.

Fermionic NNLO contributions to Bhabha Scattering

Tord Riemann, DESY, Zeuthen

based on work with:

S. Actis (DESY), M. Czakon (U. Würzburg)

and J. Gluza (Sileasian U. Katowice)

Matter to the Deepest, Ustron, Poland, 8 Sep. 2007

See also: • http://www-zeuthen.desy.de/theory/research/bhabha/

and • hep-ph/0412164,0604101 (Massive Bhabha 2-loop masters)

and • hep-ph/0609051,0704.2400 (Fermionic 2-loop corrections)

- Introduction: Two-loop corrections to Bhabha Scattering
- Heavy leptonic contributions ACGR, [arXiv:0704.2400, hep-ph], to appear in NPB
- New: hadronic contributions AGR, nearly final
- Summary

I do not cover . . .

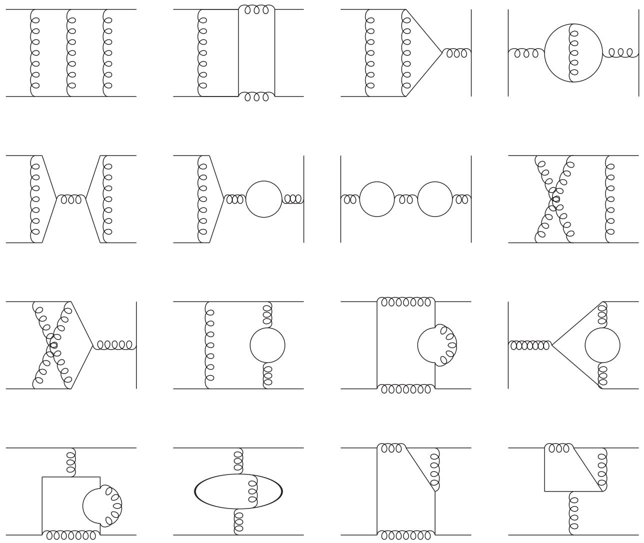
- Precision Monte-Carlos:
  $\longrightarrow$ talk by Guido Montagna
- Radiative loop corrections (with pentagon diagrams)
  $\longrightarrow$ talk by Krzysztov Kajda

$m = 0$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

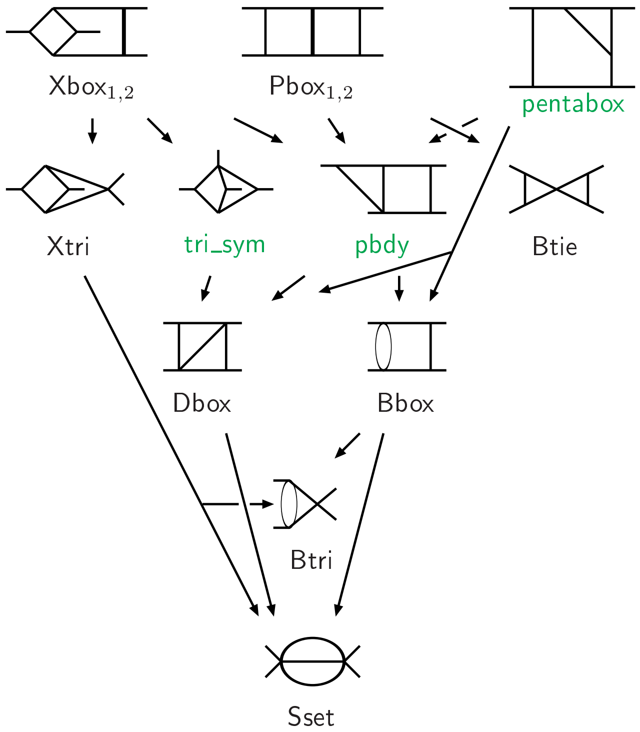
There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



In this calculation all particles massless.

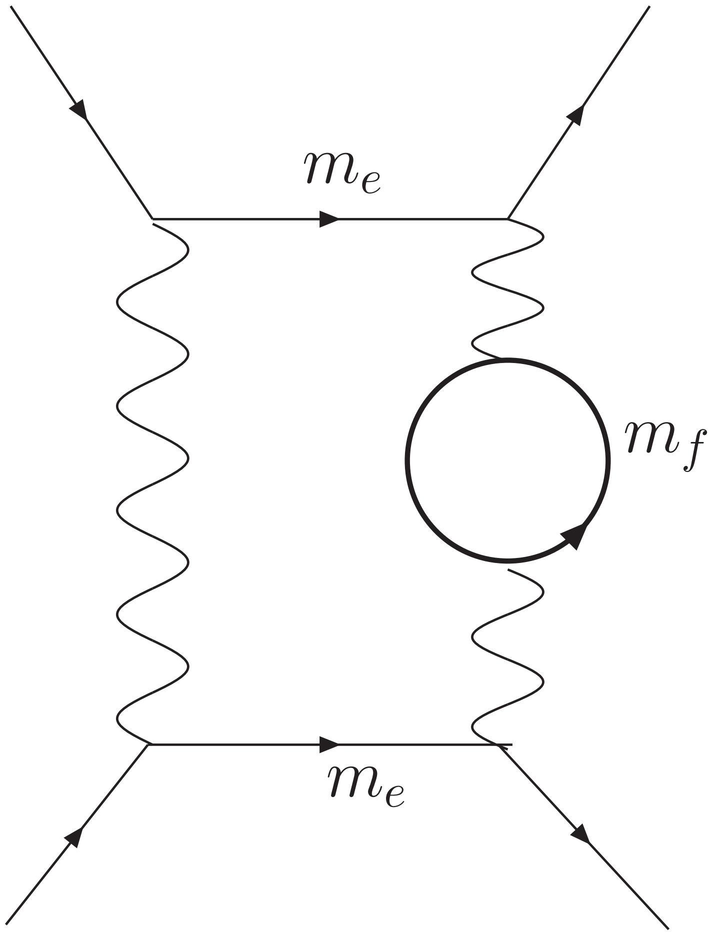
The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart



