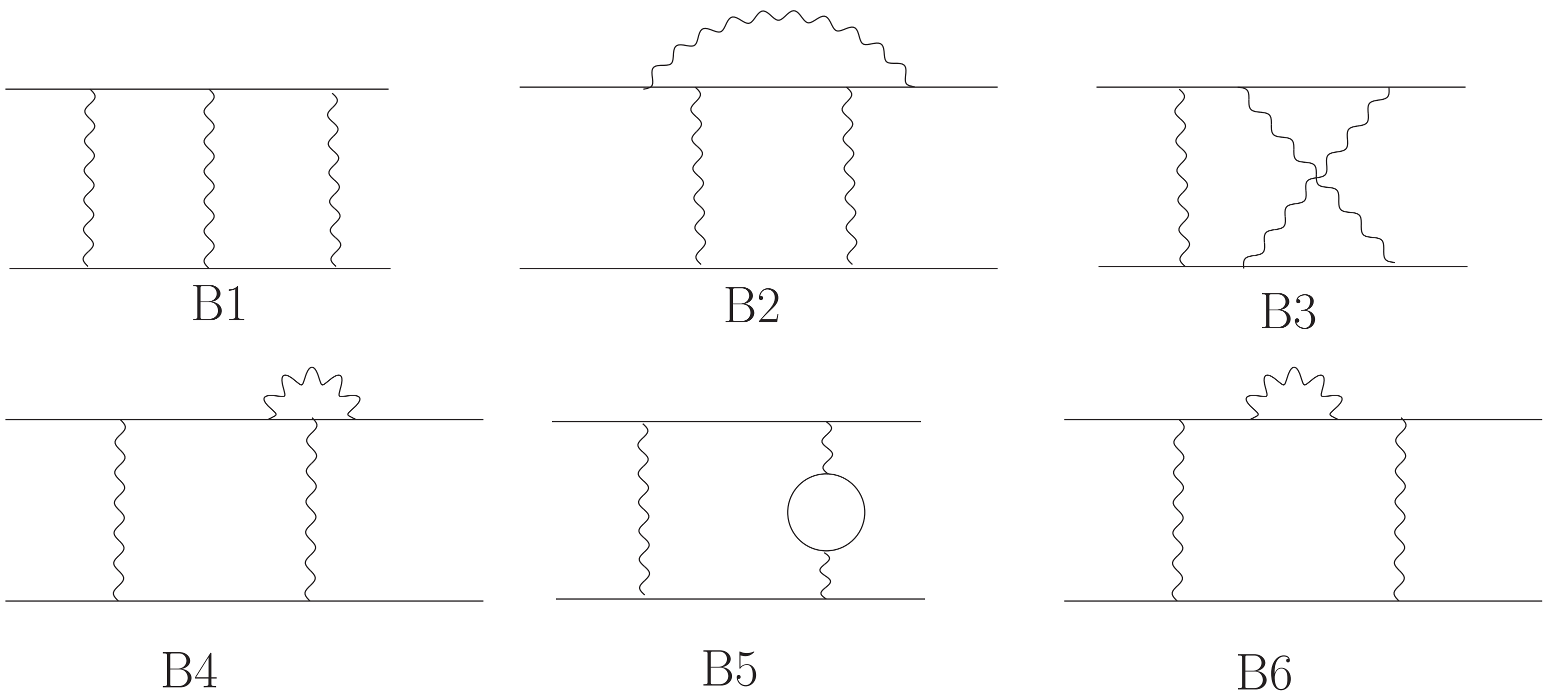
SE5

The two-loop self-energy diagrams for massive Bhabha scattering.

SE3 to SE5 are needed for the renormalization of external fermion lines.



The two-loop vertex diagrams for massive Bhabha scattering.



The two-loop box diagrams for massive Bhabha scattering.

One has to solve Feynman integrals with $L \leq 2$ loops and $N \leq 7$ internal lines:

$$G(X) = \frac{1}{\left(i\pi^{d/2}\right)^L} \int \frac{d^D k_1 \ldots d^D k_L \; X}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_j^2 - m_j^2)^{\nu_j} \ldots (q_N^2 - m_N^2)^{\nu_N}},$$

$$X = 1, k_1 P, k_1 k_2, k_2 P, \cdots$$

A completely numerical approach might also be possible Passarino 2004.

For checks in the Euclidean region this has been proven to be a powerful tool Binoth, Heinrich 2000/03

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A table of master integrals

So far, we mainly use the Laporta/Remiddi algorithm:
Derive with integration-by-parts identities a system of *algebraic* equations for the Feynman integrals and solve the system.

