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Master Integrals for Massive Bhabha Scattering

– The Double-boxes with 5 Lines –

Tord Riemann, DESY, Zeuthen

Contributed to the Workshop "Matter to the Deepest"

Ustron, 8–14 Sep 2005

A project in collaboration with
Michal Czakon Univ. Würzburg (and Katowice)
Janusz Gluza DESY (and Katowice)

See also:
- **NPB(PS) 135 (2004), hep-ph/0406203**
- **PRD 71 (2005), hep-ph/0412164**
- **http://www-zeuthen.desy.de/theory/research/bhabha/**

- **What do we need?** $\rightarrow 10^{-4}$ **for** $d\sigma/d\cos\vartheta$ **at small** ϑ
- **Scalar master integrals** $\rightarrow$ **masters with 5 lines**
- **Summary**

The Physics Needs

For more details see e.g.:
K. Mönig, "Bhabha scattering at the ILC"
talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005
/afs/ifh.de/user/m/moenig/public/www/bhabha_ilc.pdf

ILC – Need Bhabha cross-sections with 3–4 significant digits.

Why?

- **ILC:** $e^+e^- \to W^+W^-, f\bar{f}$ **with** $O(10^6)$ **events** $\to 10^{-3}$
- **GigaZ: relevant physics derived from** $Z \to$ **hadrons**, l^+l^-, **the latter with** $O(10^8)$ **events** $\to 10^{-4}$, **the systematic errors (luminosity!) influence this**
- **ILC:**$e^+e^- \to e^+e^-$, **a probe for New Physics with** $O(10^5)$ **events/year** $\to 10^{-3}$

Conclude: will need $\Delta\mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.

Some Kinematics

Need a cross-section prediction with 5 significant digits.

Perturbative orders:

$$\left(\frac{\alpha}{\pi}\right) = 2 \times 10^{-3}$$

$$\left(\frac{\alpha}{\pi}\right)^2 = 0.6 \times 10^{-5}$$

Kinematics:

$$\begin{aligned}
\sqrt{s} &= 90 \ldots 1000 \text{ GeV} \\
\vartheta &= 26 \ldots 82 \text{ mrad} \\
\cos\vartheta &\sim 0.999\,66 \ldots 0.996\,64 \\
T &= \frac{s}{2}(1-\beta^2\cos\vartheta) > 1.36 \text{ GeV}|_{GigaZ},\ 42.2 \text{ GeV}|_{ILC500}
\end{aligned}$$

Conclude:

- t-**channel exchange of** γ **dominates everything else**
- $m_e^2/s \ < \ m_e^2/T \ \leq \ 10^{-5} \ldots 10^{-7}$
- **Calculate: 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung**

Status by end of 2004

Established: 10^{-3} MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, "Theoretical error of luminosity cross section at LEP", hep-ph/0306083 [1]

- BHLUMI v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- see also: Jadach, Melles, Ward, Yost: PLB 1996, thesis Melles 1996 [2]
- NLLBHA: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- SAMBHA: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2] $\to$ Conclude:
The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:

$$m_e, m_\gamma, (d-4), E_\gamma$$

Results: Numerical comparison in all $f\bar{f}$

Bhabha $e^-e^+ \to e^-e^+\ (\gamma)$ **at LC:** $\sqrt{s} = 500\ \text{GeV}, E_{\max}(\gamma_{\text{soft}}) = \frac{\sqrt{s}}{10}$

$\cos\theta$	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\text{Born}}$ (pb)	$\left[\frac{d\sigma}{d\cos\theta}\right]_{\mathcal{O}(\alpha^3)=\text{Born+QED+weak+soft}}$	Group
−0.9999	0.21482 70434 05632 5	0.14889 12125 78083 7	aITALC
−0.9999	0.21482 70434 05632 6	0.14889 12189 28404 0	*FeynArts*
−0.9	0.21699 88288 10920 5	0.19344 50785 26863 6	aITALC
−0.9	0.21699 88288 10920 0	0.19344 50785 26862 2	*FeynArts*
−0.9	0.21699 88288 41513 1	0.19344 50785 62637 9	$m_e = 0$
+0.0	0.59814 23072 50330 3	0.54667 71794 69423 1	aITALC
+0.0	0.59814 23072 50329 4	0.54667 71794 69421 8	*FeynArts*
+0.0	0.59814 23072 88584 4	0.54667 71794 99961 4	$m_e = 0$
+0.9	$0.18916\,03223\,32270\,6 \cdot 10^3$	$0.17292\,83490\,66507\,2 \cdot 10^3$	aITALC
+0.9	$0.18916\,03223\,32270\,6 \cdot 10^3$	$0.17292\,83490\,66508\,0 \cdot 10^3$	*FeynArts*
+0.9	$0.18916\,03223\,31848\,5 \cdot 10^3$	$0.17292\,83490\,61347\,4 \cdot 10^3$	$m_e = 0$
+0.9999	$0.20842\,90676\,46142\,9 \cdot 10^9$	$0.19140\,17861\,11341\,6 \cdot 10^9$	aITALC
+0.9999	$0.20842\,90676\,46436\,4 \cdot 10^9$	$0.19140\,17861\,11979\,0 \cdot 10^9$	*FeynArts*