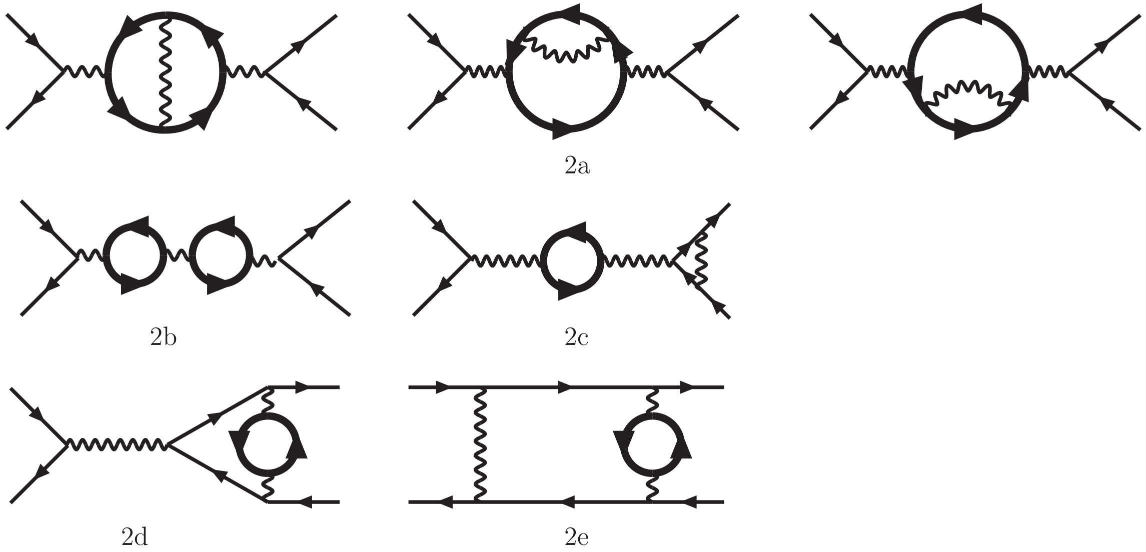
From Talk of Zvi Bern, LoopFest 2002

What is really new, is the treatment of the 2-boxes with two different fermions involved:



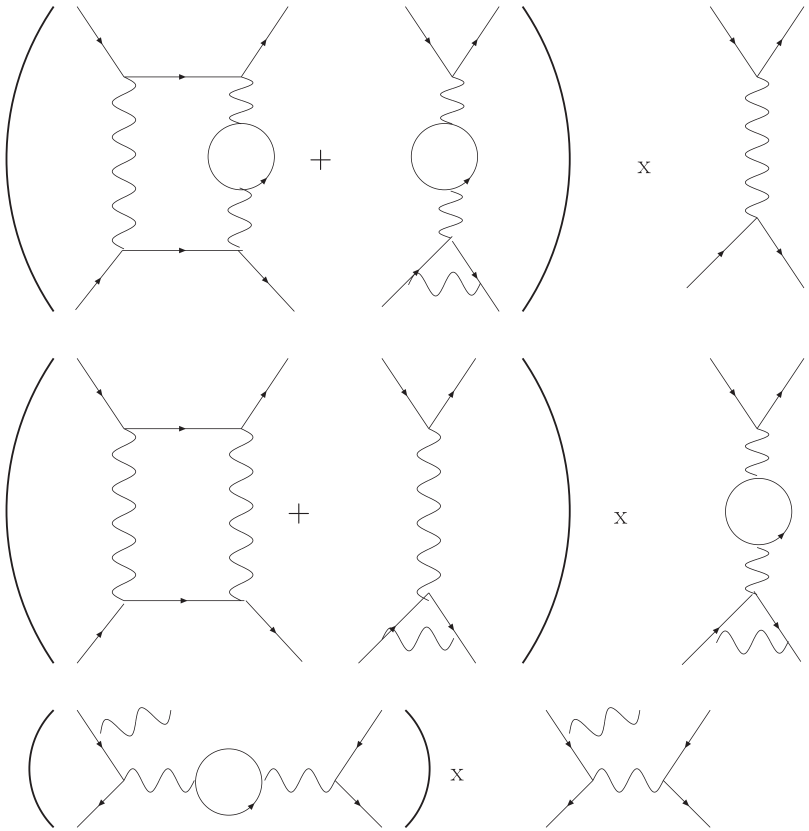
Virtual 2-loop corrections to Bhabha scattering – recent developments

- 1998 Arbuzov,Kuraev,Shaikhatdenov: $(\alpha^2 L)$ cross-section formula
- 1999,2004 Smirnov; Tausk; Heinrich,Smirnov: Few massive planar and non-planar two-loop 7-line box diagrams
- 2001 Bern,Dixon,Ghinculov: Massless photonic two-loop corrections
- 2001 Glover,Tausk,v.d.Bij: $(\alpha^2 L)$ cross-section from 2001 BDG
- 2004 Bonciani,Ferroglia,Mastrolia,Remiddi,v.d.Bij: The $n_f = 1$ SE and vertex masters and the $n_f = 1$ fermionic 2-box
- 2005 Czakon,Gluza,TR: List of all masters (33 2-boxes, 9 of them with seven lines), some 2-box masters evaluated
- 2005 Penin: $(\alpha^2 L^0)$ photonic cross-section ($n_f = 1$ fermionic box from Bonciani et al.)
- 2006 Czakon,Gluza,TR: All massive, planar $n_f = 1$ 2-boxes for $m_e^2 << s, t, u$
- 2006 Actis,Czakon,Gluza,TR: All masters for $n_f = 2$ at $m_e^2 << m_f^2 << s, t, u$
- 2007 Becher,Melnikov; Actis,Czakon,Gluza,TR: cross-section from $n_f = 2$ at $m_e^2 << m_f^2 << s, t, u$
- 2007 Actis,Gluza,TR: $n_f = 2$ cross-section with dispersion relation, $m_e^2 << m_f^2, s, t, u$, inclusion of hadronic insertions (preliminary)



Classes of Bhabha-scattering 2-loop diagrams containing at least one fermion loop.

The eight (i.e. 4 direct and 4 crossed) fermionic 2-loop box diagrams have to be combined with other diagrams for an IR-finite contribution:



After combining the 2-loop terms with the loop-by-loop terms and with soft real corrections:

$$\frac{d\sigma^{\mathrm{NNLO}}}{d\Omega} + \frac{d\sigma_{\gamma}^{\mathrm{NLO}}}{d\Omega} = \frac{d\sigma^{\mathrm{NNLO,e}}}{d\Omega} + \sum_{f \neq e} Q_f^2 \frac{d\sigma^{\mathrm{NNLO,f^2}}}{d\Omega} + \sum_{f \neq e} Q_f^4 \frac{d\sigma^{\mathrm{NNLO,f^4}}}{d\Omega}$$
$$+ \sum_{f_1, f_2 \neq e} Q_{f_1}^2 Q_{f_2}^2 \frac{d\sigma^{\mathrm{NNLO,2f}}}{d\Omega}.$$

The Box Corrections

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2\mathrm{e}\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\mathrm{I,f}}(x,y)$, where $\mathrm{i} = \mathrm{A},\mathrm{B},\mathrm{C}$,

$$\begin{aligned}
A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right],\\
A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t) &= F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right].
\end{aligned}$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

Look e.g. at $B^{(2)}_{\mathrm{A},f}(t,s)$

The interference of the box diagram of class 2e with the s-channel tree-level amplitude,

$$B_{2\mathrm{e},f} = \frac{\alpha^2}{4\,s^2}\mathrm{Re}\Big[B^{(2)}_{A,f}(s,t)\Big]$$

.

How to evaluate the $N_f = 2$ diagrams?

