First results for three loop deep inelastic structure functions in QCD

The current team

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We compute the structure functions by an old strategy of obtaining all Mellin moments as a function of the Mellin number N.

By means of an inverse Mellin transform we can obtain the results in Bjorken-x space.

Until the two loop anomalous dimensions this method could still be used in a brute force way (G.-A., L. and Y.).

For the two loop σ_L/σ_T already some integrals had to be done by recursion (K. and K.).

The two loop coefficient functions were done by a different technique (Z. and v. N.)

Problem one:

In Mellin space the solution consists of 'Harmonic Sums'. These had to be studied first in all generality.

In Bjorken-x space the proper category of functions had to be defined as 'Harmonic Polylogarithms'.

A procedure for the inverse Mellin transform had to be produced.

Problem two:

All integrals had to be reduced to a set of master integrals and integrals that could be solved directly. For the master integrals we obtained difference equations.

The worst we have had thusfar is a fourth order equation.

Such equations are solved with a FORM program once the inhomogeneous term is known (consists of 'simpler' integrals).

Problem three:
Finding the reduction algorithms.
There are many topologies and subtopologies and each have their own scheme.
Some of these schemes take months to derive. Some are done in less than a day.

Problem four:

Avoid spurious poles.

Sometimes a reduction scheme introduces extra powers of $1/\epsilon$.

This means that all integrals that they multiply must be expanded to extra powers in ϵ.

This is not always feasible.

Only when the derivation of the reduction algorithms is done with very great care, we can avoid such spurious poles.

This is the hardest part of the derivations.

Problem five:

Program the relations in a way that leads to results.

Brute force application has a tendency to give intermediate expressions that exceed disk sizes and other computer resources.

A careful tabellation of intermediate integrals solves this.

Problem six:

Manage the extremely large tables.

Problem seven:

Get all topologies and subtopologies tested and run.

The Mincer program for fixed moments is of great use here. At any moment we can give N a fixed (integer) value and continue with Mincer. This way we can debug our programs almost on a line by line basis.

Problem eight:
Keep FORM under control.
Sometimes rare bugs occur.
Sometimes new features are needed.

Status

On the level of integrals we distinguish BBB's and CBB's.
All BBB's have been programmed, tested and of the more complicated we have extensive tables.
We have (with still one exception) programs for all CBB's. The easier ones have been debugged and many have been run already. The more complicated ones are switched on one by one at the moment and are being debugged.
The last subtopology will be solved when needed (it is rather similar to another one that has been solved already, but it still requires great care).

Results

We have run all diagrams that give contributions to the n_f part of the non-singlet structure functions (both the anomalous dimensions and the coefficient functions).
We have checked the results with the numerical reconstructions based on the fixed moments that were previously published.
We found already an interesting result.
In addition we have run a few thousand other diagrams for the complete singlet calculation.

Notation:

$$
\begin{aligned}
\mathbf{N}_{\pm}\, f(N) &= f(N \pm 1) \\
\mathbf{N}_{\pm\mathbf{i}}\, f(N) &= f(N \pm i)
\end{aligned}
$$

Known results from the literature

$$
\begin{aligned}
c^{(0)}_{2,\mathrm{ns}}(N) &= 1 \\
\gamma^{(0)}_{\mathrm{ns}}(N) &= C_F(2(\mathbf{N}_- + \mathbf{N}_+)S_1 - 3)
\end{aligned}
$$

$$
\begin{aligned}
c^{(1)}_{2,\mathrm{ns}}(N) &= C_F(7\,\mathbf{N}_+ S_1 + 2S_1 - 9 + (\mathbf{N}_- + \mathbf{N}_+)[2S_{1,1} - 3S_1 - 2S_2]) \\
\gamma^{(1)}_{\mathrm{ns}}(N) &= 4\,C_A C_F\Big(2\,\mathbf{N}_+ S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3}S_1 + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{151}{18}S_1 + 2S_{1,-2} - \frac{11}{6}S_2\Big]\Big) \\
&+ 4\,C_F n_f\Big(\frac{1}{12} + \frac{4}{3}S_1 - (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{11}{9}S_1 - \frac{1}{3}S_2\Big]\Big) + 4\,{C_F}^2\Big(4S_{-3} + 2S_1 + 2S_2 - \frac{3}{8} \\
&+ \mathbf{N}_-\Big[S_2 + 2S_3\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1 + 4S_{1,-2} + 2S_{1,2} + 2S_{2,1} + S_3\Big]\Big)
\end{aligned}
$$

and the n_f part of the two loop coefficient function:

$$
\begin{aligned}
c^{(2)}_{2,\mathrm{ns}}(N) &= 4\,C_F n_f\Big((1 - \mathbf{N}_+)\Big[\frac{122}{27}S_1 + \frac{7}{6}S_{1,1}\Big] - (\mathbf{N}_- - 1)\Big[\frac{89}{108}S_1 - S_2\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{5}{6}S_3 \\
&+ \frac{13}{18}S_{1,1} + \frac{1}{3}S_{1,1,1} - \frac{2}{3}S_{2,1} - \frac{1}{3}S_{1,2}\Big] - \frac{1}{6}S_{1,1} + \frac{457}{144} - \frac{247}{108}S_1 + \frac{19}{6}\mathbf{N}_+ S_2\Big)
\end{aligned}
$$

