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ANALYTIC ε-EXPANSION OF THE SCALAR ONE-LOOP BHABHA BOX FUNCTION * **

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We derive the first three terms of the ε-expansion of the scalar one-loop Bhabha box function from a representation in terms of three generalized hypergeometric functions, which is valid in arbitrary dimensions.

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1. Introduction

One of the important problems in perturbative calculations is a precise determination of the cross section for Bhabha scattering. For this one has to determine the electroweak one-loop corrections in the Standard Model and some parametric enhanced contributions plus the complete photonic corrections to even higher orders. Here we are interested in a determination of photonic $\mathcal{O}(\alpha^2)$ corrections for this process in $d = 4 - 2\varepsilon$, $\varepsilon \to 0$, dimensions with account of the electron mass m as a regulator of infrared singularities. These corrections naturally concern the virtual two-loop matrix element, which contributes to the cross section due to its interference with the Born matrix element. Of the same order is the absolute square of the one-loop amplitude M_1. The corresponding cross section contributions have been analytically determined recently [1].

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A peculiarity of the contribution from M_1 is the necessity to determine this amplitude as a function of the parameter ε up to terms of order ε: $M_1(\varepsilon) = m_1/\varepsilon + m_0 + m_1\varepsilon$. In a series of papers, the possibility was studied to find some closed analytical expressions for one-loop 2-, 3- and 4-point functions for arbitrary dimension, external momenta and masses (in principle also complex ones) in terms of generalized hypergeometric functions with relatively simple integral representations [2–4]. The results immediately apply to a deduction of the coefficient m_1. Beyond that this representation is in particular of great importance for the development of efficient algorithms for the calculation of 5-, 6- and higher point functions since these functions may be reduced to 4- and lower point functions with "unphysical" external kinematics.

In this contribution, we explicitely perform the ε-expansion of the most complicated part: The scalar one-loop box function I_{1111} with two photons (taken here in the s-channel), as it is needed for the calculation of Bhabha scattering up to order $\mathcal{O}(\varepsilon)$. Our starting point is the analytical expression as known from [3, 4]:

$$\begin{aligned}
&\frac{(t-4m^2)}{\Gamma\left(2-\frac{d}{2}\right)} I_{1111}^{(d)} \\
&= \frac{t-4m^2}{i\pi^{d/2}\Gamma\left(2-\frac{d}{2}\right)} \int \frac{d^d k_1}{k_1^2(k_1^2+2q_4k_1)(k_1+q_1+q_4)^2(k_1^2-2q_3k_1)} \\
&= -\frac{4m^{d-4}}{s} F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\frac{t}{t-4m^2},-m^2Z\right) \\
&\quad + \frac{4m^{d-4}}{(d-3)s} F_{1;1;0}^{1;2;1}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2}, & 1; & 1; \\ \frac{d-1}{2}: & & \frac{d-2}{2}; & -;\end{matrix}\; -m^2Z, 1-\frac{4m^2}{s}\right] \\
&\quad - \frac{\sqrt{\pi}(-s)^{\frac{d-4}{2}}}{2^{d-4}m\sqrt{s}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} F_1\left(\frac{d-3}{2},1,\frac{1}{2};\frac{d-1}{2};\frac{sZ}{4},1-\frac{s}{4m^2}\right)
\end{aligned} \tag{1.1}$$

with $Z = \frac{4u}{s(4m^2-t)}$, $q_i^2 = m^2$, $(q_1+q_4)^2 = s$, $(q_1+q_2)^2 = t$ and s, t, u being the usual Mandelstam variables. Here F_1, F_2 are Appell hypergeometric functions

$$F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}} \frac{\left(\frac{1}{2}\right)_s}{(1)_s}\, x^r y^s, \tag{1.2}$$

$$F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};x,y\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{3}{2}\right)_r\left(\frac{d-2}{2}\right)_s}\, x^r y^s \tag{1.3}$$

and the Kampé de Fériet function (KdF) [5] is defined as

$$F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2},\ 1; & 1;\\ \frac{d-1}{2}: & \frac{d-2}{2}; & -;\end{matrix}\ x,y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}} \frac{\left(\frac{d-3}{2}\right)_{r}}{\left(\frac{d-2}{2}\right)_{r}}\, x^r y^s\,. \tag{1.4}$$

The ε-expansion of the generalized hypergeometric functions occurring above is not quite straight forward. In particular since there stands the factor $\Gamma\left(2-\frac{d}{2}\right) \sim \frac{1}{\varepsilon}$ in front of all of them, one needs their expansion up to order ε^2. We have to develop different techniques for each of them.

