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RADCOR 2002 - Loops and Legs
Kloster Banz

Evaluating Double and Triple (?) Boxes

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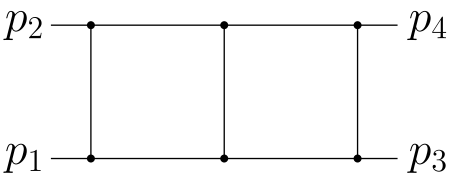
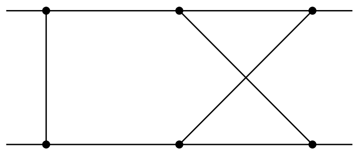




$m_i^2,\ i = 1, \ldots 7,$

$s = (p_1 + p_2)^2,\ t = (p_1 + p_3)^2,$
$p_i^2,\ i = 1, 2, 3, 4$

1. all legs off-shell, $p_i^2 \neq 0,\ i = 1, 2, 3, 4$; $m_i = 0$ for all i

The first analytical result [N.I. Ussyukina and A.I. Davydychev, Phys. Lett. **B298** (1993) 363]

$$\frac{\left(i\pi^2\right)^2}{s^2 t} C(p_1^2 p_4^2, p_2^2 p_3^2, st)\,,$$

$$\begin{aligned} C(x_1, x_2, x_3) \; &= \; \frac{1}{\lambda}\,(6\,[\mathrm{Li}_4(-\rho x) + \mathrm{Li}_4(-\rho y)] \\ &+3 \ln \frac{y}{x}\,[\mathrm{Li}_3(-\rho x) - \mathrm{Li}_3(-\rho y)] + \frac{1}{2} \ln^2 \frac{y}{x}\,[\mathrm{Li}_2(-\rho x) + \mathrm{Li}_2(-\rho y)] \\ &+\frac{1}{4} \ln^2(\rho x) \ln^2(\rho y) + \frac{\pi^2}{2} \ln(\rho x) \ln(\rho y) + \frac{\pi^2}{12} \ln^2 \frac{y}{x} + \frac{7\pi^4}{60}\bigg)\ , \end{aligned}$$

$$\lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy}\ , \quad \rho(x, y) = \frac{2}{1 - x - y + \lambda(x, y)}\ ,$$

$x = x_1/x_3,\ y = x_2/x_3$, $\mathrm{Li}_n(z)$ is polylogarithm

2. all legs on-shell, $p_i^2 = 0, \; i = 1,2,3,4$; $m_i = 0$ for all i

3. one leg off-shell, $p_1^2 = q^2 \neq 0$, $p_i^2 = 0, \; i = 2,3,4$; $m_i = 0$ for all i

4. $p_i^2 = m^2, \; i = 1,2,3,4$

Evaluation of general Feynman integrals corresponding to a given graph:

(1) reduction of general Feynman integrals of this class to master integrals, e.g., using integration by parts (IBP) [K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B **192** (1981) 159]

(2) the evaluation of these master integrals.

2. The basic master planar and non-planar on-shell massless double-boxes:

[V.A. Smirnov, Phys. Lett. **B460** (1999) 397; J.B. Tausk, Phys. Lett. **B469** (1999) 225]

(by means of Feynman parameters and MB representation)

It is natural to consider non-planar double boxes as functions of the three Mandelstam variables s, t and $u = (p_1 + p_4)^2$ not necessarily restricted by the physical condition $s + t + u = 0$

Reduction procedures for the evaluation of general double-box diagrams, with arbitrary numerators and integer powers of the propagators:

[V.A. Smirnov and O.L. Veretin, Nucl. Phys. **B566**] (the planar case); [C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, Nucl. Phys. **B580** (2000) 577] (the non-planar case)

two master integrals in [SV]:
$K^{(0)} \equiv K(1,\ldots,1,0)$ and $K(1,1,1,1,1,2,1,0)$.