Great independent agreement up to 14 digits! : limit in double precision

Previous agreement with *FeynArts*: 11 digits hep-ph/0307132**, SANC: 10 digits** hep-ph/0207156

Thanks to T. Hahn, numbers supplied with *FeynArts + FormCalc + LoopTools*

Table 2:
The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.
$M_Z = 91.16$ *GeV*, $m_t = 150$ *GeV*, $M_H = 100$ *GeV*.
Upper rows: DZ, lower rows: H.
δ_m: *largest relative deviation in per mille.*

$\sqrt{s}$ (GeV) / θ	60	89	91.16	93	200
15°	129.6 129.6	65.11 65.11	57.93 57.93	49.00 49.00	11.82 11.82
45°	1.451 1.451	1.376 1.377	1.755 1.756	.4833 .4837	11.67 11.68
60°	.4303 .4305	.6124 .6129	1.125 1.126	.2697 .2699	.03075 .03077
75°	.1717 .1718	.3627 .3630	.8718 .8720	.2232 .2233	.01072 .01072
90°	.08873 .08876	.2768 .2769	.7790 .7787	.2088 .2087	.004862 .004855
105°	.05917 .05918	.2690 .2690	.8082 .8074	.2157 .2157	.002858 .002853
120°	.04906 .04906	.3053 .3051	.9323 .9309	.2429 .2426	.002077 .002074
135°	.04671 .04672	.3626 .3624	1.111 1.109	.2838 .2833	.001743 .001742
165°	.04839 .04839	.4638 .4635	1.425 1.422	.3590 .3584	.001539 .001540
δ_m	0.6	0.8	1.8	2.0	1.7

Bhabha scattering

Bardin,Hollik,T.R., Z.PhysikC49(1991)485

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Status 2005

Know the constant term ($m_e = 0$) from 2-loop Bhabha scattering

A. Penin, Two-Loop Corrections to Bhabha Scattering, hep-ph/0501120 v.3, $\rightarrow$ PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by $D = 4 - 2\epsilon$ into the on-mass-shell renormalization with $m_e \to 0$ and IR regulation by $\lambda = m_\gamma \neq 0$

Use IR-properties of amplitudes (see Penin):

[A] Exponentiation of the IR logarithms (Sudakov 1956,...)

[B] Factorization of the collinear logarithms into expernal legs (Frenkel, Taylor 1976)

[C] Non-renormalization of the IR exponents (YFS 1961, ...)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

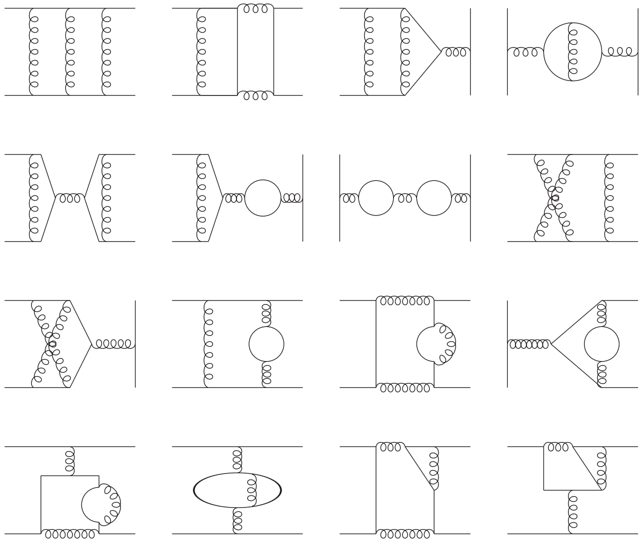
If all this is correct, the constant term in m_e is known for the MCs (but the radiative one-loops with 5-point functions).

$m = 0$

Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \to \mu^+\mu^-$, since it's closely related but has less diagrams.

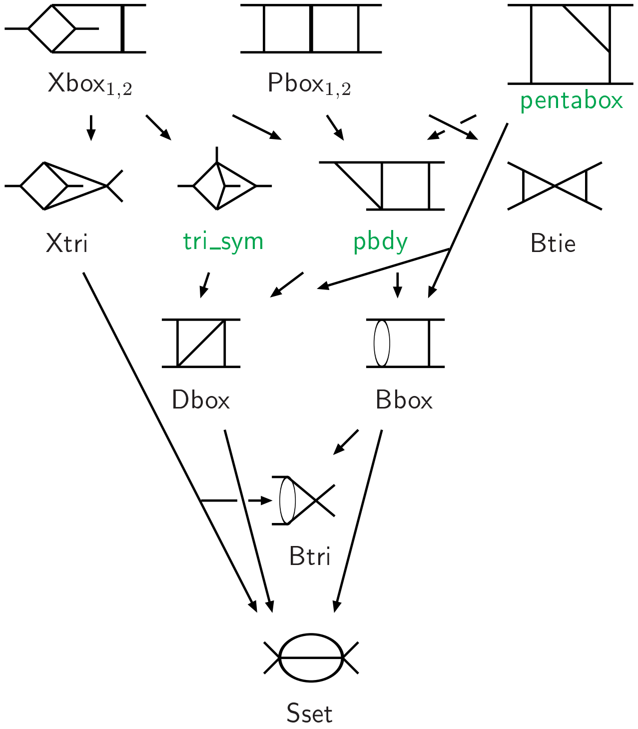
There are 47 QED diagrams contributing to $e^+e^- \to \mu^+\mu^-$.