2. Expansion of F_1

We need to know the expansion

$$F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) = F_1^0 + \varepsilon F_1^1 + \varepsilon^2 F_1^2 + \cdots \tag{2.1}$$

with the kinematics: $x = -\frac{u}{t-4m_e^2} < 0, y = 1 - \frac{s}{4m_e^2} < 0, |y| \gg 1$. In two steps we obtain a form in which one of the parameters of $F_1 \sim \varepsilon$. The transformations are the following:

$$F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right) = 2\frac{\Gamma(\frac{d-1}{2})\Gamma(\frac{6-d}{2})}{\Gamma(\frac{1}{2})(-y)^{\frac{d-3}{2}}}\,{}_2F_1\left[1,\frac{d-3}{2},\frac{3}{2};1-z\right]$$
$$+\ \frac{d-3}{(d-6)(-x)\sqrt{-y}}F_1\left(\frac{6-d}{2},1,\frac{1}{2},\frac{8-d}{2};\frac{1}{x},\frac{1}{y}\right) \tag{2.2}$$

$\left(\frac{x}{y} \equiv z = 1 - \frac{st}{(s-4m_e^2)(t-4m_e^2)}; 0 < z, 1-z < 1\right)$ and

$$F_1\left(\frac{6-d}{2},1,\frac{1}{2},\frac{8-d}{2};\frac{1}{x},\frac{1}{y}\right) = \frac{\Gamma(\frac{1}{2})\ \Gamma(\frac{8-d}{2})}{\Gamma(\frac{7-d}{2})}y^{\frac{6-d}{2}}\ {}_2F_1\left[1,\frac{6-d}{2},\frac{7-d}{2};\frac{1}{z}\right]$$
$$+\frac{(d-6)\sqrt{X-1}\ (Y-X)}{\sqrt{X}}F_1\left(1,\frac{d-4}{2},1,\frac{3}{2};X,Y\right) \tag{2.3}$$

with $X = 1-y = \frac{s}{4m_e^2} \gg 1, Y = \frac{y-1}{x-1} = 1 - \frac{t}{4m_e^2} \gg 1$ $(X > Y,\ 1-\frac{1}{Y} = \omega)$. Here we observe that the argument of the ${}_2F_1$ as well as those of the F_1 function are larger than 1, i.e. both functions are complex and the imaginary parts must cancel since the F_1 on the l.h.s. has arguments less than 0 and thus is real. For the imaginary part of the F_1 function we obtain

$$\mathrm{Im}\ F_1\left(1,\frac{d-4}{2},1,\frac{3}{2};X,Y\right) = \frac{\pi^{\frac{3}{2}}\sqrt{X}(X-1)^{\frac{3-d}{2}}}{2Y\Gamma(\frac{5-d}{2})\Gamma(\frac{d-2}{2})}\ {}_2F_1\left[1,\frac{d-3}{2},\frac{d-2}{2};z\right]. \tag{2.4}$$

Transforming ${}_2F_1\left[1,\frac{6-d}{2},\frac{7-d}{2};\frac{1}{z}\right]$ with argument $\frac{1}{z}>1$ to a ${}_2F_1$ function with argument $z<1$, one shows that the imaginary parts cancel. Transforming (the real part) further to the argument $1-z$ one finally obtains

$$\begin{aligned}F_1\left(\frac{d-3}{2},1,\frac{1}{2},\frac{d-1}{2};x,y\right)&=-(d-3)\frac{Y}{\sqrt{X}}\ \mathrm{Re}\ F_1\left(1,\frac{d-4}{2},1,\frac{3}{2};X,Y\right)\\ &+(d-3)\frac{\Gamma(\frac{d-3}{2})\Gamma(\frac{6-d}{2})}{\Gamma(\frac{1}{2})(-y)^{\frac{d-3}{2}}}\sin^2\left(\pi\frac{d}{2}\right)\ {}_2F_1\left[1,\frac{d-3}{2},\frac{3}{2};1-z\right]\\ &-(d-3)\frac{\pi}{2}\sin\left(\pi\frac{d}{2}\right)(-x)^{-\frac{d-4}{2}}\frac{1}{\sqrt{-y(1-z)}}\,. \end{aligned}\tag{2.5}$$