New results:

$$
\begin{aligned}
\gamma_{\rm ns}^{(2)}(N) \;=\; & 16\, C_A C_F n_f \Big(\frac{3}{2}\zeta_3 - \frac{5}{4} + \frac{10}{9}S_{-3} - \frac{10}{9}S_3 + \frac{4}{3}S_{1,-2} - \frac{2}{3}S_{-4} + 2S_{1,1} - \frac{25}{9}S_2 + \frac{257}{27}S_1 \\
& - \frac{2}{3}S_{-3,1} - \mathbf{N}_+\Big[S_{2,1} - \frac{2}{3}S_{3,1} - \frac{2}{3}S_4\Big] + (1-\mathbf{N}_+)\Big[\frac{23}{18}S_3 - S_2\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_{1,1} + \frac{1237}{216}S_1 \\
& + \frac{11}{18}S_3 - \frac{317}{108}S_2 + \frac{16}{9}S_{1,-2} - \frac{2}{3}S_{1,-2,1} - \frac{1}{3}S_{1,-3} - \frac{1}{2}S_{1,3} - \frac{1}{2}S_{2,1} - \frac{1}{3}S_{2,-2} + S_1\zeta_3 + \frac{1}{2}S_{3,1}\Big]\Big) \\
& + 16\, C_F {n_f}^2\Big(\frac{17}{144} - \frac{13}{27}S_1 + \frac{2}{9}S_2 + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{2}{9}S_1 - \frac{11}{54}S_2 + \frac{1}{18}S_3\Big]\Big) + 16\, {C_F}^2 n_f\Big(\frac{23}{16} - \frac{3}{2}\zeta_3 \\
& + \frac{4}{3}S_{-3,1} - \frac{59}{36}S_2 + \frac{4}{3}S_{-4} - \frac{20}{9}S_{-3} + \frac{20}{9}S_1 - \frac{8}{3}S_{1,-2} - \frac{8}{3}S_{1,1} - \frac{4}{3}S_{1,2} + \mathbf{N}_+\Big[\frac{25}{9}S_3 - \frac{4}{3}S_{3,1} \\
& - \frac{1}{3}S_4\Big] + (1-\mathbf{N}_+)\Big[\frac{67}{36}S_2 - \frac{4}{3}S_{2,1} + \frac{4}{3}S_3\Big] + (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1\zeta_3 - \frac{325}{144}S_1 - \frac{2}{3}S_{1,-3} + \frac{32}{9}S_{1,-2} \\
& - \frac{4}{3}S_{1,-2,1} + \frac{4}{3}S_{1,1} + \frac{16}{9}S_{1,2} - \frac{4}{3}S_{1,3} + \frac{11}{18}S_2 - \frac{2}{3}S_{2,-2} + \frac{10}{9}S_{2,1} + \frac{1}{2}S_4 - \frac{2}{3}S_{2,2} - \frac{8}{9}S_3\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
c^{(3)}_{2,\mathrm{ns}}(N) \;=\; & 16\, C_A C_F n_f \Big(\delta(N{-}2)\Big[\frac{5}{3}\zeta_5 - \frac{119}{300}\zeta_3\Big] + \frac{142883}{7776} - \frac{1051}{72}\zeta_3 + \frac{3}{4}\zeta_4 + \frac{83}{54}S_4 - 2S_{1,4} \\
& - \frac{4}{9}S_{-4,1} + \frac{191}{81}S_{-3} - \frac{16}{3}S_{-3,-2} + \frac{20}{27}S_{-3,1} - \frac{4}{9}S_{-3,1,1} - \frac{29}{18}S_{-2} - \frac{14}{3}S_{-2}\zeta_3 - \frac{16}{3}S_{-2,-3} \\
& + \frac{13}{3}S_{-2,-2} + \frac{8}{3}S_{-2,-2,1} + \frac{23}{9}S_{-5} + \frac{199819}{8100}S_1 - \frac{101}{18}S_{1,2} + \frac{181}{108}S_{3,1} - \frac{83}{54}S_{-4} + \frac{4132}{135}S_{1,-2} \\
& + \frac{8}{3}S_{1,-2,-2} + \frac{56}{9}S_{1,-2,1} + \frac{21463}{1080}S_{1,1} - \frac{10}{3}S_{1,1}\zeta_3 - 4S_{1,1,-3} + \frac{58}{9}S_{1,1,1} - \frac{23}{18}S_{1,2,1} + 10S_{1,-4} \\
& + \frac{8}{3}S_{1,1,1,-2} + \frac{1}{3}S_{1,1,1,2} - 4S_{2,-2,1} - \frac{1}{3}S_{1,1,2,1} - \frac{8}{3}S_{1,2,-2} + S_{1,2,2} + \frac{35}{9}S_{1,3} + \frac{5}{3}S_{1,3,1} - \frac{1}{12}S_{2,2} \\
& - \frac{36719}{16200}S_2 + \frac{32}{3}S_2\zeta_3 + 10S_{2,-3} - \frac{218}{9}S_{2,-2} + \frac{23}{18}S_{1,1,2} - \frac{263}{60}S_{2,1} - \frac{8}{3}S_{2,1,-2} - 25S_1\zeta_3 \\
& - \frac{2}{3}S_{2,3} - \frac{4537}{1620}S_3 + \frac{28}{3}S_{3,-2} - \frac{208}{9}S_{1,-3} - \frac{112}{9}\mathbf{N}_-S_{1,1,-2} + \mathbf{N}_+\Big[\frac{4}{9}S_{3,1,1} - \frac{5}{3}S_{2,1,1} + \frac{4}{9}S_{4,1} \\
& - \frac{23}{9}S_5\Big] + (\mathbf{N_{-3}} - \mathbf{N_{-2}})\Big[\frac{2}{5}S_1\zeta_3 + \frac{2}{5}S_{1,-3} - \frac{119}{450}S_{1,-2} - \frac{2}{15}S_{1,-2,1} - \frac{2}{3}S_{1,1}\zeta_3 - \frac{2}{15}S_{1,1,-2} \\
& + \frac{1}{3}S_{1,1,3} - \frac{1}{3}S_{1,3,1} + \frac{4}{15}S_{2,-2}\Big] + (\mathbf{N_{-2}} - \mathbf{N}_-)\Big[\frac{2}{5}S_3 - \frac{179}{450}S_1 + \frac{2}{3}S_1\zeta_3 + \frac{1}{5}S_{1,1} - \frac{59}{450}S_2 \\
& + \frac{1}{5}S_{2,1} - \frac{1}{3}S_{1,3} + \frac{2}{15}S_{1,-2} + \frac{1}{3}S_{3,1}\Big] + (\mathbf{N}_- - 1)\Big[4S_{1,-4} + 2S_{1,-2,-2} + \frac{2}{3}S_{1,2,2}\Big] \\
& + (1 - \mathbf{N}_+)\Big[\frac{11057}{324}S_1 - \frac{21}{2}S_1\zeta_3 + \frac{4}{9}S_{1,-3} - \frac{7}{3}S_{1,-2} - \frac{14}{3}S_{1,-2,-2} - \frac{8}{3}S_{1,-2,1} + \frac{3559}{216}S_{1,1} \\
& - 8S_{1,1}\zeta_3 - 8S_{1,1,-3} + \frac{176}{9}S_{1,1,-2} + \frac{217}{36}S_{1,1,1} + \frac{16}{3}S_{1,1,1,-2} + \frac{2}{3}S_{1,1,1,2} - \frac{7}{36}S_{1,1,2} \\
& - \frac{2}{3}S_{1,1,2,1} + \frac{2}{3}S_{1,1,3} - \frac{55}{9}S_{1,2} - \frac{16}{3}S_{1,2,-2} + \frac{7}{36}S_{1,2,1} - \frac{4}{3}S_{1,2,2} + \frac{38}{9}S_{1,3} + \frac{8}{3}S_{1,3,1} \\