In this calculation all particles massless.

The Bhabha scattering amplitude can be obtained from $e^+e^- \to \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).

Two-loop integral inheritance chart



From Talk of Zvi Bern, LoopFest 2002

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The massive 2-loop contributions

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d\cos\vartheta}(e^+e^- \to e^+e^-)$$

We are using a scheme with

(1) $m_e \neq 0$ (good with the MC's)

(2) $m_\gamma = 0$ (bad with the MC's; $\to$ Mastrolia, Remiddi 2003)

(3) dim.reg. for UV and IR divergences

Also:

Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij: all but many 2-boxes

Heinrich, Smirnov: Calculation of selected complicated Feynman integrals

.

There are few topologies only:

- **1-loop: 5**
- **2-loop self energies: 5 (3 for external legs)**
- **2-loop vertices: 5**
- **2-loop boxes: 6** $\rightarrow$ **next slide**

The many Feynman integrals may be reduced to 'few' master integrals by sophisticated methods (Remiddi-Laporta algorithm, 1996/2000 $\rightarrow$ IdSolver (Czakon 2003)).

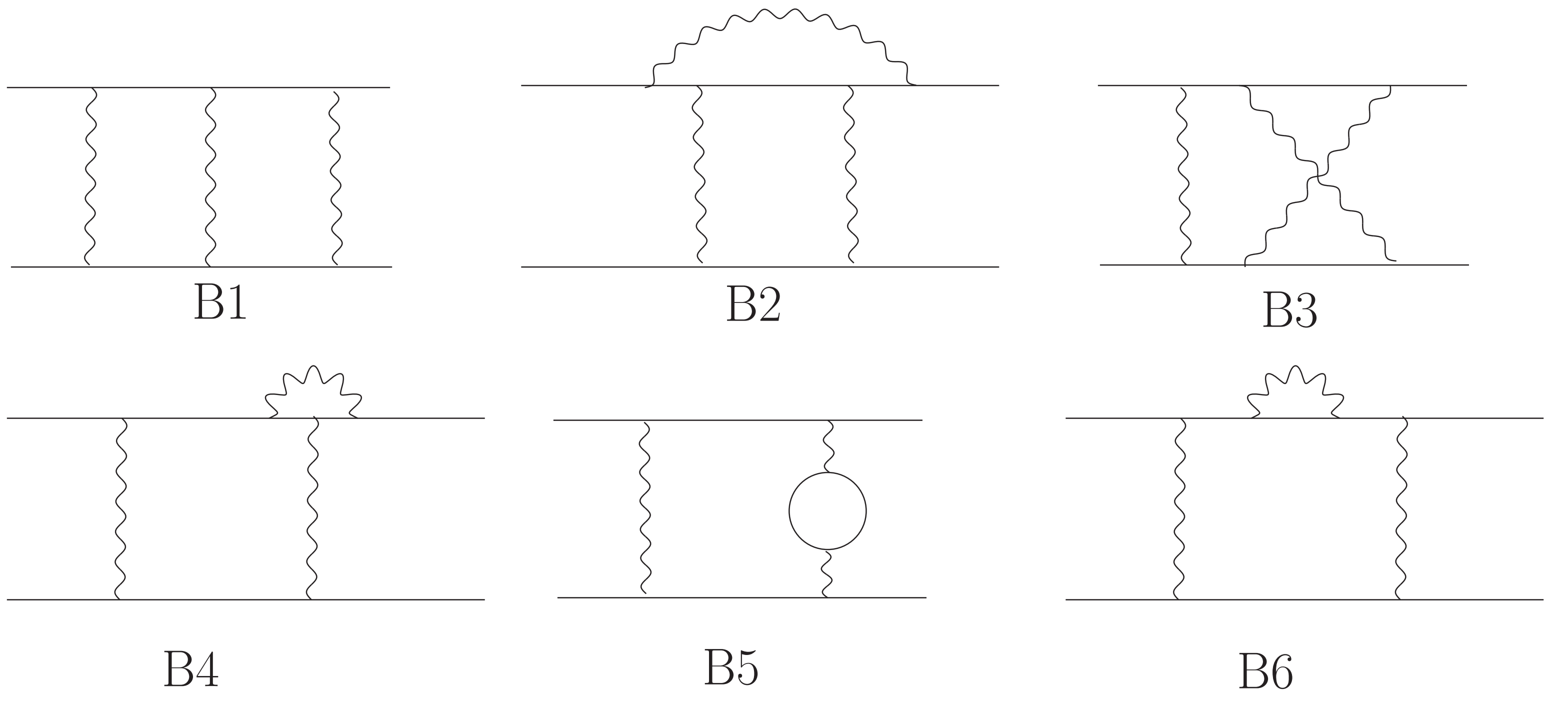
The massive diagrams (See also webpage)

Assume 3 leptonic flavors, do not look at loops in external legs.