[C. Anastasiou, J.B. Tausk and M.E. Tejeda-Yeomans, Nucl. Phys. Proc. Suppl. **89** (2000) 262]:
$K(1,\ldots,1,0)$ and $K(1,1,1,1,1,1,1,-1)$

Application to the evaluation of two-loop virtual corrections to various scattering processes in the zero-mass approximation:

[Z. Bern, L. Dixon and A. Ghinculov, Phys. Rev. **D63** (2001) 053007; C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. **B601** (2001) 318; **B601** (2001) 341; Nucl. Phys. **B605** (2001) 486; E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. **B605** (2001) 467; Z. Bern, A. De Freitas and L.J. Dixon, JHEP **0109** (2001) 037; Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov and H.L. Wong, JHEP **0111** (2001) 031. Z. Bern, A. De Freitas and L.J. Dixon, JHEP **0203** (2002) 018. C. Anastasiou, E.W.N. Glover and M.E. Tejeda-Yeomans, Nucl. Phys. **B629** (2002) 255; E.W.N. Glover, radcor/LL–02.]

3. One leg off-shell.

Two master integrals: [V.A. Smirnov, Phys. Lett. **B491** (2000) 130; **B500** (2001) 330]

Reduction algorithm (based on a Laporta's idea) and all master integrals (by use of differential equations):

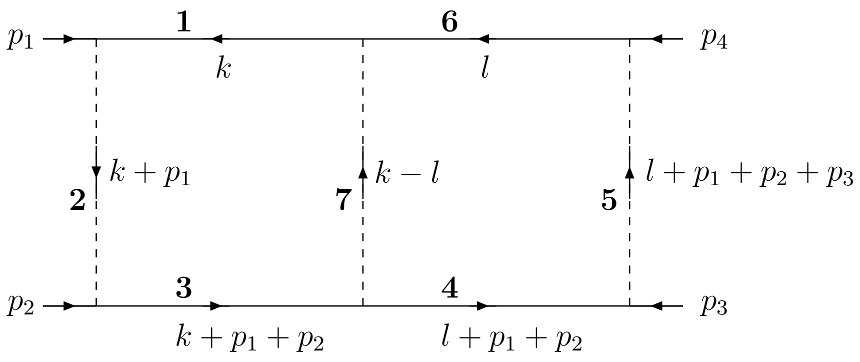
[T. Gehrmann and E. Remiddi, Nucl. Phys. **B601** (2001) 248; **B601** (2001) 287]

2dHPL!

Applications ($e^+e^- \rightarrow$ 3jets):

T. Gehrmann, radcor/LL–02

4. Motivations: A. Werthenbach, J. Fleischer, radcor/LL–02



$$
\begin{aligned}
B(a_1,\ldots,a_8;s,t,m^2;\epsilon) &= \int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{(k^2-m^2)^{a_1}[(k+p_1)^2)]^{a_2}[(k+p_1+p_2)^2-m^2]^{a_3}} \\
&\times \frac{[(k+p_1+p_2+p_3)^2]^{-a_8}}{[(l+p_1+p_2)^2-m^2)]^{a_4}[(l+p_1+p_2+p_3)^2]^{a_5}(l^2-m^2)^{a_6}[(k-l)^2]^{a_7}}
\end{aligned}
$$

$$
\begin{aligned}
B(a_1,\ldots,a_8;s,t,m^2;\epsilon) &= \frac{\left(i\pi^{d/2}\right)^2(-1)^a}{\prod_{j=2,4,5,6,7}\Gamma(a_j)\Gamma(4-a_{4567}-2\epsilon)(-s)^{a-4+2\epsilon}} \\
&\times \frac{1}{(2\pi i)^6}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=1}^{5}\mathrm{d}z_j\left(\frac{m^2}{-s}\right)^{z_1+z_5}\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)} \\
&\times \frac{\Gamma(4-a_{13}-2a_{28}-2\epsilon+z_2+z_3)\Gamma(a_{1238}-2+\epsilon+z_4+z_5)\Gamma(a_7+w-z_4)}{\Gamma(4-a_{46}-2a_{57}-2\epsilon-2w-2z_1-z_2-z_3)} \\
&\times \frac{\Gamma(a_{4567}-2+\epsilon+w+z_1-z_4)\Gamma(a_8-z_2-z_3-z_4)\Gamma(-w-z_2-z_3-z_4)}{\Gamma(4-a_{1238}-2\epsilon+w-z_4)\Gamma(a_8-w-z_2-z_3-z_4)} \\
&\times \frac{\Gamma(a_5+w+z_2+z_3+z_4)\Gamma(2-a_{567}-\epsilon-w-z_1-z_2)}{\Gamma(4-a_{13}-2a_{28}-2\epsilon+z_2+z_3-2z_5)} \\
&\times \Gamma(2-a_{457}-\epsilon-w-z_1-z_3)\Gamma(2-a_{128}-\epsilon+z_2-z_5)\Gamma(2-a_{238}-\epsilon+z_3-z_5) \\
&\times \Gamma(4-a_{46}-2a_{57}-2\epsilon-2w-z_2-z_3)\Gamma(-z_1)\Gamma(-z_5)\,,
\end{aligned}
$$