To expand up to the required order ($\sim\varepsilon^2$), it is sufficient to set $\sin\left(\pi\frac{d}{2}\right)=-\pi\varepsilon$ and

$$ {}_2F_1\left[1,\frac{d-3}{2},\frac{3}{2};1-z\right]=\frac{1}{2\sqrt{1-z}}\ln\left(\frac{1+\sqrt{1-z}}{1-\sqrt{1-z}}\right)+O(\varepsilon)\,. \tag{2.6}$$

This simplifies the expansion considerably and the Re $F_1\left(1,\frac{d-4}{2},1,\frac{3}{2};X,Y\right)$ we take from [4]. Characteristic variables appearing in the result are

$$A=\frac{\sqrt{1-\frac{1}{X}}-1}{\sqrt{1-\frac{1}{X}}+1}<0\,,\ B=\frac{\sqrt{1-\frac{1}{Y}}-1}{\sqrt{1-\frac{1}{Y}}+1}\ <\ 0 \tag{2.7}$$

and introducing $a=\sqrt{1-\frac{1}{X}}$, $b=\sqrt{1-\frac{1}{Y}}$ we can write ($1>a>b>0$) $A=\frac{a-1}{a+1}$, $B=\frac{b-1}{b+1}$ and

$$F_1^0=-\frac{m_e}{\sqrt{s}}\frac{1}{b}\ln(-B)\,, \tag{2.8}$$

yielding the correct $\frac{1}{\varepsilon}$-term of D_0 [6]. Keeping only the leading terms, collecting the contributions, yields correspondingly

$$\begin{aligned}&\frac{2}{s(t-4m_e^2)}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\varepsilon)}\Gamma(1-\varepsilon)\left(-\frac{s}{4}\right)^{-\varepsilon}\Gamma(\varepsilon)\frac{1}{b}\Bigg[\mathrm{Re}\left\{\ln(B)+\cdots\right\}\\ &-\pi^2\varepsilon^2\ln\left(\frac{1-AB}{A-B}\right)-\pi^2\varepsilon\left(1+\varepsilon\ln\left(\frac{X}{Y}-1\right)\right)+O(\varepsilon^3)\Bigg]\,,\end{aligned}\tag{2.9}$$

where the higher order terms in ε of $\mathrm{Re}\left\{\ln(B)+\cdots\right\}$ have to be taken from [4].

3. Expansion of F_2

We need to know the expansion

$$F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\omega,z\right) = F_2^0 + \varepsilon F_2^1 + \varepsilon^2 F_2^2 + \cdots \qquad (3.1)$$

with the kinematics: $\omega = \frac{t}{t-4m_e^2}, z = -4m_e^2\left(\frac{1}{s}+\frac{1}{t-4m_e^2}\right)$. At first we perform the following Euler transformation:

$$\begin{aligned} &F_2\left(\frac{d-3}{2},1,1,\frac{3}{2},\frac{d-2}{2};\omega,z\right) \\ &= (1-z)^{\frac{3-d}{2}} F_2\left(\frac{d-3}{2},1,\frac{d-4}{2},\frac{3}{2},\frac{d-2}{2};\frac{\omega}{1-z},-\frac{z}{1-z}\right). \end{aligned} \qquad (3.2)$$

The factor $(1-z)^{-\frac{d-3}{2}}$ will be dropped in what follows and is taken into account again when collecting the results. Introducing $\frac{\omega}{1-z} = \omega'$ and $-\frac{z}{1-z} = z'$, we have

$$0 \le \omega' + z' < 1\,. \qquad (3.3)$$

With $\alpha = \frac{1}{2}-\varepsilon, \beta = 1,\ \beta' = -\varepsilon,\ \gamma = \frac{3}{2}$ and $\gamma' = 1-\varepsilon$