MC,JG,TR (→ tables) , with IdSolver (M. Czakon, unpublished):

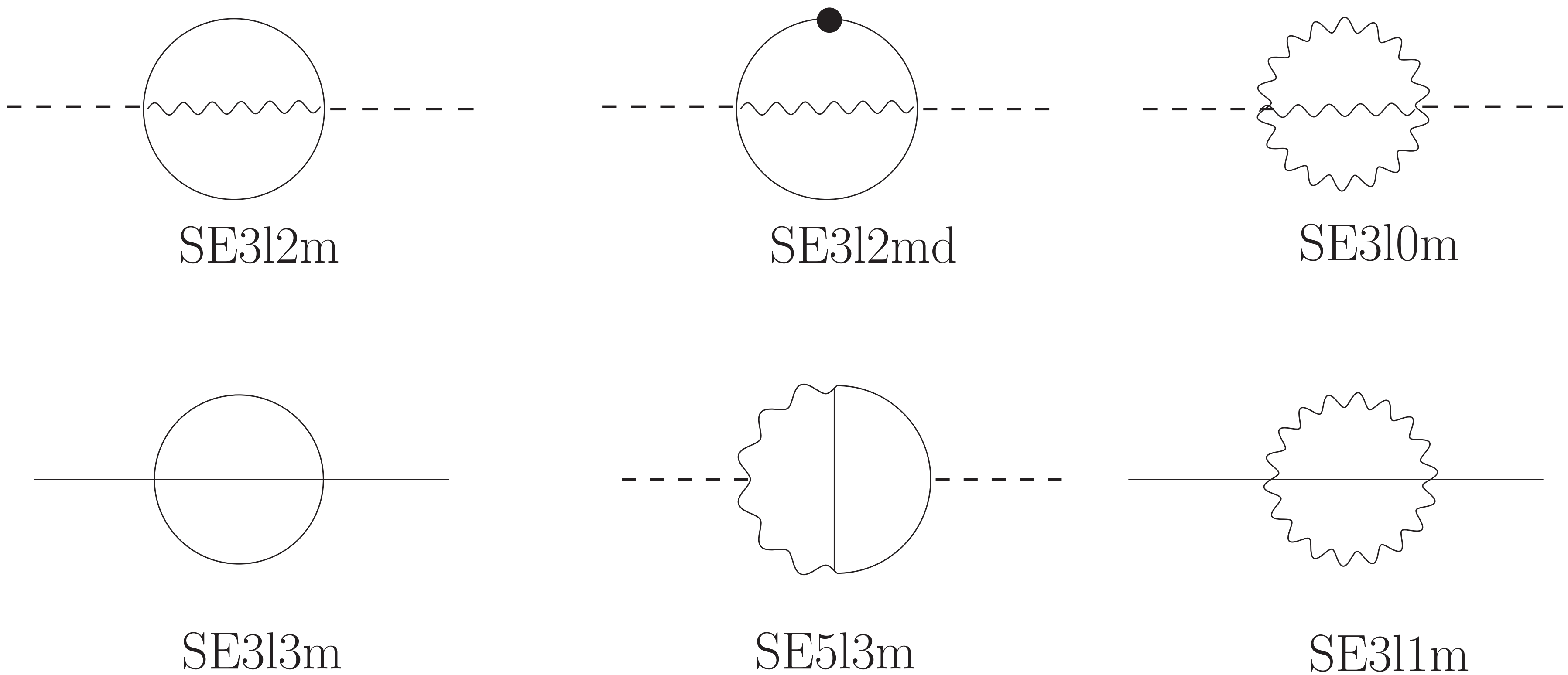
- **1-loop: 5 masters (all known)**
- **2-loop self energies: 6 masters (all known)**
- **2-loop vertices: 19 masters (all known)**
- **2-loop boxes: 33 masters → (5 published, maybe more known)**

The calculation of the master integrals is mainly done with two methods:

- **derive and solve (systems of) differential equations (with boundary conditions)**
- **derive and solve (up to 8-dimensional) Mellin-Barnes integral representations for single Feynman integrals (Heinrich, Smirnov)**

The simplest diagram is the tadpole:

$$
\begin{aligned}
\texttt{T1l1m} &= \frac{e^{\epsilon\gamma_E}}{i\pi^{D/2}} \int \frac{d^D q}{q^2-1} \\
&= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right)\epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right)\epsilon^2 + \ldots
\end{aligned}
$$



The six two-loop 2-point MIs. External solid (dashed) lines describe on (off) -shell momenta.

A class of scalar self-energy integrals:

$$\mathtt{SE3l2m}(a,b,c,d) = -\frac{e^{2\epsilon\gamma_E}}{\pi^D}\int \frac{d^Dk_1 d^Dk_2 (k_1k_2)^{-d}}{[(k_1+k_2-p)^2-m^2]^b [k_1^2]^a [k_2^2-m^2]^c}.$$

The two Master Integrals are:

$$\mathtt{SE3l2m} = \mathtt{SE3l2m}(1,1,1,0)$$

$$\mathtt{SE3l2md} = \mathtt{SE3l2m}(1,1,2,0)$$

In Bonciani et al. 2003 it is used instead as a master integral:

$$\mathtt{SE3l2mN} = \mathtt{SE3l2m}(1,1,1,-1)$$

By an algebraic relation, valid for $m^2=1$ and $p^2=s$,

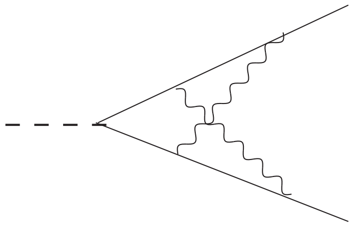
$$\mathtt{SE3l2md} = \frac{-(1+s)+\epsilon(2+s)}{s-4}\,\mathtt{SE3l2m} + \frac{2(1-\epsilon)}{s-4}\left(\mathtt{T1l1m}^2 + 3\,\mathtt{SE3l2mN}\right),$$

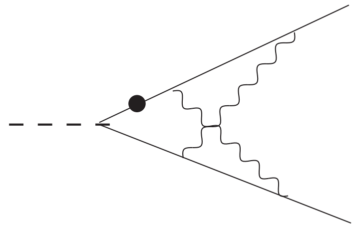
one may derive then SE3l2md.