where $a_{4567}=a_4+a_5+a_6+a_7, a_{13}=a_1+a_3$.

The master double box:

$$B^{(0)}(s,t,m^2;\epsilon)\equiv B(1,\ldots,1,0;s,t,m^2;\epsilon)\,.$$

The final result [V.A. Smirnov, Phys. Lett. **B524** (2002) 129]:

$$B^{(0)}(s,t,m^2;\epsilon) = -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^2 x^2}{s^2(-t)^{1+2\epsilon}} \left[\frac{b_2(x)}{\epsilon^2} + \frac{b_1(x)}{\epsilon} + b_{01}(x) + b_{02}(x,y) + O(\epsilon)\right] ,$$

where $x = 1/\sqrt{1-4m^2/s},\ y = 1/\sqrt{1-4m^2/t}$, and

$$\begin{aligned}
b_2(x) &= 2(m_x - p_x)^2 , \\
b_1(x) &= -8\left[\mathrm{Li}_3\left(\frac{1-x}{2}\right) + \mathrm{Li}_3\left(\frac{1+x}{2}\right) + \mathrm{Li}_3\left(\frac{-2x}{1-x}\right) + \mathrm{Li}_3\left(\frac{2x}{1+x}\right)\right] \\
&\quad +4(m_x - p_x)\left[\mathrm{Li}_2\left(\frac{1-x}{2}\right) - \mathrm{Li}_2\left(\frac{-2x}{1-x}\right)\right] - (4/3)m_x^3 + 4m_x^2 p_x \\
&\quad -6m_x p_x^2 + (2/3)p_x^3 + 4l_2(m_x p_x + p_x^2) - 2l_2^2(m_x + 3p_x) \\
&\quad -(\pi^2/3)(4l_2 - m_x - 3p_x) + (8/3)l_2^3 + 14\zeta_3 , \\
b_{01}(x) &= -8(m_x - p_x)\left[\mathrm{Li}_3(x) - \mathrm{Li}_3(-x) - \mathrm{Li}_3\left(\frac{1+x}{2}\right) + \mathrm{Li}_3\left(\frac{1-x}{2}\right)\right. \\
&\quad \left. -\mathrm{Li}_3\left(\frac{2x}{1+x}\right) + \mathrm{Li}_3\left(\frac{-2x}{1-x}\right)\right] + 4\left[\mathrm{Li}_2(x)^2 + \mathrm{Li}_2(-x)^2 + 4\mathrm{Li}_2\left(\frac{1-x}{2}\right)^2\right] \\
&\quad -8\mathrm{Li}_2(x)\,\mathrm{Li}_2(-x) + 16\mathrm{Li}_2\left(\frac{1-x}{2}\right)(\mathrm{Li}_2(x) - \mathrm{Li}_2(-x)) \\
&\quad -(4/3)[\pi^2 - 6l_2^2 + 3m_x^2 + 6m_x(2l_2 - 2l_x - p_x) + 12l_x p_x - 3p_x^2](\mathrm{Li}_2(x) - \mathrm{Li}_2(-x)) \\
&\quad -(8/3)[\pi^2 - 6l_2^2 + 6l_x p_x - 6m_x(l_x + p_x - 2l_2)]\mathrm{Li}_2\left(\frac{1-x}{2}\right) \\
&\quad +8(m_x - p_x)\left[(p_x - m_x + 2l_2)\mathrm{Li}_2\left(\frac{2x}{1+x}\right) + 2(l_x - m_x + l_2)\mathrm{Li}_2\left(\frac{-2x}{1-x}\right)\right] \\
&\quad -8(m_x - p_x)(2l_x - p_x - 5m_x + 4l_2)(-m_x p_x + l_2(m_x + p_x) - l_2^2 + \pi^2/6) \\
&\quad -(20/3)m_x^4 + (164/3)m_x^3 p_x - 40m_x^2 p_x^2 - (4/3)m_x p_x^3 - (8/3)p_x^4 \\
&\quad +8m_x l_x(m_x^2 - 3m_x p_x + 2p_x^2) \\
&\quad -4l_2(7m_x^3 + 21m_x^2 p_x - 4m_x l_x p_x - 23m_x p_x^2 + 4l_x p_x^2 - p_x^3) \\
&\quad -\pi^2((17/3)m_x^2 - (4/3)m_x l_x - 2m_x p_x + (4/3)l_x p_x - (7/3)p_x^2) \\
&\quad +l_2^2(84m_x^2 - 8m_x l_x - 16m_x p_x + 8l_x p_x - 44p_x^2) \\
&\quad -(8/3)l_2(6l_2^2 - \pi^2)(3m_x - 2p_x) - (4/3)\pi^2 l_2^2 + 4l_2^4 + \pi^4/9 ,
\end{aligned}$$