Not too many QED diagrams:

- **Born diagrams: 2**
- **1-loop diagrams: 14**
- **2-loop diagrams: 154 (with 68 double-boxes)** **interfere with Born**

The two-loop box diagrams for massive Bhabha scattering



- B5: Completely known (2004)
  Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij: hep-ph/0405275, hep-ph/0411321
  Czakon, Gluza, Riemann: *http://www-zeuthen.desy.de/.../MastersBhabha.m* (unpubl.)
- B1–B3: Few masters known (Smirnov, Heinrich 2002,2004)
- B4, B6: Not much known (Czakon et al. 2004)

The basic planar 2-box master of B1, B7l4m, was a breakthrough

The two-loop Feynman integrals

One has to solve many, very complicated Feynman integrals with $L = 2$ loops and $N \leq 7$ internal lines:

$$G(X) = \frac{1}{\left(i\pi^{d/2}\right)^2} \int \frac{d^D k_1 d^D k_2 \; X}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_j^2 - m_j^2)^{\nu_j} \ldots (q_N^2 - m_N^2)^{\nu_N}},$$

$$X = 1, (k_1 P), (k_1 k_2), (k_2 P), \cdots$$

where P is some external momentum: $p_1, ..p_4$

A completely numerical approach might be possible Passarino 2004.

For checks in the Euclidean region $(s < 0, t < 0)$ this has been proven to be a powerful tool Binoth, Heinrich 2000/03

We prefer to calculate the integrals analytically (where possible)
Derive a minimal set of so-called master integrals and algebraic expressions in terms of them for all the other Feynman integrals

.

So we need a A table of master integrals

We use IdSolver with the Laporta/Remiddi algorithm:
Derive with integration-by-parts (and Lorentz-invariance) identities a system of *algebraic* equations for the Feynman integrals and solve the system.

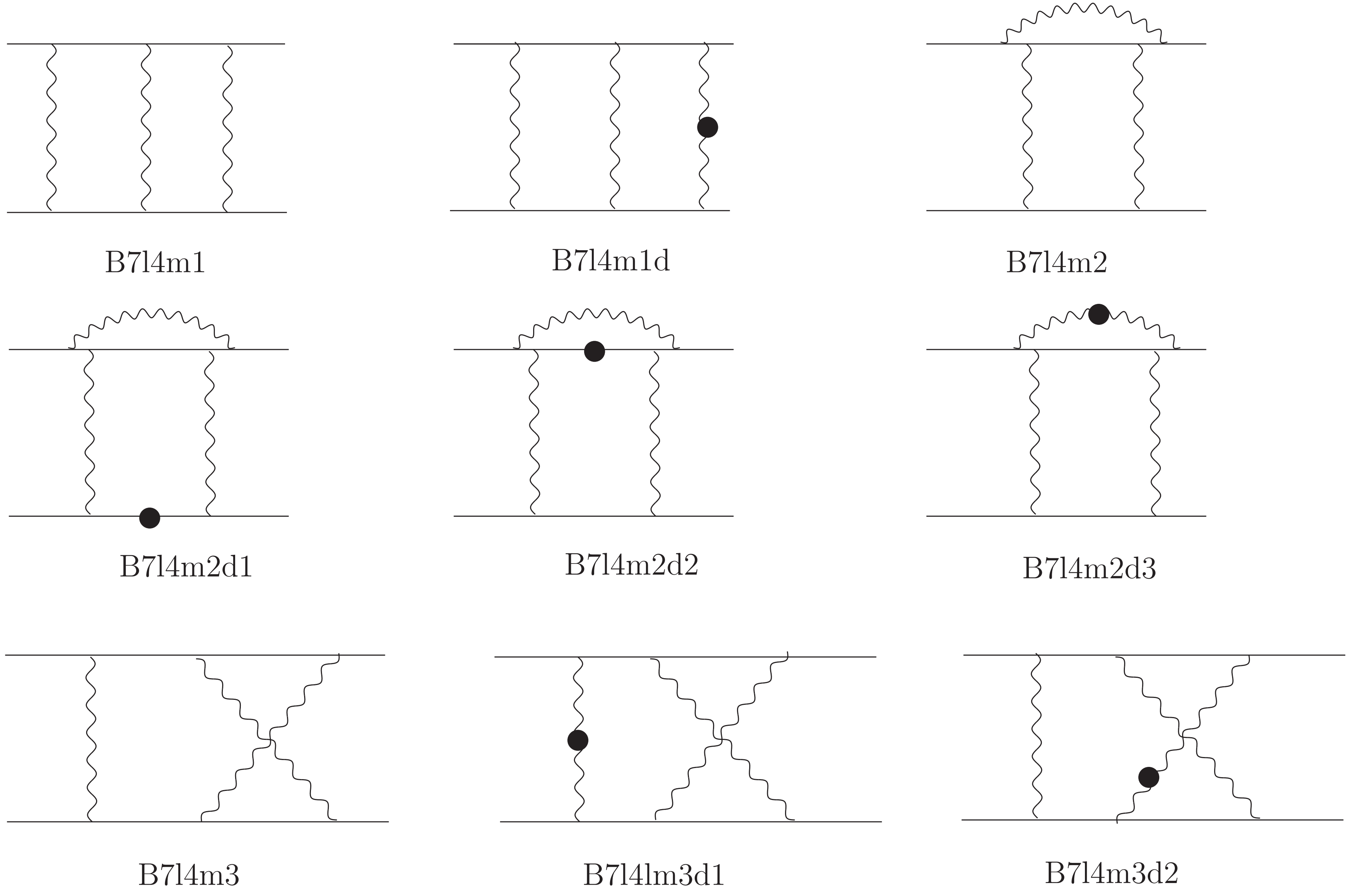
- 1-loop: 5 masters (all known)
- 2-loop self energies: 6 masters (all known)
- 2-loop vertices: 19 masters (all known)
- 2-loop boxes: 33 masters → (O(5) published, maybe more known) see table