& - 4S_{1,4} - \frac{231037}{5400}S_2 + 3S_2\zeta_3 - 6S_{2,-3} + \frac{118}{9}S_{2,-2} + \frac{20}{3}S_{2,-2,1} - \frac{793}{90}S_{2,1} - \frac{16}{3}S_{2,1,-2} \\
& + \frac{1}{6}S_{2,1,2} - \frac{4}{9}S_{2,2} - \frac{1}{6}S_{2,2,1} - 2S_{2,3} + \frac{49717}{1620}S_3 - \frac{22}{3}S_{3,-2} - \frac{47}{54}S_{3,1} - \frac{1}{6}S_{3,2} - \frac{166}{27}S_4 \\
& + \frac{5}{6}S_{4,1}\Big] + (\mathbf{N}_+ - \mathbf{N_{+2}})\Big[10S_1\zeta_3 - \frac{219}{50}S_1 + \frac{6}{5}S_{1,-2} + 4S_{1,-2,1} + \frac{24}{5}S_{1,1} + 4S_{1,1}\zeta_3 \\
& - 4S_{1,1,-2} - 2S_{1,1,3} + 3S_{1,2} - 3S_{1,3} + 2S_{1,3,1} - \frac{21}{50}S_2 - 4S_2\zeta_3 + 4S_{2,-2} - \frac{24}{5}S_{2,1} + 2S_{2,3} \\
& - \frac{3}{5}S_3 - S_{3,1} - 2S_{4,1}\Big] + (\mathbf{N_{+2}} - \mathbf{N_{+3}})\Big[3S_{2,2} - 3S_{1,1,2} - \frac{72}{5}S_1\zeta_3 + \frac{18}{5}S_{1,-3} - \frac{159}{50}S_{1,-2} \\
& - \frac{6}{5}S_{1,-2,1} - 6S_{1,1}\zeta_3 - \frac{6}{5}S_{1,1,-2} + 3S_{1,1,3} + 3S_{1,2,1} + 6S_2\zeta_3 - 3S_{1,3,1} + \frac{6}{5}S_{2,-2} + 3S_{4,1} \\
& - 3S_{2,3} - \frac{159}{50}S_3 - \frac{21}{5}S_{3,1} + \frac{18}{5}S_4\Big] + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{1711}{108}S_1\zeta_3 - \frac{5608067}{291600}S_1 - \frac{1}{2}S_1\zeta_4 \\
& - \frac{79}{9}S_{1,-4} + 4S_{1,1,-2} + \frac{392}{27}S_{1,-3} + \frac{2}{9}S_{1,-3,1} - \frac{6613}{405}S_{1,-2} - \frac{104}{27}S_{1,-2,1} + \frac{4}{9}S_{1,-2,1,1} \\
& + \frac{8}{3}S_{1,1}\zeta_3 - \frac{25511}{1620}S_{1,1} + \frac{52}{9}S_{1,1,-3} - \frac{184}{27}S_{1,1,1} - \frac{32}{9}S_{1,1,1,-2} - \frac{11}{9}S_{1,1,1,1} - \frac{17}{18}S_{1,1,1,2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{36}S_{1,1,2}+\frac{4}{9}S_{1,1,2,1}+\frac{13}{18}S_{1,1,3}+\frac{661}{108}S_{1,2}+\frac{34}{9}S_{1,2,-2}+\frac{29}{12}S_{1,2,1}+\frac{1}{2}S_{1,2,1,1}-4S_{1,3}+S_{4,1}\\
&-\frac{8}{3}S_{1,3,1}+\frac{17}{9}S_{1,4}+\frac{44537}{2700}S_2-\frac{23}{3}S_2\zeta_3-\frac{79}{9}S_{2,-3}+\frac{377}{27}S_{2,-2}+\frac{20}{9}S_{2,-2,1}+\frac{5731}{540}S_{2,1}\\
&+\frac{34}{9}S_{2,1,-2}+\frac{59}{18}S_{2,1,1}+\frac{5}{9}S_{2,1,2}-\frac{53}{18}S_{2,2}-\frac{5}{9}S_{2,2,1}+\frac{13}{18}S_{2,3}-\frac{4511}{405}S_3-\frac{67}{9}S_{3,-2}-\frac{83}{18}S_{3,1}\\
&-\frac{5}{6}S_{3,1,1}+\frac{1}{2}S_{3,2}+\frac{253}{54}S_4\Big]\Big)+16\,C_F{n_f}^2\Big(\frac{1}{18}S_{1,1,1}-\frac{9517}{7776}-\frac{1}{18}\zeta_3-\frac{757}{648}S_1-\frac{29}{108}S_{1,1}\\
&-\frac{1}{18}S_{1,2}-\frac{43}{324}S_2-\frac{1}{6}S_{2,1}+\frac{19}{54}S_3+(1-\mathbf{N}_+)\Big[\frac{13}{18}S_{2,1}-\frac{19}{18}S_3+\frac{265}{108}S_2+\frac{7}{18}S_{1,2}-\frac{7}{18}S_{1,1,1}\\
&-\frac{133}{108}S_{1,1}-\frac{1421}{648}S_1\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{5585}{5832}S_1+\frac{1}{27}S_1\zeta_3+\frac{161}{324}S_{1,1}+\frac{13}{54}S_{1,1,1}+\frac{1}{9}S_{1,1,1,1}\\
&-\frac{1}{9}S_{1,1,2}-\frac{13}{54}S_{1,2}-\frac{1}{9}S_{1,2,1}+\frac{1}{9}S_{1,3}-\frac{301}{324}S_2-\frac{29}{54}S_{2,1}-\frac{2}{9}S_{2,1,1}+\frac{2}{9}S_{2,2}+\frac{62}{81}S_3+\frac{1}{3}S_{3,1}\\
&-\frac{23}{54}S_4\Big]\Big)+16\,C_F{}^2n_f\Big(\frac{119}{150}\zeta_3\delta(N{-}2)-\frac{341}{576}+\frac{139}{6}\zeta_3-\frac{3}{4}\zeta_4-\frac{46}{9}S_{-5}+\frac{8}{9}S_{-4,1}-\frac{382}{81}S_{-3}\\
&+\frac{83}{27}S_{-4}+\frac{32}{3}S_{-3,-2}-\frac{40}{27}S_{-3,1}+\frac{8}{9}S_{-3,1,1}+\frac{29}{9}S_{-2}+\frac{28}{3}S_{-2}\zeta_3-\frac{16}{3}S_{-2,-2,1}+\frac{66367}{129600}S_1\\
&+\frac{32}{3}S_{-2,-3}-\frac{26}{3}S_{-2,-2}+\frac{301}{6}S_1\zeta_3-4S_{1,-4}+\frac{416}{9}S_{1,-3}-\frac{8264}{135}S_{1,-2}+\frac{8}{3}S_{1,-2,-2}\\
&-\frac{112}{9}S_{1,-2,1}+\frac{83}{144}S_{1,1}+\frac{20}{3}S_{1,1}\zeta_3+8S_{1,1,-3}-\frac{224}{9}S_{1,1,-2}-\frac{557}{72}S_{1,1,1}-\frac{16}{3}S_{1,1,1,-2}\\
&-\frac{2}{3}S_{1,1,1,2}-\frac{31}{18}S_{1,1,2}+\frac{2171}{216}S_{1,2}+\frac{16}{3}S_{1,2,-2}+\frac{7}{3}S_{1,2,1}+\frac{2}{3}S_{1,2,2}-\frac{235}{36}S_{1,3}+4S_{1,4}\\
&-\frac{10}{3}S_{1,3,1}-\frac{140237}{16200}S_2-22S_2\zeta_3-20S_{2,-3}+\frac{436}{9}S_{2,-2}+8S_{2,-2,1}+\frac{7133}{1080}S_{2,1}+\frac{16}{3}S_{2,1,-2}\\
&+\frac{50}{9}S_{2,1,1}-4S_{2,2}+\frac{7}{3}S_{2,3}-\frac{7627}{3240}S_3+\mathbf{N}_+\Big[S_{2,1,1,1}-S_{2,1,2}+\frac{2}{3}S_{1,1,2,1}-S_{2,2,1}-\frac{56}{3}S_{3,-2}\\