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',\omega',z') = {}_2F_1(\alpha,\beta,\gamma;\omega') + \beta' S(\alpha,\beta,\beta',\gamma,\omega',z')\ , \qquad (3.4)$$

where $\gamma' = 1+\beta'$ has been used and

$$S(\alpha,\beta,\beta',\gamma,\omega',y) = \sum_{n=1}^{\infty} \frac{(\alpha)_n}{\beta'+n}\frac{y^n}{n!}{}_2F_1(\alpha+n,\beta,\gamma;\omega')\,. \qquad (3.5)$$

In order to get rid of the denominator $\beta'+n$ we differentiate S w.r.t. y and use [7]

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\delta)_n}{n!(\delta')_n} y^n {}_2F_1(\alpha+n,\beta,\gamma;\omega') = F_2(\alpha,\beta,\delta,\gamma,\delta';\omega',y) \qquad (3.6)$$

with $\delta = \delta'$. Applying again the Euler relation,

$$\begin{aligned} F_2\left(\alpha,\beta,\delta,\gamma,\delta;\omega',y\right) &= (1-y)^{-\alpha} F_2\left(\alpha,\beta,0,\gamma,\delta;\frac{\omega'}{1-y},-\frac{y}{1-y}\right) \\ &= (1-y)^{-\alpha}{}_2F_1\left(\alpha,\beta,\gamma;\frac{\omega'}{1-y}\right), \end{aligned} \qquad (3.7)$$

we finally have ($\beta = 1, \gamma = \frac{3}{2}$ inserted)

$$S(\alpha,\beta',\omega',z') = \int\limits_{y=0}^{z'} \frac{\partial S(\alpha,\beta',\omega',y)}{\partial y} dy$$
$$= S_0\left(\frac{1}{2},0,\omega',z'\right) + \varepsilon S_1\left(\frac{1}{2},0,\omega',z'\right) + O(\varepsilon^2), \quad (3.8)$$

with

$$\frac{\partial S}{\partial y} = \frac{1}{y}\left[\frac{1}{(1-y)^\alpha}\, {}_2F_1\left(\alpha,1,\frac{3}{2};\frac{\omega'}{1-y}\right) - {}_2F_1\left(\alpha,1,\frac{3}{2};\omega'\right)\right]$$
$$+\frac{\varepsilon}{y} S\left(\frac{1}{2},0,\omega',y\right) + O(\varepsilon^2)\,. \quad (3.9)$$

To complete the ε-expansion, we need (with $z = \frac{\omega'}{1-y}$)

$${}_2F_1\left(\alpha,1,\frac{3}{2};z\right) = {}_2F_1\left(\frac{1}{2},1,\frac{3}{2};z\right)$$
$$+\varepsilon\delta^{(1)}F(z,u) + \varepsilon^2\delta^{(2)}F(z,u) + \cdots \quad (3.10)$$

with the abbreviation $u = \frac{1+\sqrt{z}}{1-\sqrt{z}}$. Explicitely $u = \frac{\sqrt{1-y}+w}{\sqrt{1-y}-w}$ with $w = \sqrt{\omega'}$,

$$\delta^{(1)}F(z,u) = \frac{1}{2\sqrt{z}}\Big[2\mathrm{Li}_2\left(-\frac{1}{u}\right) - 2\ln(u)\ln(1+u) + \frac{3}{2}\ln^2(u) + \zeta(2)\Big] \quad (3.11)$$

and

$$\delta^{(2)}F(z,u) = \frac{1}{2\sqrt{z}}\left[-4S_{1,2}\left(-\frac{1}{u}\right) - \left(\ln(u) + 2\ln\left(1+\frac{1}{u}\right)\right)\zeta(2)\right.$$
$$-4\ln\left(1+\frac{1}{u}\right)\mathrm{Li}_2\left(-\frac{1}{u}\right) + 2\ln(u)\ln^2\left(1+\frac{1}{u}\right) + \ln^2(u)\ln\left(1+\frac{1}{u}\right)$$
$$\left.+\frac{1}{6}\ln^3(u) + 2\zeta(3) + 2\mathrm{Li}_3\left(-\frac{1}{u}\right)\right]\,. \quad (3.12)$$