The calculation of the master integrals is mainly done with two methods:

- derive and solve (systems of) differential equations (with boundary conditions)
- derive and solve (up to 8-dimensional) Mellin-Barnes integral representations for single Feynman integrals

From Czakon et al., PRD 71 (2004): 4-point MIs entering basic two-loop box diagrams. An asterisk denotes one-loop MI. MIs with a dagger: know singular parts only

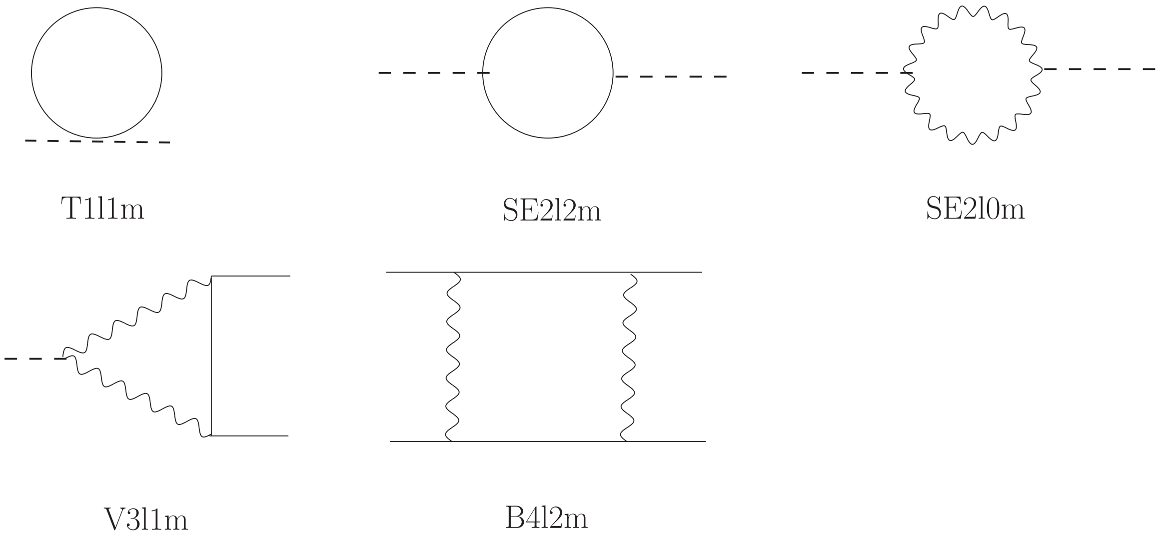
MI	B1	B2	B3	B4	B5	B6	ref.
B714m1	+	−	−	−	−	−	Smirnov:2001cm
B714m1N	+	−	−	−	−	−	Heinrich:2004iq
B714m2	−	+	−	−	−	−	Heinrich:2004iq†
B714m2[d1--d3]	−	+	−	−	−	−	
B714m3	−	−	+	−	−	−	Heinrich:2004iq†
B714m3[d1--d2]	−	−	+	−	−	−	
B613m1	+	−	+	−	−	−	
B613m1d	+	−	+	−	−	−	
B613m2	−	+	−	+	−	−	
B613m2d	−	+	−	+	−	−	
B613m3	−	−	+	−	−	−	
B613m3[d1--d5]	−	−	+	−	−	−	
B512m1	+	−	+	−	−	−	Czakon:2004tg
B512m2	−	+	−	+	−	+	Sec. IIIE1†
B512m2[d1--d2]	−	+	−	+	−	+	Sec. IIIE1†
B512m3	+	−	+	−	−	−	
B512m3[d1--d3]	+	−	+	−	−	−	Sec. IIIE1†
B513m	−	+	+	+	−	−	
B513m[d1--d3]	−	+	+	+	−	−	
B514m	−	+	+	+	+	−	Bonciani:2003cj
B514md	−	+	+	+	+	−	Sec. IIIE
B412m*	−	−	−	+	+	+	'tHooft:1972fi,Bonciani:2003cj
total = 33+1*	9	15	22	11+1*	2+1*	3+1*	



The nine two-loop box MIs with seven internal lines.

The simplest diagram is the tadpole:

$$\begin{aligned} \texttt{T1l1m} &= \frac{e^{\epsilon\gamma_E}}{i\pi^{D/2}} \int \frac{d^D q}{q^2 - 1} \\ &= \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right)\epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right)\epsilon^2 + \ldots \end{aligned}$$



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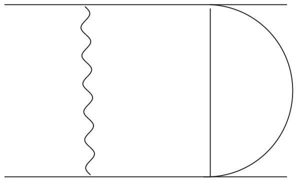
How to calculate 2-loop Bhabha masters?