$$\begin{aligned}
b_{02}(x,y) &= 2(p_x - m_x)\left\{4\left[\mathrm{Li}_3\left(\frac{1-x}{2}\right) - \mathrm{Li}_3\left(\frac{1+x}{2}\right) + \mathrm{Li}_3\left(\frac{(1-x)y}{1-xy}\right)\right.\right. \\
&\quad \left. -\mathrm{Li}_3\left(\frac{-(1+x)y}{1-xy}\right) + \mathrm{Li}_3\left(\frac{-(1-x)y}{1+xy}\right) - \mathrm{Li}_3\left(\frac{(1+x)y}{1+xy}\right)\right] \\
&\quad +2\left[\mathrm{Li}_3\left(\frac{(1+x)(1-y)}{2(1-xy)}\right) - \mathrm{Li}_3\left(\frac{(1-x)(1+y)}{2(1-xy)}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{Li}_3\left(\frac{(1-x)(1-y)}{2(1+xy)}\right)+\mathrm{Li}_3\left(\frac{(1+x)(1+y)}{2(1+xy)}\right)\Bigg]\\
&+2(m_y+p_y-m_{xy}-p_{xy})\left[2\mathrm{Li}_2\left(x\right)-2\mathrm{Li}_2\left(-x\right)+\mathrm{Li}_2\left(\frac{-2x}{1-x}\right)-\mathrm{Li}_2\left(\frac{2x}{1+x}\right)\right]\\
&+4(m_{xy}-p_{xy})(\mathrm{Li}_2\left(-y\right)-\mathrm{Li}_2\left(y\right))-4(m_x+p_x-2l_2)\mathrm{Li}_2\left(\frac{1-x}{2}\right)\\
&-4(m_{xy}-p_{xy})\mathrm{Li}_2\left(\frac{1-y}{2}\right)-4(m_x+l_y-m_{xy})\mathrm{Li}_2\left(\frac{(1-x)y}{1-xy}\right)\\
&+4(p_x+l_y-m_{xy})\mathrm{Li}_2\left(\frac{-(1+x)y}{1-xy}\right)-4(m_x+l_y-p_{xy})\mathrm{Li}_2\left(\frac{-(1-x)y}{1+xy}\right)\\
&+4(p_x+l_y-p_{xy})\mathrm{Li}_2\left(\frac{(1+x)y}{1+xy}\right)\\
&+2(m_x+p_x+m_y+p_y-2m_{xy}-2l_2)\mathrm{Li}_2\left(\frac{(1-x)(1+y)}{2(1-xy)}\right)\\
&+2(m_x+p_x+m_y+p_y-2p_{xy}-2l_2)\mathrm{Li}_2\left(\frac{(1-x)(1-y)}{2(1+xy)}\right)\\
&+2p_x^2(m_y+p_y-m_{xy}-p_{xy})+2p_x(2(m_yl_y+m_yp_y+l_yp_y)\\
&+m_{xy}(-m_y-2l_y-3p_y+3m_{xy})+p_{xy}(-3m_y-2l_y-p_y+3p_{xy}))\\
&+2m_x(2p_x+m_y-2l_y+p_y)(m_y+p_y-m_{xy}-p_{xy})-p_y^2(m_{xy}+p_{xy})\\
&+2p_y(2m_{xy}^2+p_{xy}^2)+m_y^2(2p_y-m_{xy}-p_{xy})\\
&+2m_y(p_y^2+m_{xy}^2+2p_{xy}^2-p_y(3m_{xy}+p_{xy}))-2(m_{xy}^3+p_{xy}^3)\\
&+2l_2((4m_y+4p_y-3m_{xy})m_{xy}+(2m_y+2p_y-3p_{xy})p_{xy}\\
&-2(p_x+2m_x)(m_y+p_y-m_{xy}-p_{xy})-m_y^2-4m_yp_y-p_y^2)\\
&+2l_2^2(3(m_y+p_y)-2(2m_{xy}+p_{xy}))\\
&-(\pi^2/3)(m_y+p_y-8m_{xy}+6p_{xy})\Big\}\ .
\end{aligned}
$$