3.1. Order ε of F_2

In this order we have

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',\omega',z') = {}_2F_1\left(\frac{1}{2},1,\frac{3}{2};\omega'\right)$$
$$+\varepsilon\delta^{(1)}F(\omega',u_0) - \varepsilon S\left(\frac{1}{2},0,\omega',z'\right) + O(\varepsilon^2),$$
$$\equiv F_2^0 + \varepsilon F_2^1 + O(\varepsilon^2)\,, \quad (3.13)$$

where the "scale" $u_0 = u(y=0) = \frac{1+w}{1-w} \sim \frac{s}{m_e^2}\frac{1}{1-\frac{4m_e^2}{t}}$. u_0 is large for $s \gg m_e^2$ and $-t \gg 4m_e^2$ and sets the scale for the variable u in general. Further we introduce $u_1 = u(y=z') < u_0$. Thus we can write

$$\begin{aligned}
F_2^0 &= \frac{1}{2w}\ln(u_0), \\
S\left(\frac{1}{2},0,\omega',y\right) &= \frac{1}{2w}\int\limits_{y=0}^{y}\frac{1}{y}\ln\left(\frac{u}{u_0}\right)dy \\
&= \frac{1}{2w}\int\limits_{u}^{u_0}\left[\frac{2}{u-1}-\frac{1}{u-u_0}-\frac{1}{u-\frac{1}{u_0}}\right]\ln\left(\frac{u}{u_0}\right)du \\
&\equiv \frac{1}{2w}S_0(u_0,u)
\end{aligned} \tag{3.14}$$

and $S(\frac{1}{2},0,\omega',z') = \frac{1}{2w}S_0(u_0,u_1)$. The integration yields:

$$\begin{aligned}
S_0(u_0,u) &= 2\mathrm{Li}_2\left(\frac{1}{u}\right)+\mathrm{Li}_2\left(\frac{u}{u_0}\right)-\mathrm{Li}_2\left(\frac{1}{u_0u}\right)+2\mathrm{Li}_2\left(-\frac{1}{u_0}\right)-\zeta(2) \\
&-\ln\left(\frac{u}{u_0}\right)\left[2\ln(u-1)-\ln(u_0u-1)-\ln\left(1-\frac{u}{u_0}\right)-\frac{1}{2}\ln\left(\frac{u}{u_0}\right)\right].
\end{aligned} \tag{3.15}$$

3.2. Order ε^2 of F_2

The next order can be written in the form

$$F_2^2 = \delta^{(2)}F(\omega',u_0) - S_1\left(\frac{1}{2},0,\omega',z'\right) \tag{3.16}$$

with

$$\begin{aligned}
S_1\left(\frac{1}{2},0,\omega',z'\right) \equiv S_1(u_0,u_1) &= \int\limits_{y=0}^{z'}\frac{dy}{y}\left\{\frac{1}{2w}\ln(1-y)\ln(u)\right. \\
&\left.+\delta^{(1)}F(\omega',u)-\delta^{(1)}F(\omega',u_0)+S\left(\frac{1}{2},0,\omega',y\right)\right\}.
\end{aligned} \tag{3.17}$$

The curly bracket in the above integrand finally reads

$$\{\cdots\} = \mathrm{Li}_2\left(\frac{u}{u_0}\right)-\mathrm{Li}_2(1)+\mathrm{Li}_2\left(\frac{1}{u^2}\right)-\mathrm{Li}_2\left(\frac{1}{u_0u}\right)+2\ln(u)\ln\left(\frac{u_0-1}{u-1}\right)$$

$$-2\ln(u_0+1)\ln\left(\frac{u}{u_0}\right)+\frac{3}{2}\ln(u_0u)\ln\left(\frac{u}{u_0}\right)-\ln\left(\frac{u}{u_0}\right)$$
$$\times\left[2\ln(u-1)-\ln(u_0u-1)-\ln\left(1-\frac{u}{u_0}\right)-\frac{1}{2}\ln\left(\frac{u}{u_0}\right)\right]. \quad (3.18)$$