- **Self-energies and vertices and (very few) 2-boxes:**
  use differential equations and Harmonic Polylogarithms, introduced by Remiddi, Vermaseren, plus . . .)
- **Some 7-line 2-boxes**
  use Mellin-Barnes technique, sum up multiple series, use numerical checks in Euclidean space (s, t negative)
- **For the unsolved 2-boxes:**
  Combination of both methods: present study

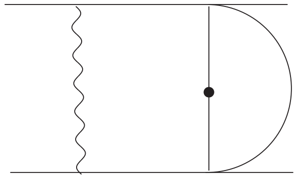
There are other methods not used here:
difference equations
pure numerical approaches
. . .

The 2-boxes with 5 lines

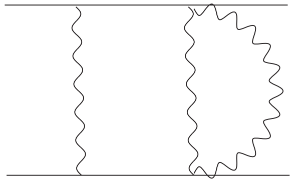
The completely known 2-boxes with 5 lines are B5l4m (Bonciani et al., Czakon et al. 2004), B5l2m1 (Czakon et al. 2004) :



B5l4m1

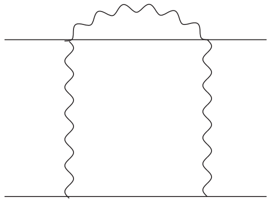


B5l4m1d1

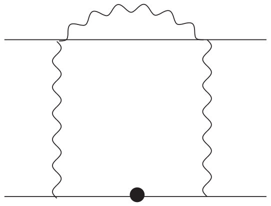


B5l2m1

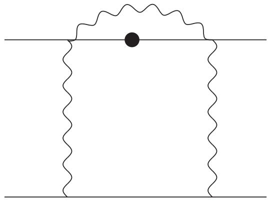
The divergent parts of the B5l2m2 and B5l2m3 type are known (Czakon et al. 2004):



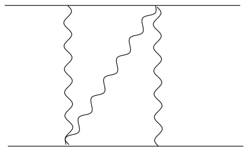
B5l2m2



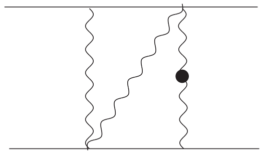
B5l2m2d1



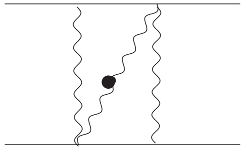
B5l2m2d2



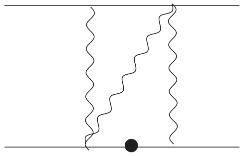
B5l2m3



B5l2m3d1

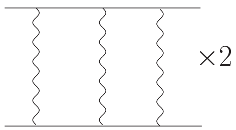


B5l2m3d2

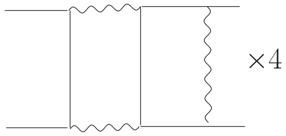


B5l2m3d3

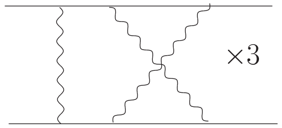
B5l3m: The divergences in $D-4=-2\epsilon$



B7l4m1

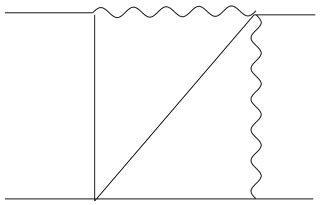


B7l4m2

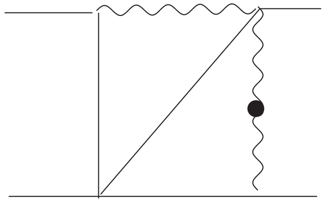


B7l4m3

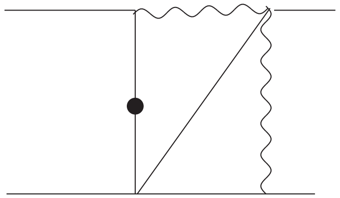
The B5l3m boxes, contribute to B2 (2nd planar 2-box) (shrink two lines...)



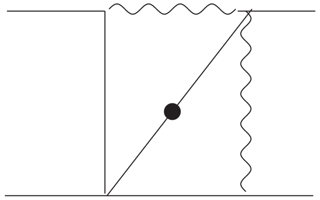
B5l3m



B5l3md1



B5l3md2



B5l3md3

The B5l3md2 topology appears twice as a master
but the B5l3md1 does not!

The B5l3m topology: Gross features

$$MB5l3m[x,y] = Sum[B5l3m[k,x,y]*ep^k, k, 0, 1]; \quad (1)$$
$$MB5l3md1[x,y] = Sum[B5l3md1[k,x,y]*ep^k, k, -2, 1]; \quad (2)$$
$$MB5l3md2[x,y] = Sum[B5l3md2[k,x,y]*ep^k, k, -2, 1]; \quad (3)$$
$$MB5l3md2a[x,y] = Sum[B5l3md2a[k,x,y]*ep^k, k, -2, 1]; \quad (4)$$
$$MB5l3md3[x,y] = Sum[B5l3md3[k,x,y]*ep^k, k, -1, 1]; \quad (5)$$

Note:

- B5l3m – the basic master is finite
- B5l3md2 – use 4-dim. MB-Representation
- B5l3md2' – the same, but $(s \leftrightarrow t)$
- B5l3md1, B5l3md3 – system of 2 coupled differential eqns
  Only BLB5l3md1 has $1/\epsilon^2$ (so decouples), and last step is the two $1/\epsilon$ coefficients of B5l3md1 and B5l3md3.
  The first one is found by algebraic manipulations (see Czakon et al. LCWS Paris 2004), the second then fulfils a diff.eqn

Differential equations

$$\frac{\partial B5l3md3[-1]}{\partial x} = \frac{1+x^2}{x(1-x^2)} B5l3md3[-1] - \frac{yH[0,y]}{(1-x^2)(1-y^2)} \tag{6}$$

with $s = -(1-x)^2/x$, $t = -(1-y)^2/y$

Solution:

$$B5l3md3[-1] = -\frac{xy}{(-1+x^2)(-1+y^2)} H[0,x]H[0,y] \tag{7}$$

with

$$H[0,x] = \ln(x) \tag{8}$$

The coefficients in the equation are of the form

$$\frac{A_1}{x-B_1} + \frac{A_2}{x-B_2} + .. \tag{9}$$

One may derive (systems of) differential equations for the masters, with inhomogeneity composed of simpler masters (Kotikov, Laporta, Remiddi)

$$\frac{\partial M_n}{\partial x} = A(x,y)\ M_n + I(x,y) \tag{10}$$

$$I(x,y) = \sum_{k=0,n-1} c_k\ M_k \tag{11}$$

Expand in ϵ $(D = 4 - 2\epsilon)$:

$$M_n = \sum_{i=-2,i_m} M_{n,i}\epsilon^i \qquad \text{etc.} \tag{12}$$

General solution for homogeneous eqn. $(M_h' = A\ M_h)$:

$$M_h'/M_h = A \tag{13}$$

$$\int (M_h'/M_h) = \ln M_h = \int A \tag{14}$$

$$= \int \sum \frac{a_i}{x - x_i} \sim \ln(x - x_i) \tag{15}$$

so:

$$M_h \sim \text{Polynomials} \tag{16}$$

Then the inhomogeneous solution is:

$$M(x,y) = M_h(x,y)\left(\text{const(y)} + \int \frac{I(x',y)}{M_h(x',y)}\right) \tag{17}$$

Result:

nested integrals over 'simple' iterated integrands

The method leads to the HPLs $H(\{a\},x)$ and GPLs $G(\{a(y)\},x)$

Harmonic Polylogarithms $H(x)$

$$H[-1,1,x] = \int_0^x \frac{dx''}{(1+x'')} \int_0^{x''} \frac{dx'}{(1-x')} \qquad (18)$$

$$= Li_2\left(\frac{1+x}{2}\right) + \ldots \qquad (19)$$

Generalized Harmonic Polylogarithms $G(x,y)$ ...

but it works only if the polynomial structure is simple enough for a solution with this class of functions

Method is absolutely 'super' if it works.

But:
one needs complete chains of masters of lower complexity, and there are systems of up to 6 (!) potentially coupled 1st order equations

.

Mellin-Barnes representations

Boos, Davydychev 1991, Smirnov 1999, Tausk 1999, Smirnov book 2004

$$\frac{1}{(A+B)^{\nu}} = \frac{B^{-\nu}}{(1-(-A/B))^{-\nu}} = \frac{B^{-\nu}}{2\pi i\Gamma(\nu)} \int\limits_{-i\infty}^{i\infty} d\sigma A^{\sigma} \; B^{-\sigma} \; \Gamma(-\sigma)\Gamma(\nu+\sigma) \tag{20}$$

Is special case of a well-known Mellin-Barnes integral for hypergeometric functions

$$\frac{1}{(1-z)^{\nu}} = {}_2F_1(\nu, b, b', z)|_{b=b'} \tag{21}$$

$$= \frac{1}{2\pi i\Gamma(\nu)} \frac{\Gamma(b')}{\Gamma(b)} \int_{-i\infty}^{+i\infty} d\sigma (-z)^{\sigma} \Gamma(\nu+\sigma)\Gamma(-\sigma) \frac{\Gamma(b+\sigma)}{\Gamma(b'+\sigma)} \tag{22}$$

with $-z = A/B$**.**

How can this be made useful here?

Introduce Feynman parameters

The momentum integrals of a Feynman diagram may be performed with Feynman parameters, one for each line.
In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:
allow for propagators with indices, $1/(k_1^2 - m_1^2)^{a_1}$ etc.

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int\limits_0^\infty \Pi_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}} \tag{23}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F = [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \tag{24}$$

$$+(m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5 \tag{25}$$

We want to apply now:

$$\int_0^1 \prod_i^4 dx_i \; x_i^{\alpha_i-1} \; \delta\left(1-x_1-x_2-x_3-x_4\right) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma\left(\alpha_1+\alpha_2+\alpha_3++\alpha_4\right)} \tag{26}$$

with coefficients α_i dependent on a_i and on F

For this, we have to apply several MB-integrals here.
And repeat the procedure for the 2nd subloop.