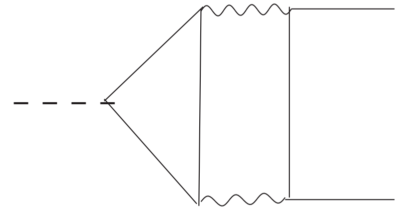
Because the integral SE3l2md is one of our masters, we reproduce it here explicitely:

$$\mathtt{SE3l2md}(x) = \frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}-\left(\frac{1-\zeta_2}{2}+\frac{1+x}{1-x}H[0,x]+\frac{1+x^2}{(1-x)^2}H[0,0,x]\right)+\mathcal{O}(\epsilon).$$

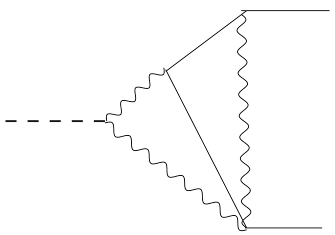
At our webpage, there is a file with all the master integrals we have determined so far.

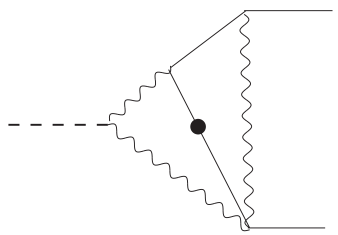

V6l4m1

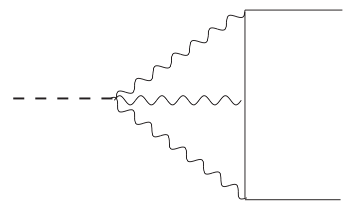

V6l4m1d

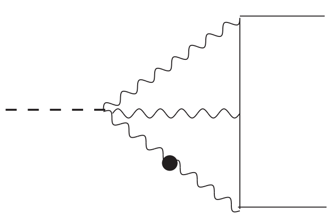

V6l4m2

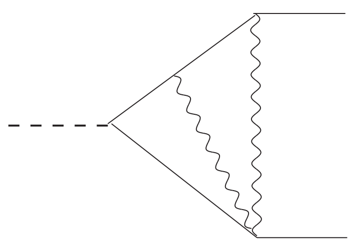
The three two-loop vertex MIs with six internal lines.

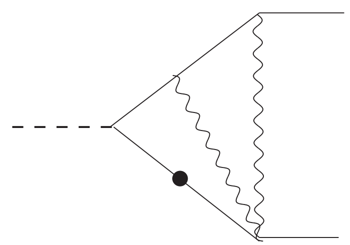

V5l2m1


V5l2m1d

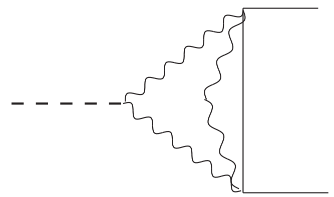

V5l2m2


V5l2m2d

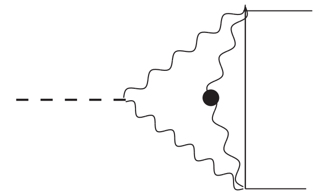

V5l3m


V5l3md

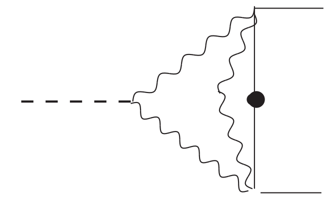
The six two-loop vertex MIs with five internal lines.