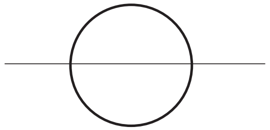
We did it in 2 ways

- **Decompose the 2-loop integrals to master integrals, solve them.**
  Here: In the limit $m_e^2 << m_f^2 << s, t, u$
  This is finished, hep-ph/07042400v2 $\longrightarrow$ **NPB, to appear**

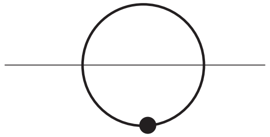
- **Alternatively, rewrite the 2-loop integrals as dispersion integrals.**
  Decompose afterwards into master integrals
  They are simpler, of one-loop type, but a numerical integration remains then.
  Advantages:
  – $m_e^2 << m_f^2, s, t, u$
  – **apply also to hadronic insertions**

The 2-loop master integrals for the $N_f = 2$ contributions

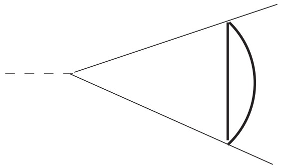
S. Actis, M. Czakon, J. Gluza, TR, 2006(publ.) / 2007(box master expansion corrected)



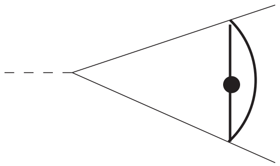
SE3l2M1m



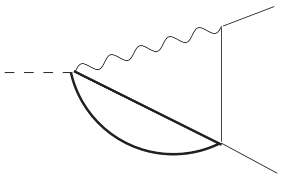
SE3l2M1md



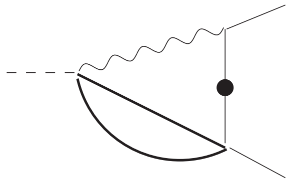
V4l2M2m



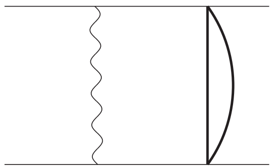
V4l2M2md



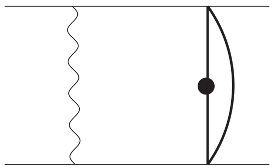
V4l2M1m



V4l2M1md



B5l2M2md



B5l2M2m

There are eight additional master integrals with two different mass scales.

The 2-box-diagrams represent a three-scale problem: $s/m_e^2, t/m_e^2, M^2/m_e^2$

There are several opportunities to evaluate the master integrals.

We used here the following:

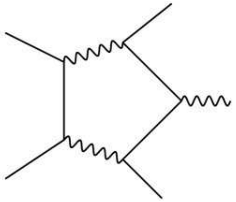
- **Feynman parameter representation**
- **derive Mellin-Barnes-representation**
  ($\longrightarrow$ with package AMBRE (public, Gluza,Kajda,TR))
- **The ϵ-expansion in $d = (4 - 2\epsilon$**
  ($\longrightarrow$ with package MB (public, Czakon))
- **Perform 2 subsequent small mass expansions**

Next slide: propaganda for AMBRE

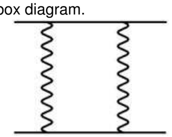
AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

To download 'right click' and 'save target as'.

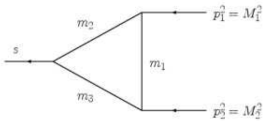
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



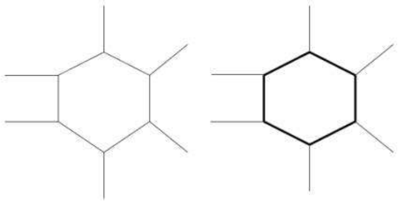
  - example3.nb - Massive QED one-loop box diagram.



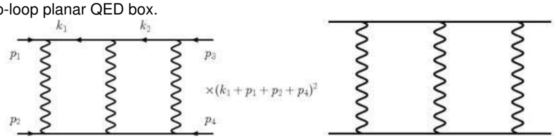
  - example4.nb - General one-loop vertex.



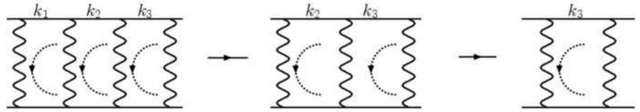
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.



  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



  - example8.nb - The loop-by-loop iterative procedure.



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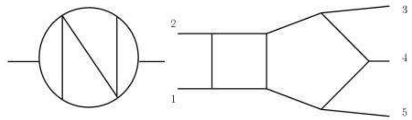

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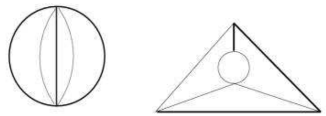

  - $ (–)& $ (& 5 / . – *) !" . – (

Self-energy master integrals:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051v2

$$L(R) \quad = \quad \ln\left(\frac{m_e^2}{M^2}\right)$$

$$\begin{aligned}
\texttt{SE3l2M1m[on shell]} \quad &= \quad M^2\, m^{-4\epsilon}\Big\{R\Big[\frac{1}{2\epsilon^2}+\frac{5}{4\epsilon}-\frac{3}{8}+\frac{\zeta_2}{2}+\frac{3}{2}L(R)-\frac{1}{2}L^2(R)\Big] \\
&+ \quad R^2\Big[\frac{11}{18}-\frac{1}{3}L(R)\Big]+\epsilon\Big[R\Big(\frac{45}{16}+\frac{5}{4}\zeta_2-\frac{\zeta_3}{3}-\frac{7}{4}L(R)+L^2(R) \\
&- \quad \frac{1}{2}L^3(R)\Big)+R^2\Big(-\frac{3}{4}+\frac{8}{9}L(R)-\frac{1}{2}L^2(R)\Big)\Big]\Big\},
\end{aligned}$$

$$\begin{aligned}
\texttt{SE3l2M1md[on shell]} \quad &= \quad m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\Big[1+2L\left(R\right)\Big]+\frac{1}{2}\left(1+\zeta_2\right)+L\left(R\right)+L^2\left(R\right) \\
&+ \quad \epsilon\Big[\frac{1}{6}\left(3+3\zeta_2-2\zeta_3\right)+\left(1+\zeta_2\right)L\left(R\right)+L^2\left(R\right)+\frac{2}{3}L^3\left(R\right)\Big] \\
&+ \quad R\Big[-\frac{3}{4}+\frac{1}{2}L(R)+\epsilon\left(\frac{7}{8}-L(R)+\frac{3}{4}L^2(R)\right)\Big] \\
&+ \quad R^2\Big[-\frac{5}{36}+\frac{1}{6}L(R)+\epsilon\Big(-\frac{5}{72}+\frac{1}{18}L(R)+\frac{1}{4}L^2(R)\Big)\Big]\Big\}.
\end{aligned}$$

Vertex master integrals:

Actis,Czakon,Gluza,TR, NPB(PS) 160 (2006) 91, hep-ph/0609051v2

$L_m(x) = \ln(-m^2/x)$ and $L_M(x) = \ln(-M^2/x)$,

$$
\begin{aligned}
\texttt{V412M1m[x]} &= m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2} + \frac{5}{2\epsilon} + \frac{1}{2}\Big[19 - 3\zeta_2 - L_m^2(x)\Big] \\
&+ \frac{M^2}{x}\Big[-2 + 4\zeta_2 - 4\zeta_3 - 2L_m(x) + 2L_M(x) - 4\zeta_2 L_M(x) \\
&+ 2L_m(x)L_M(x) - L_M^2(x) - L_m(x)L_M^2(x) + \frac{1}{3}L_M^3(x)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\texttt{V412M1md[x]} &= \frac{m^{-4\epsilon}}{m^2}\Big\{\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big[1 + \frac{1}{2}L_m(x)\Big] + 2 - \zeta_2 + L_m(x) + \frac{1}{4}L_m^2(x) \\
&+ \frac{M^2}{x}\Big[\frac{1}{\epsilon} - \frac{1}{\epsilon}L_M(x) - 1 + 3\zeta_2 + L_m(x) + L_M(x) \\
&- L_m(x)L_M(x) - \frac{1}{2}L_M^2(x)\Big]\Big\},
\end{aligned}
$$

$$
\texttt{V412M2m[x]} = m^{-4\epsilon}\Big\{\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big[\frac{5}{2} + L_m(x)\Big] + \frac{1}{2}(19 + \zeta_2) + 5\ L_m(x) + L_m^2(x)\Big\},
$$

$$
\texttt{V412M2md[x]} = \frac{m^{-4\epsilon}}{6x}\Big[12\zeta_3 - 6\zeta_2 L_M(x) - L_M^3(x)\Big],
$$

Box master integrals:

Correct Mellin-Barnes representations in Actis et al., NPB(PS) 160 (2006) 91, hep-ph/0609051v2

$$
\begin{aligned}
\texttt{B512M2m[x,y]} \quad = \quad & \frac{m^{-4\epsilon}}{x}\Big\{\frac{1}{\epsilon^2}L_m(x) + \frac{1}{\epsilon}\Big(-\zeta_2 + 2L_m(x) + \frac{1}{2}L_m^2(x) + L_m(x)L_m(y)\Big) \\
- \quad & 2\zeta_2 - 2\zeta_3 + 4L_m(x) + L_m^2(x) + \frac{1}{3}L_m^3(x) - 4\zeta_2 L_m(y) \\
+ \quad & 2L_m(x)L_m(y) + L_m(x)L_m^2(y) - \frac{1}{6}L_m^3(y) \\
- \quad & \Big(3\zeta_2 + \frac{1}{2}L_m^2(x) - L_m(x)L_m(y) + \frac{1}{2}L_m^2(y)\Big)\ln\Big(1+\frac{y}{x}\Big) \\
- \quad & \Big(L_m(x) - L_m(y)\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big) + \mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\texttt{B512M2md[x,y]} \quad = \quad & \frac{m^{-4\epsilon}}{xy}\Big\{\frac{1}{\epsilon}\Big[-L_m(x)L_m(y) + L_m(x)L(R)\Big] - 2\zeta_3 + \zeta_2 L_m(x) + 4\zeta_2 L_m(y) \\
- \quad & 2L_m(x)L_m^2(y) + \frac{1}{6}L_m^3(y) - 2\zeta_2 L(R) + 2L_m(x)L_m(y)L(R) - \frac{1}{6}L^3(R) \\
+ \quad & \Big(3\zeta_2 + \frac{1}{2}L_m^2(x) - L_m(x)L_m(y) + \frac{1}{2}L_m^2(y)\Big)\ln\Big(1+\frac{y}{x}\Big) \\
+ \quad & \Big(L_m(x) - L_m(y)\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big) - \mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
B^{(2)}_{\mathrm{A},f}(x,y) \;=\;& \frac{1}{\epsilon}\,\frac{2}{3}\Big(\frac{x^2}{y}+2x+y\Big)\Big[\frac{5}{3}-L(R_f)+L_e(y)\Big]L_e(x)\\
+\;& \frac{1}{3}\,\frac{x^2}{y}\Big\{2\Big(\frac{131}{27}-10\,\zeta_2-2\,\zeta_3\Big)-2\Big(\frac{25}{9}-6\,\zeta_2\Big)L(R_f)+\frac{7}{6}L^2(R_f)\\
-\;& \frac{1}{3}L^3(R_f)+\Big[\frac{82}{9}-2\,\zeta_2-\frac{4}{3}L(R_f)\Big]L_e(x)-2\Big[\frac{1}{3}+8\,\zeta_2-\frac{1}{2}L(R_f)\Big]L_e(y)\\
-\;& \Big[\frac{23}{6}-2\,L(R_f)\Big]L_e^2(y)+4\Big[2-L(R_f)\Big]L_e(x)\,L_e(y)-4\Big[\frac{5}{12}L_e^3(y)\\
-\;& L_e(x)\,L_e^2(y)\Big]-\Big[6\,\zeta_2+\ln^2\Big(\frac{y}{x}\Big)\Big]\ln\Big(1+\frac{y}{x}\Big)-2\,\ln\Big(\frac{y}{x}\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big)\\
+\;& 2\,\mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}+\frac{x}{3}\Big\{2\Big(\frac{262}{27}-9\,\zeta_2-4\,\zeta_3\Big)-4\Big(\frac{25}{9}-3\,\zeta_2\Big)L(R_f)\\
+\;& \frac{7}{3}L^2(R_f)-\frac{2}{3}L^3(R_f)+2\Big[\frac{121}{9}-\frac{10}{3}L(R_f)\Big]L_e(x)-2\Big[\frac{10}{3}+12\,\zeta_2\\
-\;& 2\,L(R_f)\Big]L_e(y)+\Big[\frac{13}{3}-2\,L(R_f)\Big]L_e^2(x)-\Big[\frac{16}{3}-2\,L(R_f)\Big]L_e^2(y)\\
+\;& 2\Big[\frac{17}{3}-2\,L(R_f)\Big]L_e(x)\,L_e(y)+\frac{2}{3}L_e^3(x)\\
+\;& 6\,L_e(x)\,L_e^2(y)-2\,L_e^3(y)-2\Big[6\,\zeta_2+\ln^2\Big(\frac{y}{x}\Big)\Big]\ln\Big(1+\frac{y}{x}\Big)\\
-\;& 4\,\ln\Big(\frac{y}{x}\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big)+4\,\mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}+\frac{y}{3}\Big\{2\Big(\frac{131}{27}-7\zeta_2-2\zeta_3\Big)\\
-\;& 2\Big(\frac{25}{9}-3\zeta_2\Big)L(R_f)+\frac{7}{6}L^2(R_f)-\frac{1}{3}L^3(R_f)+\Big[\frac{130}{9}-\frac{10}{3}L(R_f)\Big]L_e(x)\\
-\;& \Big[6+12\zeta_2-3L(R_f)\Big]L_e(y)+\Big[\frac{5}{3}-L(R_f)\Big]L_e^2(x)-\Big[\frac{25}{6}-L(R_f)\Big]L_e^2(y)\\
+\;& 2\Big[\frac{10}{3}-L(R_f)\Big]L_e(x)L_e(y)+\frac{1}{3}L_e^3(x)-L_e^3(y)+3L_e(x)L_e^2(y)\\
-\;& \Big[6\,\zeta_2+\ln^2\Big(\frac{y}{x}\Big)\Big]\ln\Big(1+\frac{y}{x}\Big)-2\ln\Big(\frac{y}{x}\Big)\mathrm{Li}_2\Big(-\frac{y}{x}\Big)+2\,\mathrm{Li}_3\Big(-\frac{y}{x}\Big)\Big\}
\end{aligned}
$$

$B_{2\mathrm{e},f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	188758	5200.08	284.711
μ	1635.62	1686.88	130.579
τ			39.5554

Table 1: Finite part of $B_{2\mathrm{e},f}$ in nanobarns at a scattering angle $\theta = 3^\circ$.