For the 2nd planar 2-box, B7l4m2, one gets (Smirnov book 4.73):

$$B_{\mathsf{pl},2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[\frac{m^2}{-s}\right]^{z_5+z_6} \left[\frac{-t}{-s}\right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})} \tag{27}$$

with $a = a_1 + ... + a_7$ and

$$z_i = \text{const} + i\,\Im m(z_i) \tag{28}$$

$$d = 4 - 2\epsilon \tag{29}$$

$$\text{const} = \frac{(i\pi^{d/2})^2(-1)^a(-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d-a_{4567})} \tag{30}$$

The integrand includes e.g.:

$$\Gamma_2 = \Gamma(-z_2) \tag{31}$$

$$\Gamma_4 = \Gamma(-z_4) \tag{32}$$

$$\Gamma_7 = \Gamma(a_4 + z_2 + z_4) \tag{33}$$

$$\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4) \tag{34}$$

$$\ldots \tag{35}$$

We now derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and $a_4 = 0$.
The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0 \tag{36}$$

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

$\to$ Start with the z_2, z_4 integrations by
taking the residues for closing the integration contours to the right:

$$I_{2,4} = \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \tag{37}$$

$$= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,..} \frac{-(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) \tag{38}$$

$$= \sum_{n,m=0,1,..} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \to_{a=0} 1 \tag{39}$$

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$
we got a 4-fold Mellin-Barnes integral for B5l3m
with $24 - 3 = 21$ z_i-dependent Γ-functions which may yield residua within four-fold sums.

As mentioned:

This formula has to be calculated now explicitely for the case

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \tag{40}$$

(B5l3md2 is a dotted master, with index $a_2 = 2$, all others are one)

Next tasks:

- Find a region of definiteness of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \tag{41}$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

Here this means:

$$
\begin{aligned}
B5l3md2 \quad &\rightarrow \quad MB(\text{4-dim,fin}) + MB_3(\text{3-dim,fin}) && (42)\\
&+ \quad MB_{36}(\text{2-dim}, \epsilon^{-1}, fin) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1,fin}) && (43)\\
&+ \quad MB_5(\text{3-dim,fin}) && (44)
\end{aligned}
$$

After these preparations e.g.:

$$
\begin{aligned}
MB_{365}(\text{1-dim}, \epsilon^{-2}) \quad &\sim \quad \frac{1}{\epsilon^2}\int dz_6 \frac{(-s)^{(z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} && (45)\\
&\sim \quad \frac{1}{\epsilon^2}\sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} && (46)\\
&= \quad -\frac{1}{\epsilon^2}\frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} && (47)\\
&= \quad \frac{1}{\epsilon^2}\frac{x}{4(1-x^2)}H[0,x] && (48)
\end{aligned}
$$

Here were residua at $z_6 = -n-1, n = 0, 1, ..$ taken

The divergent parts of the masters B5l3m are:

```
B5l3m[-2,x_,y_] = B5l3m[-1,x_,y_] = 0;


B5l3md1[-2,x_,y_] = ((-1 + x)^2*y*(-1 + y^2 + 2*y*H[0, y]))/
                    (8*x*(-1 + y)*(1 + y)^3);
B5l3md1[-1,x_,y_] = ((y*(6*(-1 + x - x^2 + x^3)*H[0, x]*
  (-1 + y^2 + 2*y*H[0, y]) - 6*(1 + x)*(-2 - 2*x^2 + 2*y^2 + 2*x^2*y^2 +
  y*z2 - 2*x*y*z2 + x^2*y*z2 + 2*(-2*x - y + 2*x*y - x^2*y - 2*x*y^2 +
  (-1 + x)^2*y*H[-1, -y] + 3*(-1 + x)^2*y*H[-1, y])*H[0, y] -
  6*(-1 + x)^2*y*H[0, -1, y] - 4*y*H[0, 0, y] + 8*x*y*H[0, 0, y] -
  4*x^2*y*H[0, 0, y] + 2*y*H[0, 1, y] - 4*x*y*H[0, 1, y] +
  2*x^2*y*H[0, 1, y])))/(24*x*(1 + x)*(-1 + y)*(1 + y)^3));


B5l3md2[-2,x_,y_] = -x/(1 - x^2)/4 H[0, x];
B5l3md2[-1,x_,y_] = ((x*(2*(1 + y^2)*H[0, x]*H[0, y] -
   (-1 + y^2)*(z2 + 6*H[-1, 0, x] - 4*H[0, 0, x] -
       2*H[1, 0, x])))/(4*(-1 + x^2)*(-1 + y^2)));


B5l3md2a[-a,x_,y_] = B5l3md2[-a,y,x],   a=-2,-1;


B5l3md3[-2,x_,y_] = 0;
B5l3md3[-1,x_,y_] = -((x*y*H[0, x]*H[0, y])/((-1 + x^2)*(-1 + y^2)));
```

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Summary

Recent essential progress for the massive 2-box master integral determination:

- A complete list of masters (2004)
- Huge files with algebraic relations for all the reducible Feynman integrals needed for the interferences of boxes with Born (not complete but fully understood)
- Underway: Determination of all 2-box masters in a systematic approach
  use Generalized Harmonic Polylogarithms, introduced by Remiddi, Vermaseren
  plus . . .)
- An unsolved problem is the systematic summation of the massive multiple sums after the MB-integral evaluation

Most important conclusion from the efforts of the last year:

It is possible to do the complete massive 2-loop calculation with present computers and human resources.