Here $\zeta_3=\zeta(3)$, $\zeta(z)$ is the Riemann zeta function, $l_z=\ln z$ for $z=x,y,2$, $p_z=\ln(1+z)$ and $m_z=\ln(1-z)$ for $z=x,y,xy$.

No two-dimensional harmonic polylogarithms!

Agreement with the leading power behaviour in the (Sudakov) limit $m^2 \ll |s|, |t|$ obtained by use of the strategy of expansion by regions

[M. Beneke and V.A. Smirnov, Nucl. Phys. **B522** (1998) 321; V.A. Smirnov, *Applied Asymptotic Expansions in Momenta and Masses* (Springer, Berlin, Heidelberg, 2001)]

$$\begin{aligned} B^{(0)}(s,t,m^2;\epsilon) &= -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^2}{s^2(-t)^{1+2\epsilon}} \left\{ 2\frac{L^2}{\epsilon^2} - \left[(2/3)L^3 + (\pi^2/3)L + 2\zeta_3\right]\frac{1}{\epsilon} \right. \\ & \quad -(2/3)L^4 + 2\ln(t/s)L^3 - 2(\ln^2(t/s) + 4\pi^2/3)L^2 \\ & + \left[4\mathrm{Li}_3(-t/s) - 4\ln(t/s)\mathrm{Li}_2(-t/s) + (2/3)\ln^3(t/s) - 2\ln(1+t/s)\ln^2(t/s) \right. \\ & \left. \left. +(8\pi^2/3)\ln(t/s) - 2\pi^2\ln(1+t/s) + 10\zeta_3\right] L + \pi^4/36 \right\} + O(m^2L^3, \epsilon), \end{aligned}$$

where $L = \ln(-m^2/s)$

Confirmed also numerically by T. Binoth and G. Heinrich

Perspectives

Sooner or later we shall achieve the limit in the process of analytical evaluation of Feynman integrals so that we shall be forced to proceed only numerically (see, e.g., G. Passarino, Nucl. Phys. **B619** (2001) 257).

New numerical methods: e.g., [T. Binoth and G. Heinrich, Nucl. Phys. **B585** (2000) 741; G. Heinrich, radcor/LL–02; G. Passarino, Nucl. Phys. **B619**; G. Passarino and S. Uccirati, Nucl. Phys. **B629** (2002) 97. A. Ghinculov and Y. Yao, Phys. Rev. D **63** (2001) 054510; Nucl. Phys. B **516** (1998) 385; S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087; Phys. Lett. **B504** (2001) 188]

However, the dramatic progress in the field of analytical evaluation of Feynman integrals shows that we have not yet exhausted our abilities.

Master integrals and a reduction procedure.