$B_{2\mathrm{e},f}$ [nb] / $\sqrt{s}$ [GeV]	10	91	500
e	143.162	3.23102	0.160582
μ	61.3875	1.79381	0.0995184
τ	10.0105	0.935319	0.0639576
t			-0.00256757

Table 2: Finite part of $B_{2\mathrm{e},f}$ in nanobarns at a scattering angle $\theta = 90^\circ$.

$\sqrt{s}$ [GeV]	10	91	500
e	-124.237	-254.293	-400.574
μ	-4.8036	-29.1057	-70.1032
τ		-2.08719	-13.4901

Table 3: Real part for the vertex form factor.

$$\frac{d\sigma^{\mathrm{NNLO,f^2}}}{d\Omega} = \frac{\alpha^2}{s}\left\{\sigma_1^{\mathrm{NNLO,f^2}} + \sigma_2^{\mathrm{NNLO,f^2}} \ln\left(\frac{2\omega}{\sqrt{s}}\right)\right\}$$

The $\sigma_1^{\mathrm{NNLO,f^2}}$ is the main result of this study:

$$
\begin{aligned}
\sigma_1^{\mathrm{NNLO,f^2}} \quad &= \quad \frac{\left(1-x+x^2\right)^2}{3x^2}\Big\{-\frac{1}{3}\Big[\ln^3\left(\frac{s}{m_e^2}\right)+\ln^3\left(R_f\right)\Big]+\ln^2\left(\frac{s}{m_e^2}\right)\Big[\frac{55}{6}-\ln\left(R_f\right)\\
&+ \quad \ln\left(1-x\right)-\ln\left(x\right)\Big]+\ln\left(\frac{s}{m_e^2}\right)\Big[-\frac{589}{18}+\frac{37}{3}\ln\left(R_f\right)-\ln^2\left(R_f\right)\\
&- \quad 2\ln\left(R_f\right)\Big(\ln\left(x\right)-\ln\left(1-x\right)\Big)-8\mathrm{Li}_2\left(x\right)\Big]+\frac{4795}{108}-\frac{409}{18}\ln\left(R_f\right)+\frac{19}{6}\ln^2\left(R_f\right)\\
&- \quad \ln^2\left(R_f\right)\Big(\ln\left(x\right)-\ln\left(1-x\right)\Big)-8\ln\left(R_f\right)\mathrm{Li}_2\left(x\right)+\frac{40}{3}\mathrm{Li}_2\left(x\right)\Big\}\\
&+ \quad \ln\left(\frac{s}{m_e^2}\right)\Big[\zeta_2\Big(-\frac{2}{3x^2}+\frac{4}{3x}+\frac{11}{2}-\frac{23}{3}x+\frac{16}{3}x^2\Big)+\ln^2\left(x\right)\Big(-\frac{1}{3x^2}+\frac{17}{12x}\\
&- \quad \frac{5}{4}-\frac{x}{12}+\frac{2}{3}x^2\Big)+\ln^2\left(1-x\right)\Big(-\frac{2}{3x^2}+\frac{11}{6x}-\frac{5}{2}+\frac{11}{6}x-\frac{2}{3}x^2\Big)\\
&+ \quad \ln\left(x\right)\ln\left(1-x\right)\Big(\frac{2}{3x^2}-\frac{4}{3x}-\frac{1}{2}+\frac{5}{3}x-\frac{4}{3}x^2\Big)+\ln\left(x\right)\Big(\frac{55}{9x^2}-\frac{83}{9x}+\frac{65}{6}\\
&- \quad \frac{85}{18}x+\frac{10}{9}x^2\Big)+\frac{1}{3}\ln\left(1-x\right)\Big(-\frac{10}{3x^2}+\frac{31}{6x}-10+\frac{31}{6}x-\frac{10}{3}x^2\Big)\Big]\\
&+ \quad \frac{1}{3}\ln^3\left(x\right)\Big(-\frac{1}{3x^2}+\frac{31}{12x}-\frac{11}{6}-\frac{x}{6}+\frac{x^2}{3}\Big)+\frac{1}{3}\ln^3\left(1-x\right)\Big(-\frac{1}{3x^2}+\frac{1}{x}\\
&- \quad \frac{4}{3}+x-\frac{x^2}{3}\Big)+\ln^2\left(x\right)\ln\left(1-x\right)\Big(-\frac{1}{3x^2}+\frac{1}{3x}-\frac{4}{3}+x-\frac{x^2}{3}\Big)\\
&+ \quad \frac{1}{3}\ln\left(x\right)\ln^2\left(1-x\right)\Big(-\frac{1}{x^2}+\frac{2}{x}-\frac{7}{4}+\frac{x}{2}\Big)+\ln^2\left(x\right)\Big[\frac{55}{18x^2}-\frac{46}{9x}+\cdots\cdots\cdots
\end{aligned}
$$