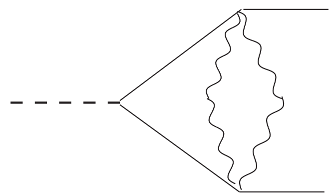
V4l1m1



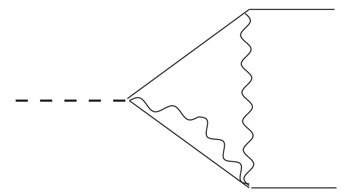
V4l1m1d1



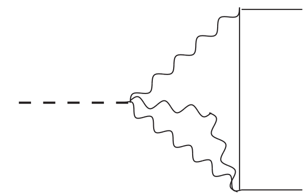
V4l1m1d2



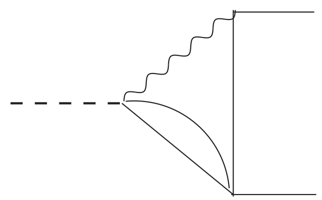
V4l2m1



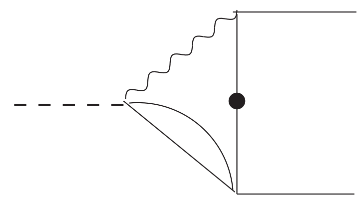
V4l2m2



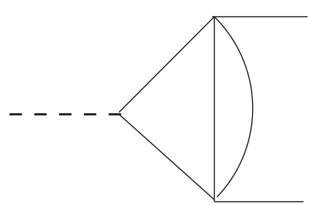
V4l1m2



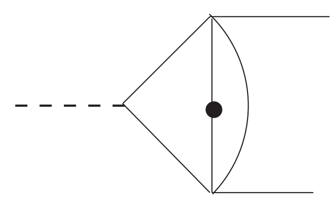
V4l3m



V4l3md

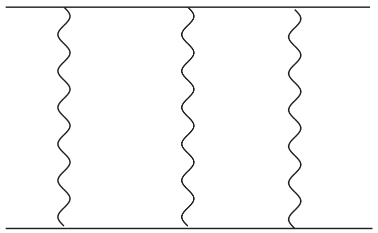


V4l4m

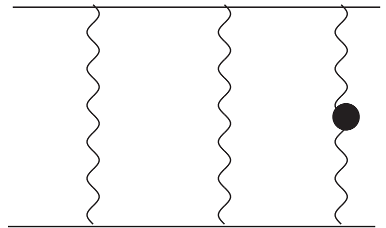


V4l4md

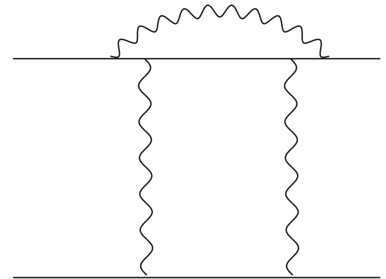
The ten two-loop vertex MIs with four internal lines.