Methods to solve the reduction problem:

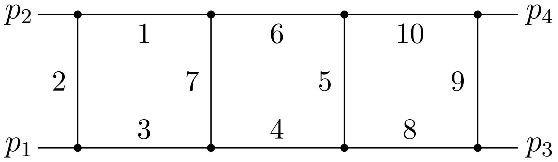
shifting dimension [Tarasov];

[Gehrmann & Remiddi];

non-recursive solutions of recurrence relations [P.A. Baikov, Phys. Lett. **B385** (1996) 404; Nucl. Instrum. Methods A **389** (1997) 347; Phys. Lett. **B474** (2000) 385; radcor/LL–02]

or, somehow else

$\#\text{loops} + \#\text{legs} = 3 + 4 = 7 \gg 1.$



Analytical result for general off-shell L-loop ladder planar diagram [N.I. Ussyukina and A.I. Davydychev, **B305** (1993) 136]

The general planar triple box Feynman integral

$$
\begin{aligned}
T(a_1, \ldots, a_{10}; s, t; \epsilon) \;=\; & \int\!\!\int\!\!\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l\, \mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}} \\
& \times \frac{1}{[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}} \\
& \times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}} \,,
\end{aligned}
$$

The master planar triple box $T^{(0)}(s, t; \epsilon) \equiv T(1, \ldots, 1; s, t; \epsilon)$

Three-loop BFKL analysis (virtual part) $\rightarrow$
Regge limit $t/s \rightarrow 0$

Expansion by regions
Leading power: only (1c-1c-1c) and (2c-2c-2c) regions contribute, with the leading power behaviour $1/t$

(2c-2c-2c) $\rightarrow$

$$
\begin{aligned}
T^{(0),(2c-2c-2c)}(s, t; \epsilon) \;=\; & \int\!\!\int\!\!\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l\, \mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}[2p_1 k + s]^{a_3}} \\
& \times \frac{1}{[2p_1 l + s]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}} \\
& \times \frac{1}{[2p_1 r + s]^{a_8}[(r+p_2+\tilde{p})^2]^{a_9}[r^2]^{a_{10}}}
\end{aligned}
$$

with $\tilde{p}^2 = t,\; 2p_1\tilde{p} = 0,\; 2p_2\tilde{p} = -t$

MB representation of the general planar triple box

$$
\begin{aligned}
T(a_1,\ldots,a_8;s,t,m^2;\epsilon) \; &= \; \frac{\left(i\pi^{d/2}\right)^3(-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}} \\
&\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)} \\
&\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)\Gamma(-z_5)\Gamma(-z_6)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)} \\
&\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7) \\
&\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7) \\
&\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7) \\
&\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7) \\
&\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7) \\
&\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,
\end{aligned}
$$

where $a_{589(10)} = a_5+a_8+a_9+a_{10}, a_{13} = a_1+a_3$, etc.

The master triple box:

$$
\begin{aligned}
T^{(0)}(s,t;\epsilon) = \frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}}\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
\times\frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
\times\frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
\times\Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
\times\Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
\times\Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)\,.
\end{aligned}
$$

NB: $\Gamma(-2\epsilon)$ in the denominator

The asymptotic expansion in the Regge limit $\to$ poles in the variable w that come from a $-w$ dependence of the gamma functions.

$\Gamma(-w) \to$ (h-h-h) contribution (order t^0)

The (1c-1c-1c) and (2c-2c-2c) contributions are generated by
$\Gamma(-\epsilon+z_2)\Gamma(-\epsilon-w-z_2+z_6+z_7)\Gamma(-1-\epsilon-z_6-z_7)$
$\Gamma(-\epsilon+z_3)\Gamma(-\epsilon-w-z_3+z_5+z_7)\Gamma(-1-\epsilon-z_5-z_7)$