$$
\begin{aligned}
\cdots\cdots + \quad & \frac{14}{3} - \frac{4}{9}x - \frac{10}{9}x^2 + \ln(R_f)\left(-\frac{1}{3x^2} + \frac{17}{12x} - \frac{5}{4} - \frac{x}{12} + \frac{2}{3}x^2\right)\Big] \\
+ \quad & \ln^2(1-x)\Big[\frac{10}{9x^2} - \frac{29}{9x} + \frac{9}{2} - \frac{29}{9}x + \frac{10}{9}x^2 + \ln(R_f)\Big(-\frac{2}{3x^2} + \frac{11}{6x} \\
- \quad & \frac{5}{2} + \frac{11}{6}x - \frac{2}{3}x^2\Big)\Big] + \ln(x)\ln(1-x)\Big[-\frac{10}{9x^2} + \frac{37}{18x} + \frac{1}{2} - \frac{25}{9}x \\
+ \quad & \frac{20}{9}x^2 + \ln(R_f)\left(\frac{2}{3x^2} - \frac{4}{3x} - \frac{1}{2} + \frac{5}{3}x - \frac{4}{3}x^2\right)\Big] + \ln(x)\Big[-\frac{589}{54x^2} + \frac{1753}{108x} \\
- \quad & \frac{701}{36} + \frac{925}{108}x - \frac{56}{27}x^2 + \mathrm{Li}_2(x)\left(-\frac{4}{x^2} + \frac{19}{3x} - 7 + 3x - \frac{2}{3}x^2\right) \\
+ \quad & \ln(R_f)\left(\frac{37}{9x^2} - \frac{56}{9x} + \frac{47}{6} - \frac{67}{18}x + \frac{10}{9}x^2\right) + \zeta_2\Big(-\frac{2}{3x^2} + \frac{4}{x} - \frac{1}{6} \\
- \quad & \frac{10}{3}x + 2x^2\Big)\Big] + \ln(1-x)\Big[\frac{56}{27x^2} - \frac{161}{54x} + \frac{56}{9} - \frac{161}{54}x + \frac{56}{27}x^2 \\
+ \quad & \ln(R_f)\left(-\frac{10}{9x^2} + \frac{31}{18x} - \frac{10}{3} + \frac{31}{18}x - \frac{10}{9}x^2\right) + \zeta_2\Big(-\frac{2}{x^2} + \frac{20}{3x} - \frac{32}{3} + \frac{20}{3}x \\
- \quad & 2x^2\Big)\Big] + \mathrm{Li}_3(x)\left(\frac{4}{3x^2} - \frac{7}{3x} + 3 - \frac{5}{3}x + \frac{2}{3}x^2\right) + \frac{2}{3}S_{1,2}(x)\Big(-\frac{1}{x^2} + \frac{1}{x} \\
- \quad & x + x^2\Big) + \zeta_2\Big[\frac{19}{9x^2} - \frac{13}{18x} - \frac{43}{3} + \frac{311}{18}x - \frac{98}{9}x^2 + \ln(R_f)\Big(-\frac{2}{3x^2} + \frac{4}{3x} \\
+ \quad & \frac{11}{2} - \frac{23}{3}x + \frac{16}{3}x^2\Big)\Big] + \zeta_3\left(-\frac{4}{3x^2} + \frac{3}{x} - 5 + \frac{11}{3}x - 2x^2\right)
\end{aligned}
$$

$d\sigma$ / $d\Omega$ [nb] | $\sqrt{s}$ [GeV]	10	91	500
LO QED	440873	5323.91	176.349
LO `Zfitter`	440875	5331.5	176.283
NNLO (e)	-1397.35	-35.8374	-1.88151
NNLO $(e + \mu)$	-1394.74	-43.1888	-2.41643
NNLO $(e + \mu + \tau)$			-2.55179
NNLO photonic	9564.09	251.661	12.7943

$d\sigma$ / $d\Omega$ [nb] | $\sqrt{s}$ [GeV]	10	91	500
LO QED [Eq. (**??**)]	0.466409	0.00563228	0.000186564
LO `Zfitter`	0.468499	0.127292	0.0000854731
NNLO (e)	-0.00453987	-0.0000919387	-4.28105 $\cdot 10^{-6}$
NNLO $(e + \mu)$	-0.00570942	-0.000122796	-5.90469 $\cdot 10^{-6}$
NNLO $(e + \mu + \tau)$	-0.00586082	-0.000135449	-6.7059 $\cdot 10^{-6}$
NNLO $(e + \mu + \tau + t)$			-6.6927 $\cdot 10^{-6}$
NNLO photonic	0.0358755	0.000655126	0.0000284063

Table 4: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 3^\circ$ and $\theta = 90^\circ$. Empty entries are related to cases where the high-energy approximation cannot be applied.

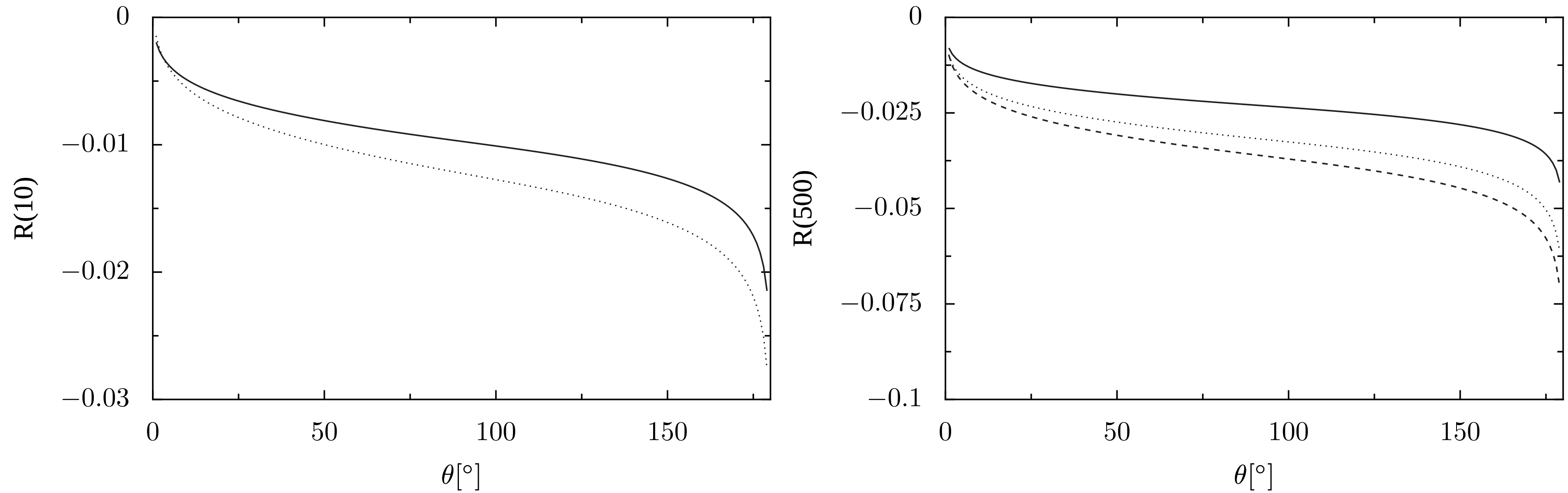

Figure 1: Ratio of the fermionic NNLO corrections to the differential cross section respect to the tree-level result for $\sqrt{s} = 10$ GeV and $\sqrt{s} = 500$ GeV. Solid line: electron-loop contributions, a dotted one the sum of electron- and muon-loop ones, and a dashed one includes also τ leptons.