&-\frac{329}{27}S_{3,1}-\frac{29}{9}S_{3,1,1}+\frac{7}{3}S_{3,2}+\frac{401}{54}S_4+\frac{53}{18}S_{4,1}+\frac{1}{18}S_5\Big]+(\mathbf{N}_{-\mathbf{3}}-\mathbf{N}_{-\mathbf{2}})\Big[\frac{119}{225}S_{1,-2}\\
&-\frac{4}{5}S_1\zeta_3+\frac{4}{15}S_{1,-2,1}-\frac{4}{5}S_{1,-3}+\frac{4}{15}S_{1,1,-2}-\frac{8}{15}S_{2,-2}\Big]+(\mathbf{N}_{-\mathbf{2}}-\mathbf{N}_-)\Big[\frac{179}{225}S_1-\frac{4}{15}S_{1,-2}\\
&+\frac{4}{15}S_{1,1}+\frac{59}{225}S_2+\frac{4}{15}S_{2,1}-\frac{4}{5}S_3\Big]+\frac{15439}{1440}(\mathbf{N}_--1)S_{1,1}+(\mathbf{N}_+-\mathbf{N}_{+\mathbf{2}})\Big[\frac{219}{25}S_1-\frac{12}{5}S_{1,-2}\\
&+4S_1\zeta_3-8S_{1,-2,1}-\frac{13}{5}S_{1,1}+8S_{1,1,-2}+2S_{1,1,2}-5S_{1,2}-2S_{1,2,1}-\frac{154}{25}S_2+\frac{36}{5}S_3+10S_{3,1}\\
&+\frac{13}{5}S_{2,1}-2S_{2,2}-8S_{2,-2}\Big]+(\mathbf{N}_{+\mathbf{2}}-\mathbf{N}_{+\mathbf{3}})\Big[5S_{1,1,2}-\frac{12}{5}S_{2,-2}-5S_{2,2}+\frac{37}{5}S_{3,1}+\frac{159}{25}S_{1,-2}\\
&+\frac{159}{25}S_3-5S_{1,2,1}+\frac{12}{5}S_{1,1,-2}+\frac{12}{5}S_{1,-2,1}+\frac{114}{5}S_1\zeta_3-\frac{36}{5}S_{1,-3}-\frac{36}{5}S_4\Big]+(1-\mathbf{N}_+)\Big[4S_4\\
&-\frac{10337}{576}S_1-8S_{1,-4}-\frac{8}{9}S_{1,-3}+\frac{14}{3}S_{1,-2}+\frac{16}{3}S_{1,-2,-2}+\frac{16}{3}S_{1,-2,1}+\frac{40}{3}S_{1,1}\zeta_3+16S_{1,1,-3}\\
&-\frac{76157}{4320}S_{1,1}-\frac{128}{9}S_{1,1,-2}+\frac{179}{72}S_{1,1,1}-\frac{32}{3}S_{1,1,1,-2}+\frac{14}{3}S_{1,1,1,1}-\frac{4}{3}S_{1,1,1,2}+\frac{565897}{16200}S_2\\
&+2S_{1,1,2,1}-\frac{295}{72}S_{1,2}+\frac{32}{3}S_{1,2,-2}-\frac{17}{3}S_{1,2,1}+\frac{4}{3}S_{1,2,2}-\frac{115}{36}S_{1,3}+8S_{1,4}-\frac{29}{6}S_{1,1,2}-\frac{2}{3}S_2\zeta_3
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{20}{3}S_{1,3,1}+12S_{2,-3}-\frac{236}{9}S_{2,-2}-\frac{40}{3}S_{2,-2,1}-\frac{4663}{1080}S_{2,1}+\frac{32}{3}S_{2,1,-2}-\frac{37}{3}S_{2,1,1}-\frac{22817}{810}S_3\\
&+\frac{32}{3}S_{2,2}+\frac{5}{3}S_{2,3}-4S_{3,-2}+\frac{65}{18}S_{3,1}+\frac{85}{6}S_1\zeta_3\Big]+(\mathbf{N}_-+\mathbf{N}_+)\Big[\frac{86}{9}S_{1,-4}-\frac{91}{36}S_5-4S_{3,1,1}\\
&-\frac{8}{9}S_{1,-2,1,1}+\frac{13226}{405}S_{1,-2}-\frac{104}{9}S_{1,1,-3}+\frac{152}{9}S_{1,1,-2}-\frac{784}{27}S_{1,-3}-4S_{1,-2,-2}+\frac{41929}{129600}S_1\\
&-\frac{244}{9}S_1\zeta_3+\frac{1}{2}S_1\zeta_4+\frac{208}{27}S_{1,-2,1}-\frac{4}{9}S_{1,-3,1}-\frac{13}{3}S_{1,1}\zeta_3-\frac{7}{8}S_{1,1,1}+\frac{64}{9}S_{1,1,1,-2}-\frac{35}{6}S_{1,1,1,1}\\
&-\frac{10}{3}S_{1,1,1,1,1}+\frac{44}{9}S_{1,1,1,2}+\frac{25}{3}S_{1,1,2}+3S_{1,1,2,1}-\frac{91}{18}S_{1,1,3}+\frac{577}{648}S_{1,2}+\frac{40}{3}S_2\zeta_3+\frac{107}{18}S_{1,2,1}\\
&+\frac{34}{9}S_{1,2,1,1}-\frac{79151}{64800}S_2-\frac{68}{9}S_{1,2,-2}-\frac{239}{54}S_{2,2}-\frac{49}{9}S_{1,2,2}-\frac{275}{108}S_{1,3}+\frac{73}{18}S_{2,1,1,1}-\frac{31}{6}S_{2,1,2}\\
&+\frac{158}{9}S_{2,-3}-\frac{754}{27}S_{2,-2}-\frac{40}{9}S_{2,-2,1}+\frac{133}{36}S_{4,1}-\frac{403}{810}S_{2,1}-\frac{68}{9}S_{2,1,-2}+\frac{79}{18}S_{2,1,1}-\frac{67}{18}S_{2,2,1}\\
&+\frac{17}{18}S_{1,4}+\frac{5}{72}S_4-\frac{5}{18}S_{1,3,1}+\frac{25}{9}S_{2,3}+\frac{7871}{3240}S_3+\frac{134}{9}S_{3,-2}-\frac{241}{72}S_{3,1}+\frac{38}{9}S_{3,2}\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
c_{L,\mathrm{ns}}^{(3)}(N) \;=\; & 16\, C_A C_F n_f \Big(\delta(N{-}2)\Big[\frac{20}{3}\zeta_5 - \frac{149}{75}\zeta_3\Big] + (\mathbf{N_{-3}} - \mathbf{N_{-2}})\Big[\frac{16}{15}S_{2,-2} - \frac{4}{3}S_{1,3,1} + \frac{4}{3}S_{1,1,3} \\
& -\frac{8}{3}S_{1,1}\zeta_3 - \frac{8}{15}S_{1,1,-2} - \frac{8}{15}S_{1,-2,1} + \frac{8}{5}S_{1,-3} - \frac{298}{225}S_{1,-2} + \frac{8}{5}S_1\zeta_3\Big] + (\mathbf{N_{-2}} - \mathbf{N}_-)\Big[\frac{4}{3}S_{1,1}\zeta_3 \\
& -\frac{418}{225}S_1 + \frac{8}{5}S_3 + \frac{8}{15}S_{1,-2} + \frac{4}{3}S_{1,-2,1} - \frac{4}{3}S_{1,1,-2} + \frac{4}{5}S_{1,1} - \frac{178}{225}S_2 + \frac{4}{3}S_{3,1} + 4S_1\zeta_3 + \frac{4}{5}S_{2,1} \\
& -\frac{2}{3}S_{1,1,3} - \frac{4}{3}S_{1,3} + \frac{2}{3}S_{1,3,1}\Big] + (\mathbf{N}_- - 1)\Big[\frac{16}{3}S_{2,-2} - \frac{13033}{1350}S_1 + 8S_1\zeta_3 - \frac{518}{45}S_{1,-2} - \frac{2}{5}S_{2,1} \\
& -\frac{8}{3}S_{1,-2,1} - \frac{254}{45}S_{1,1} - \frac{2}{15}S_3 + 8S_{1,-3} + \frac{76}{25}S_2 - \frac{8}{3}S_{1,1,-2}\Big] + (\mathbf{N}_+ - \mathbf{N_{+2}})\Big[\frac{8}{3}S_{1,-2,1} - \frac{8}{3}S_2\zeta_3 \\