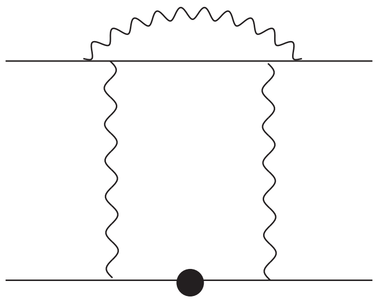
B7l4m1



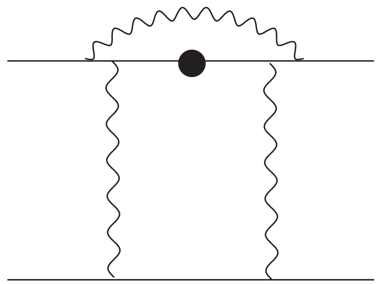
B7l4m1d



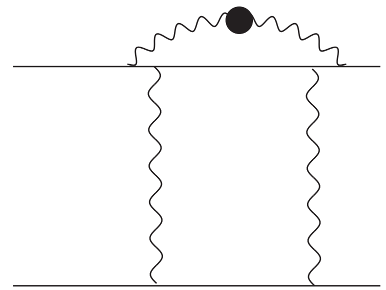
B7l4m2



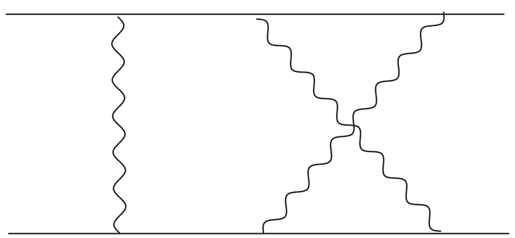
B7l4m2d1



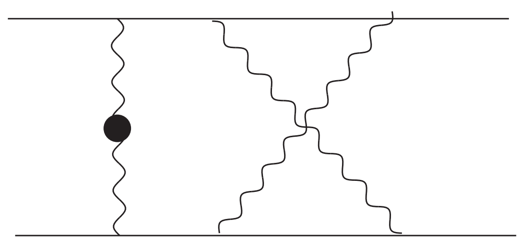
B7l4m2d2



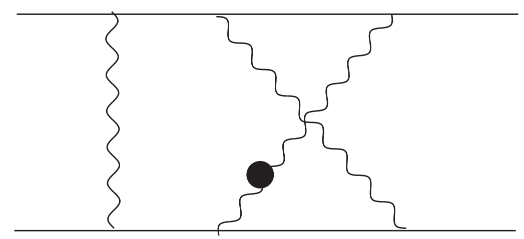
B7l4m2d3



B7l4m3

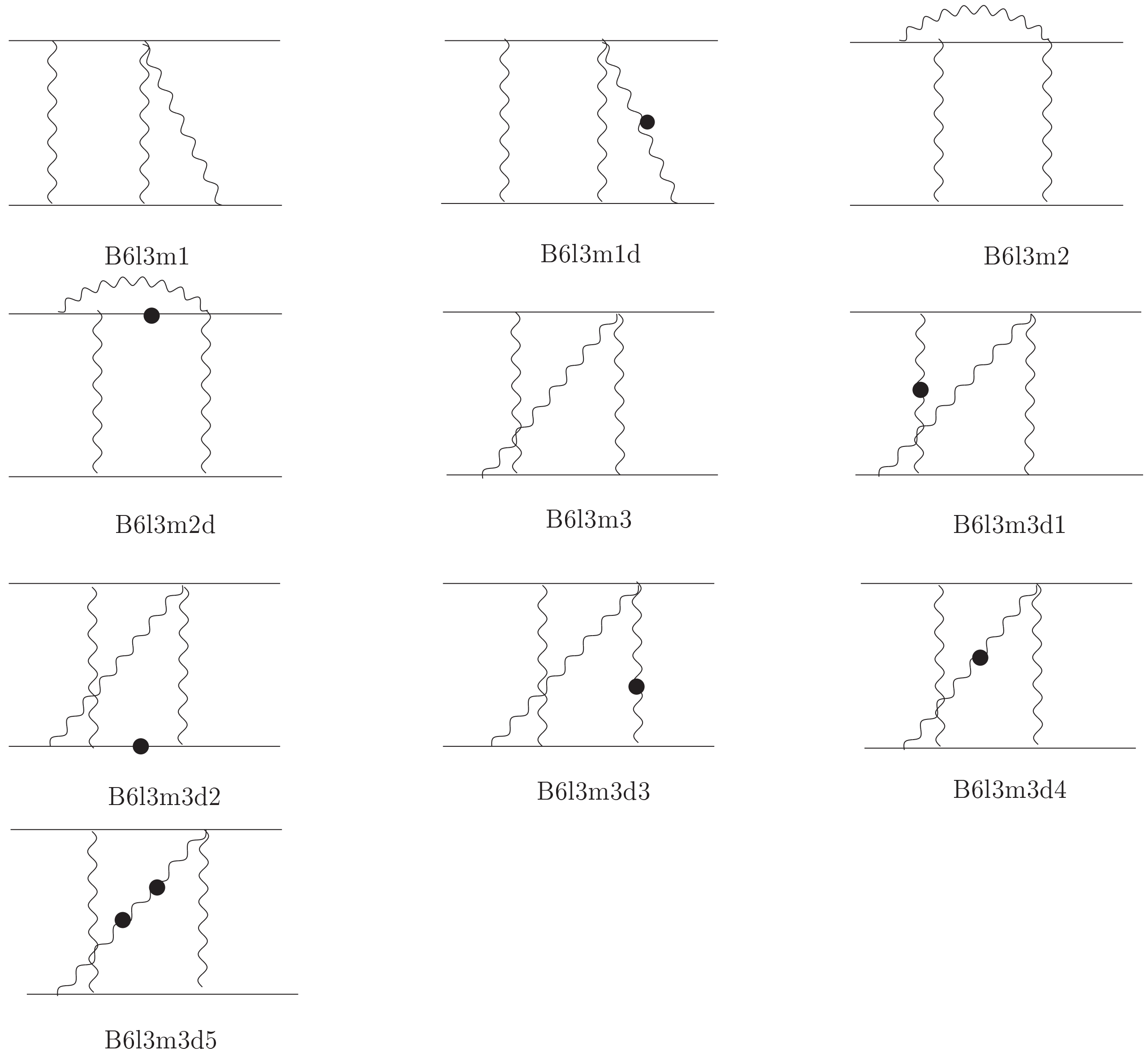