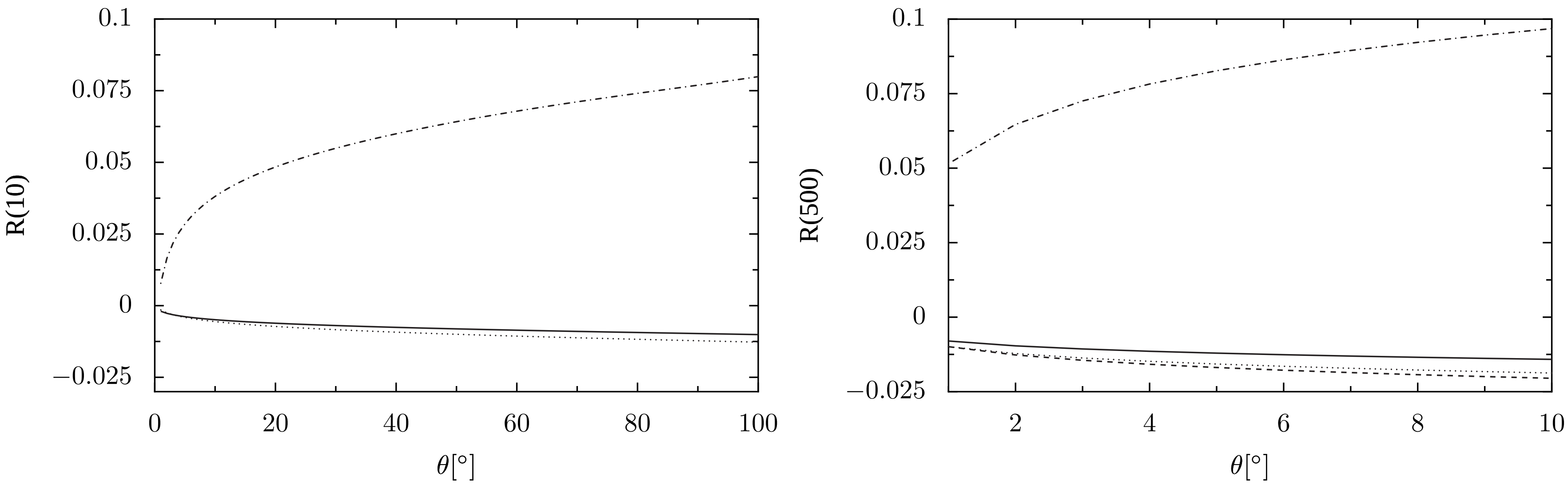

Figure 2: Here also with the photonic contributions of Arbuzov et al., Glover et al., Penin (dash-dotted lines).

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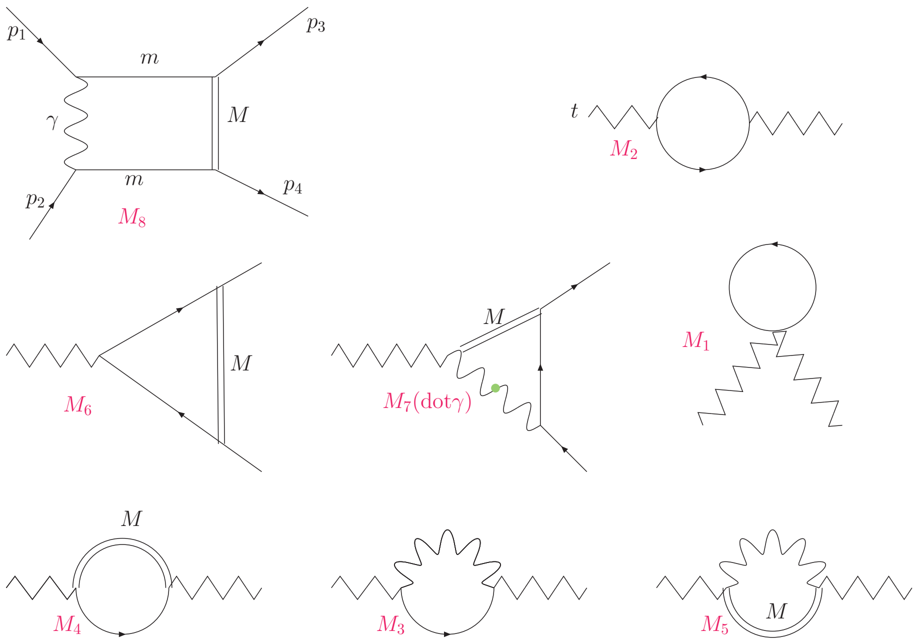
The dispersion master integrals for the $N_f = 2$ contributions

S. Actis, J. Gluza, TR, 2007 (unpubl.)

There are three box kernel functions, depending on m_e, m_f, s, t.with $m_e^2 << z = m_f^2, s, t$.
They are IR-divergent.
The eight master integrals for the 2-loop boxes are:



The Box Corrections (repeated here from above)

The contribution of the renormalized two-loop box diagrams of class 2e is given by

$$\frac{d\sigma^{2\mathrm{e}\times\mathrm{tree}}}{d\Omega} = \frac{\alpha^2}{2\,s}\left[\frac{1}{s}\,A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) + \frac{1}{t}\,A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t)\right]$$

Here the auxiliary functions can be conveniently expressed through three independent form factors $B^{(2)}_{\mathrm{I,f}}(x,y)$, where $\mathrm{i} = \mathrm{A},\mathrm{B},\mathrm{C}$,

$$A_1^{2\mathrm{e}\times\mathrm{tree}}(s,t) = F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{A},f}(s,t) + B^{(2)}_{\mathrm{B},f}(t,s) + B^{(2)}_{\mathrm{C},f}(u,t) - B^{(2)}_{\mathrm{B},f}(u,s)\right],$$

$$A_2^{2\mathrm{e}\times\mathrm{tree}}(s,t) = F_\epsilon^2 \sum_f Q_f^2\,\mathrm{Re}\left[B^{(2)}_{\mathrm{B},f}(s,t) + B^{(2)}_{\mathrm{A},f}(t,s) - B^{(2)}_{\mathrm{B},f}(u,t) + B^{(2)}_{\mathrm{C},f}(u,s)\right].$$

The normalization factor is

$$F_\epsilon = \left(\frac{m_e^2\,\pi\,e^{\gamma_E}}{\mu^2}\right)^{-\epsilon}$$

Look e.g. at $B^{(2)}_{\mathrm{C},f}(t,s)$ for hadrons:

$$B^{(2)}_{\mathrm{C},had}(t,s) = \int_{4M_\pi^2}^{\infty} \frac{dz}{z} R_{had}(z) K_C(s,t,z)$$

And similarly for leptons:

$$
\begin{aligned}
4M_\pi^2 &\longrightarrow 4m_l^2 \\
R_{had}(z) &\longrightarrow R_{lep}(z) \sim \sqrt{1-\frac{4m_l^2}{z}}\left(1+\frac{2m_l^2}{z}\right) + \epsilon R_{lep}^{\epsilon}(z)
\end{aligned}
$$