& -\frac{7}{25}S_2 - \frac{8}{3}S_{1,1,-2} - 2S_{1,3} + \frac{4}{3}S_{1,3,1} - \frac{4}{3}S_{1,1,3} + 2S_{1,2} + \frac{16}{5}S_{1,1} + \frac{8}{3}S_{1,1}\zeta_3 + \frac{4}{5}S_{1,-2} - \frac{16}{5}S_{2,1} \\
& +\frac{20}{3}S_1\zeta_3 - \frac{73}{25}S_1 - \frac{2}{5}S_3 - \frac{2}{3}S_{3,1} - \frac{4}{3}S_{4,1} + \frac{4}{3}S_{2,3} + \frac{8}{3}S_{2,-2}\Big] + (\mathbf{N_{+2}} - \mathbf{N_{+3}})\Big[\frac{4}{5}S_{2,-2} + 2S_{2,2} \\
& -\frac{4}{5}S_{1,-2,1} - 2S_{2,3} - \frac{14}{5}S_{3,1} - \frac{53}{25}S_3 + \frac{12}{5}S_4 + 2S_{4,1} - \frac{48}{5}S_1\zeta_3 + \frac{12}{5}S_{1,-3} + 2S_{1,1,3} - 4S_{1,1}\zeta_3 \\
& -\frac{4}{5}S_{1,1,-2} - \frac{53}{25}S_{1,-2} + 2S_{1,2,1} - 2S_{1,3,1} + 4S_2\zeta_3 - 2S_{1,1,2}\Big] + (1 - \mathbf{N}_+)\Big[\frac{125599}{4050}S_1 - \frac{40}{3}S_1\zeta_3 \\
& +4S_{1,-4} - \frac{68}{9}S_{1,-3} + \frac{413}{45}S_{1,-2} - \frac{8}{3}S_{1,-2,-2} + \frac{4469}{270}S_{1,1} - 8S_{1,1}\zeta_3 - 8S_{1,1,-3} + \frac{88}{9}S_{1,1,-2} \\
& +\frac{40}{9}S_{1,1,1} + \frac{16}{3}S_{1,1,1,-2} + \frac{2}{3}S_{1,1,1,2} + \frac{2}{3}S_{1,1,2} - \frac{2}{3}S_{1,1,2,1} + \frac{2}{3}S_{1,1,3} - \frac{40}{9}S_{1,2} - \frac{16}{3}S_{1,2,-2} \\
& -\frac{10}{3}S_4 - \frac{2}{3}S_{1,2,1} - \frac{2}{3}S_{1,2,2} + \frac{2}{3}S_{3,1} + \frac{8}{3}S_{1,3,1} - 4S_{1,4} - \frac{20953}{675}S_2 + 8S_2\zeta_3 + \frac{4}{3}S_{2,-3} - \frac{2}{3}S_{2,-2} \\
& +\frac{8}{3}S_{2,-2,1} - \frac{238}{45}S_{2,1} - \frac{16}{3}S_{2,1,-2} - 2S_{2,3} + \frac{824}{45}S_3 + \frac{38}{9}S_{1,3} + \frac{2}{3}S_{4,1}\Big]\Big) \\
& +16\, C_F {n_f}^2\Big((\mathbf{N}_- - 1)\Big[\frac{19}{27}S_1 + \frac{2}{9}S_{1,1} - \frac{2}{9}S_2\Big] + (1 - \mathbf{N}_+)\Big[\frac{2}{9}S_{1,2} - \frac{317}{162}S_1 - \frac{25}{27}S_{1,1} - \frac{2}{9}S_{1,1,1} \\
& +\frac{50}{27}S_2 + \frac{4}{9}S_{2,1} - \frac{2}{3}S_3\Big]\Big) + 16\, {C_F}^2 n_f\Big(\frac{298}{75}\zeta_3\delta(N{-}2) + (\mathbf{N_{-3}} - \mathbf{N_{-2}})\Big[\frac{16}{15}S_{1,1,-2} - \frac{16}{5}S_{1,-3} \\
& +\frac{16}{15}S_{1,-2,1} + \frac{596}{225}S_{1,-2} - \frac{32}{15}S_{2,-2} - \frac{16}{5}S_1\zeta_3\Big] + (\mathbf{N_{-2}} - \mathbf{N}_-)\Big[\frac{16}{15}S_{2,1} - \frac{16}{5}S_3 + \frac{356}{225}S_2 \\
& +\frac{836}{225}S_1 - \frac{8}{3}S_1\zeta_3 + \frac{16}{15}S_{1,1} - \frac{8}{3}S_{1,-2,1} + \frac{8}{3}S_{1,1,-2} - \frac{16}{15}S_{1,-2}\Big] + (\mathbf{N}_- - 1)\Big[\frac{2141}{1350}S_1 - 16S_1\zeta_3 \\
& -16S_{1,-3} + \frac{1036}{45}S_{1,-2} + \frac{16}{3}S_{1,-2,1} + \frac{73}{45}S_{1,1} + \frac{16}{3}S_{1,1,-2} - \frac{5}{3}S_{1,1,1} + \frac{5}{3}S_{1,2} + \frac{232}{225}S_2 + \frac{1}{10}S_3 \\
& -\frac{32}{3}S_{2,-2} + \frac{14}{5}S_{2,1}\Big] + (\mathbf{N}_+ - \mathbf{N_{+2}})\Big[\frac{146}{25}S_1 - \frac{10}{3}S_{1,2} - \frac{308}{75}S_2 + \frac{20}{3}S_{3,1} - \frac{4}{3}S_{2,2} + \frac{8}{3}S_1\zeta_3 \\
& -\frac{16}{3}S_{2,-2} - \frac{4}{3}S_{1,2,1} + \frac{16}{3}S_{1,1,-2} + \frac{4}{3}S_{1,1,2} - \frac{8}{5}S_{1,-2} - \frac{16}{3}S_{1,-2,1} - \frac{26}{15}S_{1,1} + \frac{24}{5}S_3 + \frac{26}{15}S_{2,1}\Big] \\
& +(\mathbf{N_{+2}} - \mathbf{N_{+3}})\Big[\frac{76}{5}S_1\zeta_3 + \frac{106}{25}S_{1,-2} + \frac{8}{5}S_{1,-2,1} - \frac{24}{5}S_{1,-3} + \frac{10}{3}S_{1,1,2} + \frac{8}{5}S_{1,1,-2} - \frac{10}{3}S_{1,2,1} \\
& -\frac{8}{5}S_{2,-2} + \frac{74}{15}S_{3,1} - \frac{24}{5}S_4 + \frac{106}{25}S_3 - \frac{10}{3}S_{2,2}\Big] + (1 - \mathbf{N}_+)\Big[\frac{74}{3}S_1\zeta_3 - \frac{173653}{10800}S_1 - 8S_{1,-4}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{136}{9}S_{1,-3}-\frac{826}{45}S_{1,-2}+\frac{16}{3}S_{1,-2,-2}-\frac{1609}{135}S_{1,1}+\frac{40}{3}S_{1,1}\zeta_3+16S_{1,1,-3}-\frac{176}{9}S_{1,1,-2}\\
&+\frac{7}{18}S_{1,1,1}-\frac{32}{3}S_{1,1,1,-2}+\frac{8}{3}S_{1,1,1,1}-\frac{4}{3}S_{1,1,1,2}+8S_{1,4}-4S_{1,1,2}+\frac{4}{3}S_{1,1,2,1}-\frac{23}{18}S_{1,2}+\frac{17}{6}S_4\\
&+\frac{32}{3}S_{1,2,-2}-2S_{1,2,1}+\frac{4}{3}S_{1,2,2}-\frac{49}{9}S_{1,3}-\frac{20}{3}S_{1,3,1}+\frac{30737}{1350}S_2-\frac{40}{3}S_2\zeta_3-\frac{8}{3}S_{2,-3}+\frac{4}{3}S_{2,-2}\\
&-\frac{16}{3}S_{2,-2,1}-\frac{17}{10}S_{2,1}+\frac{32}{3}S_{2,1,-2}-\frac{13}{3}S_{2,1,1}+\frac{13}{3}S_{2,2}+\frac{8}{3}S_{2,3}-\frac{913}{45}S_3+\frac{13}{3}S_{3,1}\Big]\Big)
\end{aligned}
$$