B7l4lm3d1

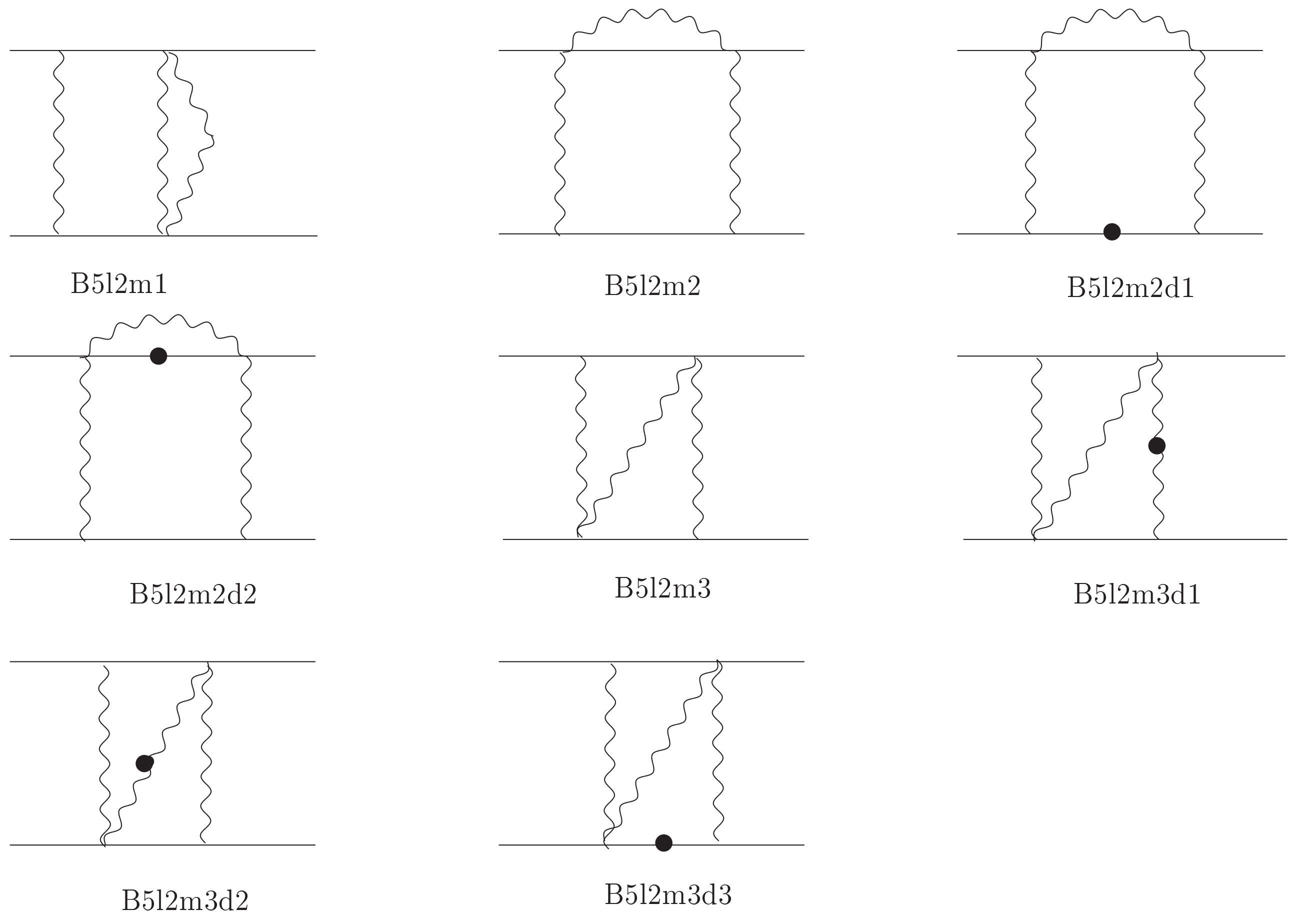


B7l4m3d2

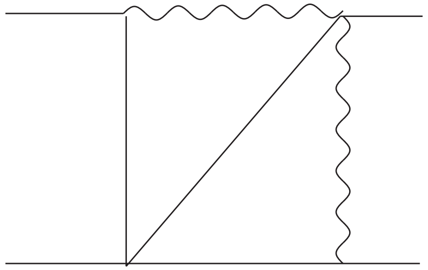
The nine two-loop box MIs with seven internal lines.



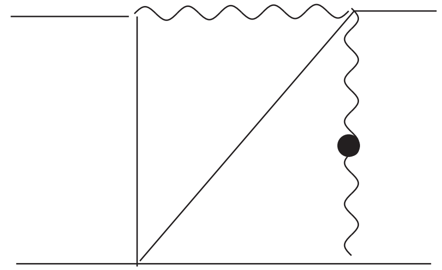
The ten two-loop box MIs with six internal lines.



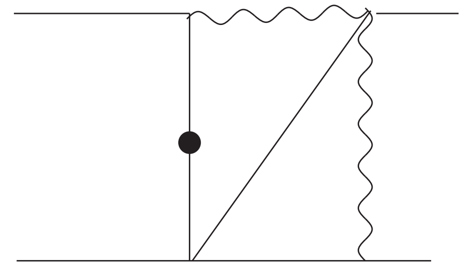
The 14 two-loop box MIs with five internal lines, 1/2.