The LO asymptotic behaviour $\to \Gamma(-1-3\epsilon-w)$

The Regge asymptotics of the planar triple box:

$$
\begin{aligned}
T^{(0)}(s,t;\epsilon) \;&=\; -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}}\left\{\frac{16}{9\epsilon^6}-\frac{5L}{3\epsilon^5}-\frac{3\pi^2}{2\epsilon^4}-\left[\frac{11\pi^2}{12}L+\frac{131\zeta(3)}{9}\right]\frac{1}{\epsilon^3}\right.\\
&+\left[\frac{49\zeta(3)}{3}L-\frac{1411\pi^4}{1080}\right]\frac{1}{\epsilon^2}-\left[\frac{503\pi^4}{1440}L-\frac{73\pi^2\zeta(3)}{4}+\frac{301\zeta(5)}{15}\right]\frac{1}{\epsilon}\\
&\left.+\left[\frac{223\pi^2\zeta(3)}{12}+149\zeta(5)\right]L-\frac{624607\pi^6}{544320}+\frac{167\zeta(3)^2}{9}+O(\epsilon)\right\},
\end{aligned}
$$

where $L=\ln s/t$

The unexpanded triple box is represented as

$$T^{(0)}(s,t;\epsilon) = -\frac{\left(i\pi^{d/2}\mathrm{e}^{-\gamma_{\mathrm{E}}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} c_j(x,L)\frac{1}{\epsilon^j}$$

where, up to $1/\epsilon^2$ terms, we have

$$c_6 = \frac{16}{9}\,, \quad c_5 = -\frac{5L}{3}\,, \quad c_4 = -\frac{3\pi^2}{2}\,,$$
$$c_3 = 3(\mathrm{Li}_3\,(-x) + L\,\mathrm{Li}_2\,(-x)) - \frac{3}{2}(L^2+\pi^2)\ln(1+x) - \frac{11}{12}L\pi^2 - \frac{131}{9}\zeta(3)\,,$$
$$c_2 = 3(S_{2,2}(-x) + L\,S_{1,2}(-x)) - 51\mathrm{Li}_4\,(-x)$$
$$+3\ln(1+x)(\mathrm{Li}_3\,(-x) + L\,\mathrm{Li}_2\,(-x)) - 37L\,\mathrm{Li}_3\,(-x) - \frac{23}{2}L^2\mathrm{Li}_2\,(-x)$$
$$-8\pi^2\mathrm{Li}_2\,(-x) - \frac{3}{4}(L^2+\pi^2)\ln^2(1+x)$$
$$-3\ln(1+x)\zeta(3) + \frac{3}{2}L^3\ln(1+x) + L\pi^2\ln(1+x) - \frac{1411}{1080}\pi^4 + \frac{49}{3}L\zeta(3)\,,$$

with $x = t/s$ (so that $L = -\ln x$)

c_6, c_5, c_4 and c_3 have been confirmed by a numerical check with the help of numerical integration in the space of alpha parameters [T. Binoth and G. Heinrich, Nucl. Phys. **B585** (2000) 741]

A hypothesis: c_1 and c_0 are expressed in terms of harmonic polylogarithms with parameters 0 and 1.

$H(a_1, a_2, \ldots, a_n; z)$, $a_i = 1, 0, -1$

[E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000) 725]

$$H(a_1, a_2, \ldots, a_n; z) = \int_0^z f(a_1; x) H(a_2, \ldots, a_n; x)\,,$$

where

$f(\pm 1; z) = 1/(1\mp z)$, $f(0; z) = 1/z$, $H(\pm 1; z) = \mp\ln(1\mp z)$, $H(0; z) = \ln z$

$$S_i(n)=\sum_{j=1}^{n} j^{-i}\,,\quad S_{ik}(n)=\sum_{j=1}^{n} j^{-i}S_k(j)\,,$$
$$S_{ikl}(n)=\sum_{j=1}^{n} j^{-i}S_{kl}(j)\,,\quad S_{iklm}(n)=\sum_{j=1}^{n} j^{-i}S_{klm}(j)$$