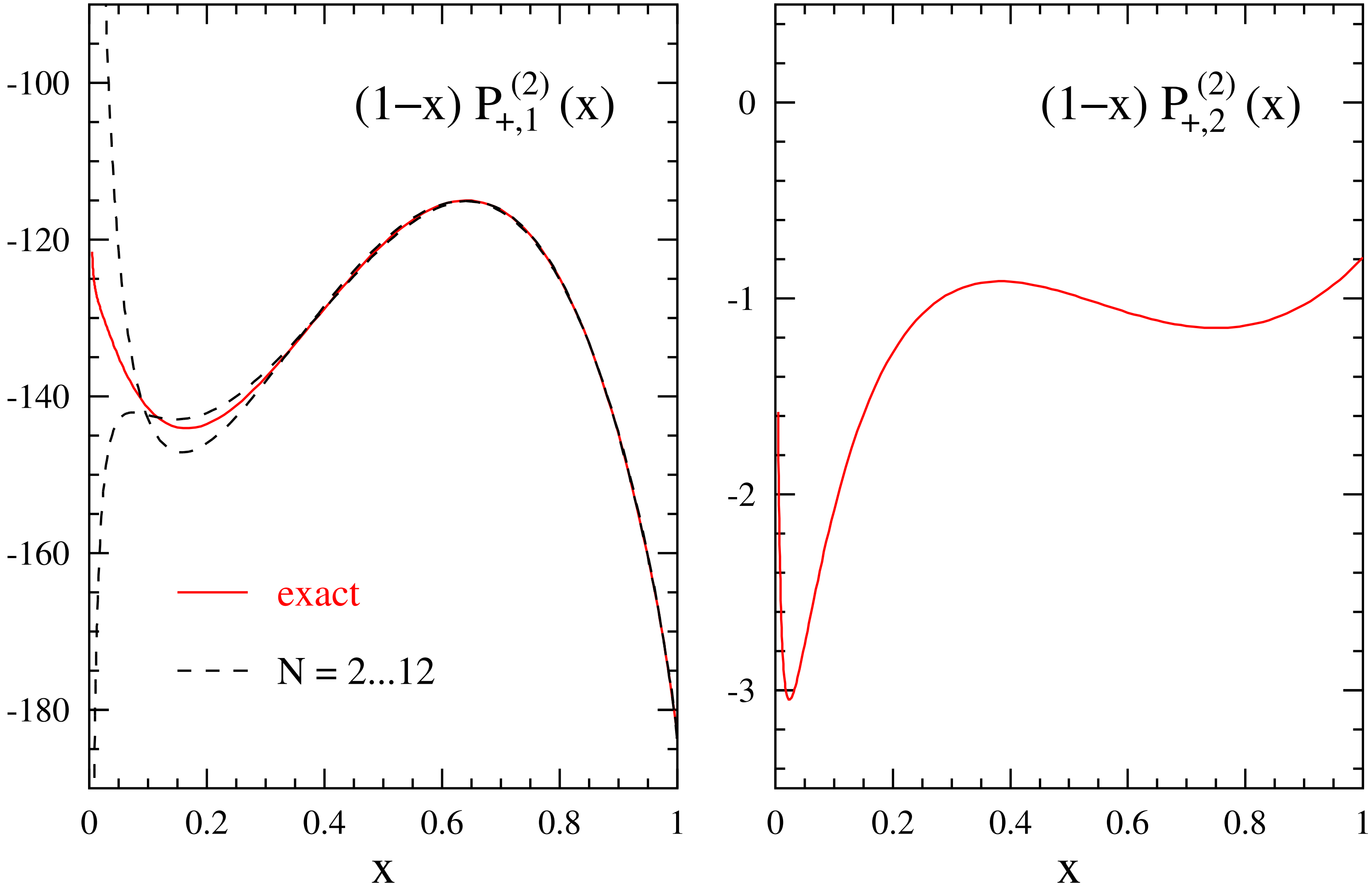
In Bjorken-x space the anomalous dimension becomes:

$$
\begin{aligned}
P^{(2)}_{\rm ns}(x) \;=\; & 16\,C_A C_F n_f\Big(p_{\rm qq}(x)\Big[\frac{5}{9}\zeta_2-\frac{209}{216}-\frac{3}{2}\zeta_3-\frac{167}{108}\ln(x)+\frac{1}{3}\ln(x)\zeta_2-\frac{1}{4}\ln^2(x)\ln(1-x)\\
&-\frac{7}{12}\ln^2(x)-\frac{1}{18}\ln^3(x)-\frac{1}{2}\ln(x){\rm Li}_2(x)+\frac{1}{3}{\rm Li}_3(x)\Big]+p_{\rm qq}(-x)\Big[\frac{1}{2}\zeta_3-\frac{5}{9}\zeta_2-\frac{2}{3}\ln(1+x)\zeta_2\\
&+\frac{1}{6}\ln(x)\zeta_2-\frac{10}{9}\ln(x)\ln(1+x)+\frac{5}{18}\ln^2(x)-\frac{1}{6}\ln^2(x)\ln(1+x)+\frac{1}{18}\ln^3(x)-\frac{10}{9}{\rm Li}_2(-x)\\
&-\frac{1}{3}{\rm Li}_3(-x)-\frac{1}{3}{\rm Li}_3(x)+\frac{2}{3}{\rm H}_{-1,0,1}(x)\Big]+(1+x)\Big[\frac{1}{6}\zeta_2+\frac{1}{2}\ln(x)-\frac{1}{2}{\rm Li}_2(x)-\frac{2}{3}{\rm Li}_2(-x)\\
&-\frac{2}{3}\ln(x)\ln(1+x)+\frac{1}{24}\ln^2(x)\Big]+(1-x)\Big[\frac{1}{3}\zeta_2-\frac{257}{54}+\ln(1-x)-\frac{17}{9}\ln(x)-\frac{1}{24}\ln^2(x)\Big]\\
&+\delta(1-x)\Big[\frac{5}{4}-\frac{167}{54}\zeta_2+\frac{1}{20}{\zeta_2}^2+\frac{25}{18}\zeta_3\Big]\Big)+16\,C_F {n_f}^2\Big(p_{\rm qq}(x)\Big[\frac{5}{54}\ln(x)-\frac{1}{54}+\frac{1}{36}\ln^2(x)\Big]\\
&+(1-x)\Big[\frac{13}{54}+\frac{1}{9}\ln(x)\Big]-\delta(1-x)\Big[\frac{17}{144}-\frac{5}{27}\zeta_2+\frac{1}{9}\zeta_3\Big]\Big)+16\,{C_F}^2 n_f\Big(p_{\rm qq}(x)\Big[\frac{5}{3}\zeta_3-\frac{55}{48}\\
&+\frac{5}{24}\ln(x)+\frac{1}{3}\ln(x)\zeta_2+\frac{10}{9}\ln(x)\ln(1-x)+\frac{1}{4}\ln^2(x)+\frac{2}{3}\ln^2(x)\ln(1-x)+\frac{2}{3}\ln(x){\rm Li}_2(x)\\
&-\frac{2}{3}{\rm Li}_3(x)-\frac{1}{18}\ln^3(x)\Big]+p_{\rm qq}(-x)\Big[\frac{10}{9}\zeta_2-\zeta_3+\frac{4}{3}\ln(1+x)\zeta_2-\frac{1}{3}\ln(x)\zeta_2-\frac{5}{9}\ln^2(x)\\
&+\frac{20}{9}\ln(x)\ln(1+x)-\frac{1}{9}\ln^3(x)+\frac{1}{3}\ln^2(x)\ln(1+x)+\frac{20}{9}{\rm Li}_2(-x)+\frac{2}{3}{\rm Li}_3(-x)+\frac{2}{3}{\rm Li}_3(x)\\
&-\frac{4}{3}{\rm H}_{-1,0,1}(x)\Big]+(1+x)\Big[\frac{7}{36}\ln^2(x)-\frac{67}{72}\ln(x)+\frac{4}{3}\ln(x)\ln(1+x)+\frac{1}{12}\ln^3(x)+\frac{2}{3}{\rm Li}_2(x)\\
&+\frac{4}{3}{\rm Li}_2(-x)\Big]+(1-x)\Big[\frac{1}{9}\ln(x)-\frac{10}{9}-\frac{4}{3}\ln(1-x)+\frac{2}{3}\ln(x)\ln(1-x)-\frac{1}{3}\ln^2(x)\Big]\\
&-\delta(1-x)\Big[\frac{23}{16}-\frac{5}{12}\zeta_2-\frac{29}{30}{\zeta_2}^2+\frac{17}{6}\zeta_3\Big]\Big)
\end{aligned}
$$