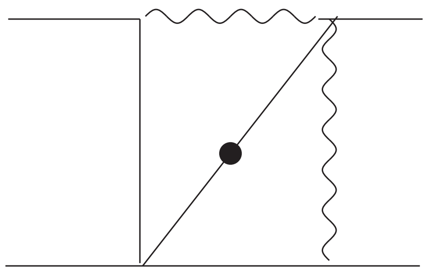
B5l3m



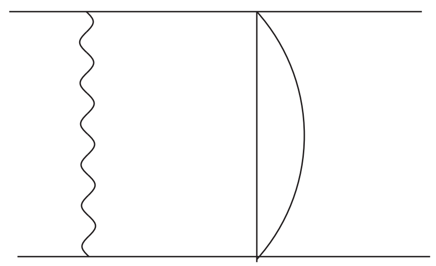
B5l3md1



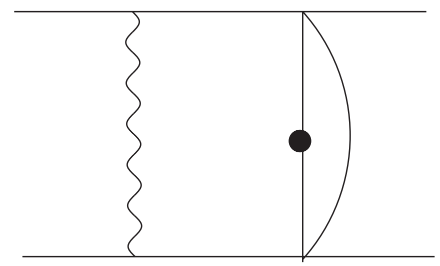
B5l3md2



B5l3md3

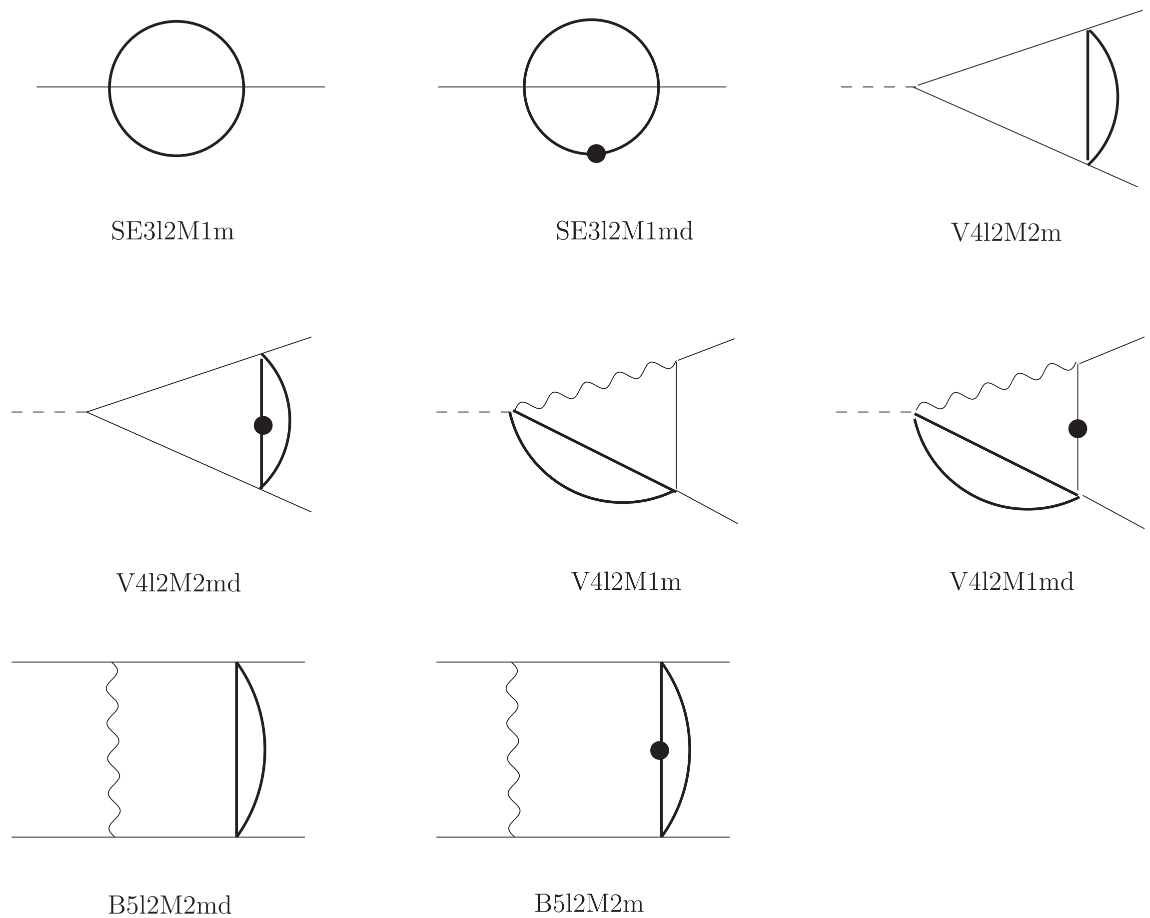


B5l4m



B5l4md

The 14 two-loop box MIs with five internal lines, 2/2.



The eight additional master integrals with two different mass scales.

Table 1: The number of (two-loop + one-loop) master integrals needed to calculate the two-loop vertex diagrams and box diagrams

Diagram	V1	V2	V3	V4	V5	B1	B2	B3	B4	B5	B6
tadpole MI	0+1	0+1	0+1	0+1	0+1	0+1	0+1	0+1	0+1	0+1	0+1
2-point MIs	3+1	4+1	4+1	1+1	3+1	4+1	5+2	5+1	4+2	3+2	3+2
3-point MIs	4+0	10+0	5+0	2+0	1+0	7+0	11+1	13+0	10+1	4+1	4+1
Box type MIs						9+0	15+0	22+0	11+1	2+1	3+1
total	7+2	14+2	9+2	3+2	4+2	20+2	31+4	40+2	25+5	9+5	10+5
net	9	16	11	5	6	22	35	42	30	14	15

Table 2: The one-loop and two-loop MIs needed for the evaluation of the self-energy diagrams

MI	SE1	SE2	SE3	SE4	SE5	ref.
T1l1m*	+	+	+	+	+	'tHooft:1972fi
SE2l2m*	+	+	−	−	−	'tHooft:1972fi
SE3l1m	−	−	+	−	−	oms:Fleischer:1999tu
SE3l2m	+	+	−	−	−	Fleischer:1999hp,Bonciani:2003te
SE3l2md	+	+	−	−	−	Fleischer:1999hp,CzGR
SE3l3m	−	−	−	+	+	oms:Broadhurst:1990ei,Broadhurst:1991fi,Fleischer:1999tu

Table 3: The 1-,2-,and 3-point MIs entering 2-loop vertex and box diagrams. Stars: 1-loop MIs.

MI	B1	B2	B3	B4	B5	B6	V1	V2	V3	V4	V5	ref.
T1l1m*	+	+	+	+	+	+	+	+	+	+	+	'tHooft:1972fi
SE2l0m*	−	+	−	+	+	+	−	−	−	−	−	'tHooft:1972fi
SE2l2m*	+	+	+	+	+	+	+	+	+	+	+	'tHooft:1972fi
SE3l0m	+	−	+	−	−	−	−	−	−	−	−	'tHooft:1972fi,Surguladze:1989ez,Bonciani:2003hc
SE3l1m	+	+	+	+	−	+	+	+	+	−	+	oms:Fleischer:1999tu
SE3l2m	+	+	+	+	+	+	+	+	+	−	+	Fleischer:1999hp,Bonciani:2003te
SE3l2md	+	+	+	+	+	+	+	+	+	−	+	Fleischer:1999hp,Sec. ??
SE3l3m	−	+	+	+	+	−	−	+	+	+	−	oms: Broadhurst:1990ei,Broadhurst:1991fi,Fleischer:1999tu
SE5l3m	−	+	−	−	−	−	−	−	−	−	−	Broadhurst:1990ei,Fleischer:1998nb,Davydychev:2003mv,Bonciani:2003hc
V3l1m*	−	+	−	+	+	+	−	−	−	−	−	'tHooft:1972fi
V4l1m1	−	+	−	+	−	+	−	−	−	−	−	Bonciani:2003hc
V4l1m1[d1--d2]	−	+	−	+	−	+	−	−	−	−	−	Bonciani:2003hc
V4l1m2	+	−	+	−	−	−	−	−	−	−	−	Bonciani:2003hc
V4l2m1	+	−	+	−	−	−	+	+	−	−	−	Bonciani:2003hc
V4l2m2	+	+	+	+	−	+	+	+	+	−	+	Bonciani:2003te
V4l3m	−	+	+	+	+	−	−	+	+	−	−	Bonciani:2003te
V4l3md	−	+	+	+	+	−	−	+	+	−	−	Bonciani:2003te
V4l4m	−	+	+	+	+	−	−	+	+	+	−	Bonciani:2003te
V4l4md	−	+	+	+	+	−	−	+	+	+	−	Bonciani:2003te
V5l2m1	−	+	−	+	−	−	−	−	−	−	−	Czakon:2004tg
V5l2m1d	−	+	−	+	−	−	−	−	−	−	−	Czakon:2004tg
V5l2m2	+	−	+	−	−	−	−	−	−	−	−	Czakon:2004tg
V5l2m2d	+	−	+	−	−	−	−	−	−	−	−	Czakon:2004tg
V5l3m	+	−	+	−	−	−	+	+	−	−	−	Bonciani:2003te
V5l3md	+	−	+	−	−	−	+	+	−	−	−	Bonciani:2003te
V6l4m1	−	−	+	−	−	−	−	+	−	−	−	Bonciani:2003te
V6l4m1d	−	−	+	−	−	−	−	+	−	−	−	Bonciani:2003te
V6l4m2	−	+	−	−	−	−	−	−	−	−	−	Bonciani:2003hc
total = 25+4*	11+2*	16+4*	16+4*	14+4*	7+4*	7+4*	7+2*	14+2*	9+2*	3+2*	4+2*	