There is no problem to perform the final integration, but the expressions blow up considerably. Therefore we confine ourselves here to the leading terms only by considering u (u_0) as large and drop the small quantities. Then the integral can be written in the simplified form ($\frac{u}{u_0}=v$ the new integration variable)

$$S_1(u_0,u_1)=\frac{1}{2w}\int\limits_{v=r}^{1}\left[\frac{1}{v}+\frac{1}{1-v}\right]\Big\{\mathrm{Li}_2(v)-\mathrm{Li}_2(1)-\ln^2(v)-\ln(u_0)\ln(v)$$
$$+\ln(v)\ln(1-v)\Big\}=\frac{1}{2w}\Big[-\mathrm{Li}_3(1-r)+\ln(1-r)\mathrm{Li}_2(r)$$
$$+\left(\ln(u_0)+\ln(1-r)\right)\mathrm{Li}_2(1-r)-\ln(r)\mathrm{Li}_2(r)+\frac{1}{3}\ln^3(r)$$
$$+\left(\frac{1}{2}\ln(u_0)-\ln(1-r)\right)\ln^2(r)+\ln^2(1-r)\ln(r)-\zeta(2)\ln\left(\frac{1}{r}-1\right)\Big] \quad (3.19)$$

with $0<r=\frac{u_1}{u_0}<1$. If one wants higher precision, it is easier to expand in $\frac{1}{u}(\frac{1}{u_0})$ instead of performing all integrals analytically, which is possible nevertheless. Collecting the results, we have

$$-\frac{2(m_e^2)^{-\varepsilon}}{s(t-4m_e^2)}\left(1-z\right)^{\varepsilon}\Gamma(\varepsilon)\frac{1}{\sqrt{\omega}}\left(\ln(u_0)+\cdots\right). \quad (3.20)$$

4. Expansion of the Kampé de Fériet function

We need to know the expansion

$$F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2},\ 1; & 1; \\ \frac{d-1}{2}: & \frac{d-2}{2}; & -;\end{matrix}\ x,y\right]=K^0+\varepsilon K^1+\varepsilon^2K^2+\cdots \quad (4.1)$$

with the kinematics: $x=-4m_e^2\left(\frac{1}{s}+\frac{1}{t-4m_e^2}\right), y=1-\frac{4m_e^2}{s}$. In this case we begin with the integral representation of the KdF function:

$$F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}: & \frac{d-3}{2},\ 1; & 1; \\ \frac{d-1}{2}: & \frac{d-2}{2}; & -;\end{matrix}\ x,y\right]$$
$$=\frac{d-3}{2}\int\limits_0^1\frac{dt\ t^{\frac{d-5}{2}}}{1-t\ y}\ {}_2F_1\left(1,\frac{d-3}{2},\frac{d-2}{2},x\ t\right). \quad (4.2)$$

Again we perform a shift such that one of the parameters of the ${}_2F_1 \sim \varepsilon$:

$$
\begin{aligned}
{}_2F_1\left(1, \frac{d-3}{2}, \frac{d-2}{2}, x\ t\right) &= (1-x\ t)^{-\frac{1}{2}}\ {}_2F_1\left(-\varepsilon, \frac{1}{2}, 1-\varepsilon, x\ t\right) \\
&= (1-x\ t)^{-\frac{1}{2}}\left[1-\varepsilon S(x\ t)\right]
\end{aligned}
\tag{4.3}
$$

with

$$
S(x\ t) = \sum_{n=1}^{\infty} \frac{1}{n-\varepsilon} \frac{(\frac{1}{2})_n}{n!} (x\ t)^n \,. \tag{4.4}
$$

The ε-expansion of $S(x\ t)$ can again be obtained by first differentiating S:

$$
\begin{aligned}
\frac{\partial S}{\partial x} &= \frac{1}{x}\left(\frac{1}{\sqrt{1-x\ t}} - 1\right) + \frac{\varepsilon}{x} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(\frac{1}{2})_n}{n!} (x\ t)^n + O(\varepsilon^2) \\
&= \frac{1}{x}\left(\frac{1}{\sqrt{1-x\ t}} - 1\right) + \frac{\varepsilon}{x} S(x\ t)|_{\varepsilon=0} + O(\varepsilon^2) \,.
\end{aligned}
\tag{4.5}
$$