Get:

$$
B^{(2)}_{\mathrm{C},lep}(t,s) = \int_{4m_l^2}^{\infty} \frac{dz}{z} R_{lep}(z) K_C(s,t,z)
$$

$$
\begin{aligned}
K_C(x,y;z) &= F_\epsilon \sum_{i=1}^{8} c_{Ci} M_i(s,t,z) \\
&= \frac{1}{3\,m_e^2\,(y-z)} \left\{ 2\,\frac{F_\epsilon}{\epsilon}\,x^2\,L_x + 4\,\zeta_2\,x^2 \left(\frac{z}{y}-2\right) - 2\,\left(x^2+y^2+x\,y\right)\,L_x \right. \\
&+ x^2\left(\frac{z}{y}-1\right) L_y + 2\,x^2\left(\frac{z}{y}-1\right) L_y^2 + 4\,x^2\,L_x\,L_y + x^2\left(\frac{z}{y}-1\right)\,\ln\!\left(\frac{z}{m_e^2}\right) \\
&- 2\,x^2\left(\frac{z}{y}-\frac{1}{2}\right)\,\ln^2\!\left(\frac{z}{m_e^2}\right) + 4\,x^2\left(\frac{z}{y}-1\right)\,\ln\!\left(\frac{z}{m_e^2}\right)\,\ln\!\left(1-\frac{z}{y}\right) \\
&+ 2\,x^2\,\ln\!\left(\frac{z}{m_e^2}\right) L_x - x^2\left(\frac{z}{y}+\frac{y}{z}-2\right)\,\ln\!\left(1-\frac{z}{y}\right) - 4\,x^2\,\ln\!\left(1-\frac{z}{y}\right) L_x \\
&+ \left. 4\,x^2\left(\frac{z}{y}-1\right)\mathrm{Li}_2\!\left(\frac{z}{y}\right) - 2\,x^2\,\mathrm{Li}_2\!\left(1+\frac{z}{x}\right)\right\}.
\end{aligned}
$$

The contributing masters are:

$$M_1 = N\int d^Dk\frac{1}{k^2-m^2}, \tag{1}$$

$$M_2 = N\int d^Dk\frac{1}{(k^2-m^2)[(k-p_1-p_2)^2-m^2]}, \tag{2}$$

$$M_3 = N\int d^Dk\frac{1}{k^2(k-p_1+p_3)^2}, \tag{3}$$

$$M_4 = N\int d^Dk\frac{1}{(k^2-m^2)[(k-p_3)^2-z]}, \tag{4}$$

$$M_5 = N\int d^Dk\frac{1}{(k^2-z)(k-p_1+p_3)^2}, \tag{5}$$

$$M_6 = N\int d^Dk\frac{1}{(k^2-z)[(k+p_3)^2-m^2][(k+p_3-p_1-p_2)^2-m^2]}, \tag{6}$$

$$M_7 = N\int d^Dk\frac{1}{(k^2-z)[(k+p_3)^2-m^2](k+p_3-p_1)^2}, \tag{7}$$

$$M_8 = N\int d^Dk\frac{1}{(k^2-z)[(k+p_3)^2-m^2](k+p_3-p_1)^2[(k+p_3-p_1-p_2)^2-m^2]}, \tag{8}$$

where

$$F_\epsilon = N = m^{2\epsilon}\frac{e^{\gamma}\epsilon}{i\pi^{2-\epsilon}}. \tag{9}$$

and e.g. the box integral $M_8 = Bo1$ is:

```
Bo1 = (4*ep*z2 + 2*Log[-(m^2/t)] - 4*ep*Log[me]*Log[-(m^2/t)] -
      4*ep*Log[1 - m^2/t]*Log[-(m^2/t)] + 3*ep*Log[-(m^2/t)]^2 -
      2*Log[-(me^2/t)] + 4*ep*Log[me]*Log[-(me^2/t)] +
      4*ep*Log[1 - m^2/t]*Log[-(me^2/t)] - 2*ep*Log[-(m^2/t)]*
       Log[-(me^2/t)] - ep*Log[-(me^2/t)]^2 + Log[-(m^2/s)]*
       (4*ep*Log[me] + 4*ep*Log[1 - m^2/t] - 2*(1 + ep*Log[-(m^2/t)] +
          ep*Log[-(me^2/t)])) + 2*ep*PolyLog[2, (m^2 + s)/s])/
     (2*ep*s*(m^2 - t))
```

$\mathrm{d}\sigma$ / $\mathrm{d}\Omega$ [nb] | $\sqrt{s}$ [GeV]	1	10	91	500
LO QED	46.6409	0.466409	0.00563228	0.000186564
LO `Zfitter`	46.643	0.468499	0.127292	0.0000854731
NNLO (e)	-0.230927	-0.00453987	-0.0000919387	-4.28105 $\cdot\ 10^{-6}$
NNLO ($e+\mu$) "	-0.256679	-0.00570942	-0.000122796	-5.90469 $\cdot\ 10^{-6}$
NNLO ($e+\mu+\tau$) "		-0.00586082	-0.000135449	-6.7059 $\cdot\ 10^{-6}$
NNLO ($e+\mu+\tau+t$) "				-6.6927 $\cdot\ 10^{-6}$
NNLO photonic	2.07476	0.0358755	0.000655126	0.0000284063
NNLO IR e	-0.19927	-0.00359349	-0.0000672264	-2.95317$\cdot 10^{-6}$
NNLO IR μ (analytic)	-0.0314292	-0.00134635	-0.0000335037	-1.66781 $\cdot 10^{-6}$
NNLO IR μ (dispersion)	-0.0333538	-0.00134663	-0.0000335037	-1.66781 $\cdot 10^{-6}$
NNLO IR τ (analytic)		-0.00021027	-0.0000162977	-1.00877 $\cdot 10^{-6}$
NNLO IR τ (dispersion)		-0.000272634	-0.0000163119	-1.00878 $\cdot 10^{-6}$

Table 5: Numerical values for the NNLO corrections to the differential cross section respect to the solid angle. Results are expressed in nanobarns for a scattering angle $\theta = 90°$. Empty entries are related to cases where the high-energy approximation cannot be applied.

Using R_{had}

This is a topic by itself, because R_{had} is basically unpublished.

```
N.N.1:
Fuer R(s) mit Fehlern, Kontinuum + Resonanzen haben wir nur unsere interne
Arbeitsversion.

N.N.2:
This procedure is a follow up of complicated programs, which unfortunately
do not exist in a really user-friendly form.

N.N.3:
I understand that for your problem it is probably too
cumbersome (and time-consuming) to use the data.

N.N.4:
es hat etwas gedauert, bis ich in meinen alten Verzeichnissen auf einer
1994er Vax am MPI fuendig geworden bin.
```

So, finally, we might reproduce the old estimates given for the vertex dispersion relation in Kniehl, Krawczyk, Kühn, Stuart (1988) $\longrightarrow$ soon we have final numerics

Summary

- We determined the $N_f = 2$ contributions to 2-loop Bhabha scattering
- The contribution is small, but non-negligible at the scale 10^{-4} ($\rightarrow$ No LEP influencing)
- Agreement for $m_e^2 << m_l^2 << s, t, u$ with:
  "Two-loop QED corrections to Bhabha scattering"
  Thomas Becher, Kirill Melnikov, arXiv:0704.3582 [hep-ph], JHEP
- Status: Now a nearly complete knowledge of the NNLO corrections to Bhabha scattering
  To be determined yet:
  $\longrightarrow$ Hadronic $N_f = 2$ contributions (non-perturbative)
  $\longrightarrow$ Leptonic case where $m_l^2 \sim s, t, u$ (also with dispersion)
  $\longrightarrow$ 1-loop diagrams with real photon emission, interfering with real (Born) radiation, including 5-point functions
  The latter was studied already by Arbuzov, Kuraev, Shaitchatdenov (1998, small photon mass) see talk by K. Kajda