4-point MIs entering basic two-loop box diagrams in Figure 17. An asterisk denotes one-loop MI. MIs with a dagger are not known completely analytically; see text.

MI	B1	B2	B3	B4	B5	B6	ref.
`B714m1`	+	−	−	−	−	−	Smirnov:2001cm
`B714m1N`	+	−	−	−	−	−	Heinrich:2004iq
`B714m2`	−	+	−	−	−	−	Heinrich:2004iq†
`B714m2[d1--d3]`	−	+	−	−	−	−	
`B714m3`	−	−	+	−	−	−	Heinrich:2004iq†
`B714m3[d1--d2]`	−	−	+	−	−	−	
`B613m1`	+	−	+	−	−	−	
`B613m1d`	+	−	+	−	−	−	
`B613m2`	−	+	−	+	−	−	
`B613m2d`	−	+	−	+	−	−	
`B613m3`	−	−	+	−	−	−	
`B613m3[d1--d5]`	−	−	+	−	−	−	
`B512m1`	+	−	+	−	−	−	Czakon:2004tg
`B512m2`	−	+	−	+	−	+	Sec. **??**†
`B512m2[d1--d2]`	−	+	−	+	−	+	Sec. **??**†
`B512m3`	+	−	+	−	−	−	
`B512m3[d1--d3]`	+	−	+	−	−	−	Sec. **??**†
`B513m`	−	+	+	+	−	−	
`B513m[d1--d3]`	−	+	+	+	−	−	
`B514m`	−	+	+	+	+	−	Bonciani:2003cj
`B514md`	−	+	+	+	+	−	Sec. **??**
`B412m`*	−	−	−	+	+	+	'tHooft:1972fi,Bonciani:2003cj
total = 33+1*	9	15	22	11+1*	2+1*	3+1*	

The two-loop master integrals for several flavors. An asterisk denotes a one-loop MI.

MI	SE5f	V4f	B5f	ref.
`T1l1m`*	+	+	+	'tHooft:1972fi
`SE2l2m`*	−	+	+	'tHooft:1972fi
`SE2l0m`*	−	−	+	'tHooft:1972fi
`SE3l1m`	−	−	+	oms:Fleischer:1999tu
`SE3l2m`	−	−	+	Fleischer:1999hp,Bonciani:2003te
`SE3l2md`	−	−	+	Fleischer:1999hp,Sec. **??**
`SE3l2M1m`	+	+	+	oms:Argeri:2002wz
`SE3l2M1md`	+	+	+	oms:Argeri:2002wz
`V3l1m`*	−	−	+	'tHooft:1972fi
`V4l2M1m`	−	−	+	
`V4l2M1md`	−	−	+	
`V4l2M2m`	−	+	+	
`V4l2M2md`	−	+	+	
`B4l2m`*	−	−	+	'tHooft:1972fi,Bonciani:2003cj
`B5l2M2m`	−	−	+	
`B5l2M2md`	−	−	+	

.

Summary

Finally: The approach with extrapolations from $m_e = m_\gamma = 0$ was successful
Penin, Bern, Dixon, Ghinculov, Glover, Tausk, van der Bij.

Recent essential progress for the massive master integral determination:

- A complete list of masters
- Determination of many masters in a systematic approach
  (use Harmonic Polylogarithms, introduced by Remiddi, Vermaseren, plus ...)

So far:
Not for all technical problems it is completely understood how to solve them in an operational manner.

It seems to be possible to do the complete massive 2-loop calculation with present computers and human resources.

Offer by drinking-a-beer Frits Berends, NL, in 1989

Okey, I need

One million Swiss francs

and

five – or ten? – Russians

and I will do this

Since then (i.e. in 15 years), there are few more than a dozen articles in spires with "Bhabha . . . two loop" in the title.

And Frits Berends retired quite recently.