In order to obtain S to $O(\varepsilon)$, we need the following integrals:

$$
S(x\ t)|_{\varepsilon=0} = \int_0^x \frac{dx}{x}\left(\frac{1}{\sqrt{1-x\ t}} - 1\right) = 2\ln\left(1+\frac{1}{v}\right) \tag{4.6}
$$

and

$$
\int_0^x \frac{dx}{x} S(x\ t)|_{\varepsilon=0} = -2\mathrm{Li}_2\left(-\frac{1}{v}\right) - 2\ln^2\left(1+\frac{1}{v}\right) \,, \tag{4.7}
$$

where we introduced $v = \frac{1+\sqrt{1-x\ t}}{1-\sqrt{1-x\ t}}$. Thus the above integral reads

$$
\int_0^1 \frac{dt\ t^{-\frac{1}{2}}}{1-t\ y} \frac{1}{\sqrt{1-x\ t}} \left\{1-\varepsilon\ln\left(\frac{4}{v\ x}\right) - \varepsilon^2\left[-2\mathrm{Li}_2\left(-\frac{1}{v}\right) - \frac{1}{2}\ln^2\left(\frac{4}{v\ x}\right)\right]\right\} . \tag{4.8}
$$

After a variable transformation we can write

$$
\begin{aligned}
\int_0^1 \frac{dt\ t^{-\frac{1}{2}}}{1-t\ y} \frac{1}{\sqrt{1-x\ t}} f(v) &= -\frac{1}{\sqrt{y-x}} \int_0^1 dt \left\{ b_1\left[\frac{1}{1+b_1\ t} + \frac{1}{1-b_1\ t}\right]\right. \\
&\quad \left. -b_2\left[\frac{1}{1+b_2\ t} + \frac{1}{1-b_2\ t}\right]\right\} f\left(\frac{v_1}{t^2}\right)
\end{aligned}
\tag{4.9}
$$

with $v_1 = v(t=1) = \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$, $b_1 = \frac{1}{\sqrt{v_0\ v_1}}$ and $b_2 = \sqrt{\frac{v_0}{v_1}}$ with $v_0 = \frac{1+\sqrt{1-\frac{x}{y}}}{1-\sqrt{1-\frac{x}{y}}}$. v_0 and v_1 both being large, results in $b_1 \ll 1$ and $b_2 < 1$ but very close to 1. Taking again the attitude to keep only leading contributions, the b_1-contribution can be dropped. The Li_2-function in the second order term can be written as

$$\mathrm{Li}_2\left(-\frac{t^2}{v_1}\right) = 2\left[\mathrm{Li}_2\left(i\frac{t}{\sqrt{v_1}}\right) + \mathrm{Li}_2\left(-i\frac{t}{\sqrt{v_1}}\right)\right] \tag{4.10}$$

so that integration is possible. We do get,however, relatively complicated complex conjugate contributions. On the other hand since $v_1 \gg 1$ this contribution is small from the very beginning and can be well approximated by expanding the Li_2-function. Here it is dropped alltogether. Thus we are left with the following contributions:

$$K^0 = \frac{d-3}{2\sqrt{\omega}}\ \ln(u_0)\,, \tag{4.11}$$

$$K^1 = \frac{d-3}{2\sqrt{\omega}}\left[\ln\left(\frac{1+b_2}{1-b_2}\right)\ln\left(\frac{v_1\ x}{4}\right) + 2\left(\mathrm{Li}_2(b_2) - \mathrm{Li}_2(-b_2)\right)\right]\,, \tag{4.12}$$

$$\begin{aligned} K^2 \;=\; & \frac{d-3}{2\sqrt{\omega}}\left[\frac{1}{2}\ln\left(\frac{1+b_2}{1-b_2}\right)\ln^2\left(\frac{v_1\ x}{4}\right) + 2\ln\left(\frac{v_1\ x}{4}\right)\left(\mathrm{Li}_2(b_2) - \mathrm{Li}_2(-b_2)\right)\right. \\ & \left. +4\left(\mathrm{Li}_3(b_2) - \mathrm{Li}_3(-b_2)\right)\right]. \end{aligned} \tag{4.13}$$

Collecting the results we obtain

$$\frac{2(m_e^2)^{-\varepsilon}}{s(t-4m_e^2)}\Gamma(\varepsilon)\frac{1}{\sqrt{\omega}}\left(\ln(u_0) + \cdots\right)\,. \tag{4.14}$$

We see that the $\frac{1}{\varepsilon}$-term cancels against the one from the F_2 contribution.