where we have used

$$p_{\rm qq}(x) \;=\; 2\,(1-x)^{-1}-1-x$$

and all divergences for $x \to 1$ are understood in the sense of +-distributions.



-100
-120
-140
-160
-180
$(1-x)\, P^{(2)}_{+,1}(x)$
exact
N = 2...12
0
0.2
0.4
0.6
0.8
1
x
0
-1
-2
-3
$(1-x)\, P^{(2)}_{+,2}(x)$
0
0.2
0.4
0.6
0.8
1
x

Implications for threshold resummation

The parameter A_3 is the coefficient of $1/(1-x)_+$ in $P^{(2)}_{\rm ns}(x)$. From the powers of $\ln(1-x)$ we can determine the factors B_2 and $D_2^{\rm DIS}$.

$$
\begin{aligned}
A_3\Big|_{n_f} &= C_A C_F n_f \left[-\frac{836}{27} + \frac{160}{9}\,\zeta_2 - \frac{112}{3}\,\zeta_3\right] + C_F^2 n_f \left[-\frac{110}{3} + 32\,\zeta_3\right] + C_F n_f^2 \left[-\frac{16}{27}\right] \\
B_2 &= C_F^2 \left[-\frac{3}{2} - 24\,\zeta_3 + 12\,\zeta_2\right] + C_F C_A \left[-\frac{3155}{54} + 40\,\zeta_3 + \frac{44}{3}\,\zeta_2\right] \\
&\quad + C_F n_f \left[\frac{247}{27} - \frac{8}{3}\,\zeta_2\right] \\
D_2^{\rm DIS} &= 0
\end{aligned}
$$

Future

We will keep working to finish first the complete non-singlet and after that the singlet. This will still take quite an amount of computer time.