5. Expansion of the scalar box function with Feynman parameters

In order to have an independent check of the above results, we derived a Feynman parameter integral representation for the ε-expansion. We follow closely [8], where the scalar four-point integral was treated with a finite photon mass in $d = 4$ dimensions.

The function to be calculated is, in LoopTools notations [9]:

$$\begin{aligned} J &= D_0(m^2,m^2,m^2,m^2 \mid t,s \mid m^2,0,m^2,0) \\ &= \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^dk}{k^2(k^2+2kq_4)(k^2-2kq_3)(k+q_1+q_4)^2} \,. \end{aligned} \tag{5.1}$$

A constant transforms the normalization of D_0 to that of $I^{(d)}_{1111}$:

$$D_0 \;=\; (4\pi\mu^2)^{\varepsilon}\; I^{(d)}_{1111}. \tag{5.2}$$

The infrared singularity may be isolated in a 3-point function C_0 by redefining

$$J \;=\; \frac{2}{s}\,(F + C_0)\,, \tag{5.3}$$

with

$$C_0 \;=\; C_0(t,\mu^2,m^2 \mid m^2,\mu^2,0) = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^dk}{k^2(k^2+2kq_4)(k^2-2kq_3)}\,, \tag{5.4}$$

and with

$$F = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^dk\;(s/4-k^2)}{k^2(k^2+2kq_4)(k^2-2kq_3)(k+q_1+q_4)^2} \tag{5.5}$$

being a finite scalar four point function.

The ε-expansions may be easily derived now starting from

$$C_0 \;=\; \Gamma(\varepsilon) \int_0^1 \frac{dx}{2p_x^2} \left[\frac{4\pi\mu^2}{p_x^2}\right]^{\varepsilon}, \tag{5.6}$$

$$\begin{aligned} F &= \Gamma(2+\varepsilon) \int_0^1 dx dy dz \frac{y^2 z}{(M^2)^2} \left(\frac{1}{2} yzs + \frac{1-2\varepsilon}{1+\varepsilon} M^2\right) \left[\frac{4\pi\mu^2}{p_x^2}\right]^{\varepsilon} \\ &= \Gamma(2+\varepsilon)\,[I_0 + \varepsilon I_1 + \varepsilon I_L] + \ldots\,, \end{aligned} \tag{5.7}$$

with

$$p_x^2 \;=\; -x(1-x)t + m^2 - i\epsilon\,, \tag{5.8}$$

$$M^2 \;=\; y[yz^2p_x^2 - (1-y)(1-z)s]\,. \tag{5.9}$$

Thus, the four-point function may be represented as follows:

$$I_0 = -\int_0^1 \frac{dx}{2p_x^2} \ln(-A) , \tag{5.10}$$

$$I_1 = -3\int_0^1 \frac{dx}{p_x^2} dz \left[\frac{z}{N(z)} + \frac{Az(1-z)}{N(z)^2} \ln \frac{z^2}{(1-z)(-A)} \right] , \tag{5.11}$$

with

$$A = \frac{s}{p_x^2}, \tag{5.12}$$

$$N(z) = z^2 + (1-z)A. \tag{5.13}$$

The last function will be given here in short as a three-fold integral. But it is evident that the y-integration leads to simple integrals in terms of dilogarithms or simpler functions:

$$I_L = \int_0^1 \frac{dx}{p_x^2} dz dy \left[\frac{Ay}{2z^2 K(y)^2} + \frac{y}{zK(y)} \right] \ln \frac{4\pi\mu^2 A}{z^2 y K(y) s} \tag{5.14}$$

with

$$K(y) = y - (1-y)\frac{1-z}{z^2} A . \tag{5.15}$$

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