Series with at least $1/n^2$ dependence:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^6} &= \frac{\pi^6}{945}\,,\\
\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^5} &= \frac{\pi^6}{1260}-\frac{\zeta(3)^2}{2}\,,\\
\sum_{n=1}^{\infty} S_2(n-1)\frac{1}{n^4} &= -4\frac{\pi^6}{2835}+\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_1(n-1)^2\frac{1}{n^4} &= \frac{37\pi^6}{22680}-\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_3(n-1)\frac{1}{n^3} &= -\frac{\pi^6}{1890}+\frac{\zeta(3)^2}{2}\,,\\
\sum_{n=1}^{\infty} S_4(n-1)\frac{1}{n^2} &= \frac{5\pi^6}{2268}-\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_{13}(n-1)\frac{1}{n^2} &= \frac{61\pi^6}{45360}\,,\\
\sum_{n=1}^{\infty} S_2(n-1)^2\frac{1}{n^2} &= \frac{59\pi^6}{22680}-\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_1(n-1)^3\frac{1}{n^3} &= -\frac{11\pi^6}{5040}+2\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{1}{n^3} &= -\frac{121\pi^6}{45360}+2\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_{12}(n-1)\frac{1}{n^3} &= \frac{41\pi^6}{22680}-\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1)\frac{1}{n^2} &= \frac{167\pi^6}{45360}-\frac{3\zeta(3)^2}{2}\,,\\
\sum_{n=1}^{\infty} S_1(n-1)^2S_2(n-1)\frac{1}{n^2} &= \frac{23\pi^6}{3780}-\zeta(3)^2\,,\\
\sum_{n=1}^{\infty} S_1(n-1)^4\frac{1}{n^2} &= \frac{859\pi^6}{22680}+3\zeta(3)^2\,,
\end{aligned}$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_{112}(n-1)\frac{1}{n^2} &= \frac{17\pi^6}{4536} - \zeta(3)^2\,, \\
\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1)\frac{1}{n^2} &= \frac{313\pi^6}{45360} - 2\zeta(3)^2\,, \\
\sum_{n=1}^{\infty} \frac{1}{n^5} &= \zeta(5)\,, \\
\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^4} &= 2\zeta(5) - \frac{\pi^2\zeta(3)}{6}\,, \\
\sum_{n=1}^{\infty} S_2(n-1)\frac{1}{n^3} &= \frac{\pi^2\zeta(3)}{2} - \frac{11\zeta(5)}{2}\,, \\
\sum_{n=1}^{\infty} S_1(n-1)^2\frac{1}{n^3} &= \frac{\pi^2\zeta(3)}{6} - \frac{3\zeta(5)}{2}\,, \\
\sum_{n=1}^{\infty} S_3(n-1)\frac{1}{n^2} &= \frac{9\zeta(5)}{2} - \frac{\pi^2\zeta(3)}{3}\,, \\
\sum_{n=1}^{\infty} S_1(n-1)^3\frac{1}{n^2} &= \frac{\pi^2\zeta(3)}{6} + \frac{15\zeta(5)}{2}\,, \\
\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{1}{n^2} &= \frac{7\zeta(5)}{2} - \frac{\pi^2\zeta(3)}{6}\,, \\
\sum_{n=1}^{\infty} S_{12}(n-1)\frac{1}{n^2} &= 9\zeta(5) - \frac{2\pi^2\zeta(3)}{3}\,, \\
\sum_{n=1}^{\infty} S_2(n-1)\frac{1}{n^2} &= \frac{\pi^4}{120}\,, \\
\sum_{n=1}^{\infty} S_1(n-1)^2\frac{1}{n^2} &= \frac{11\pi^4}{360}\,, \\
\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^3} &= \frac{\pi^4}{360}\,, \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}\,, \\
\sum_{n=1}^{\infty} S_1(n-1)\frac{1}{n^2} &= \zeta(3)\,, \\
\sum_{n=1}^{\infty} \frac{1}{n^3} &= \zeta(3)\,, \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}\,.
\end{aligned}
$$

Series with the factor $1/n$ where the convergence is provided by other factors:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \psi''''(n+1)\frac{1}{n} &= -\frac{2\pi^6}{105} + 12\zeta(3)^2\,, \\
\sum_{n=1}^{\infty} \psi'''(n+1)S_1(n)\frac{1}{n} &= \frac{\pi^6}{1512}\,, \\
\sum_{n=1}^{\infty} \psi''(n+1)S_1(n)^2\frac{1}{n} &= \frac{\pi^6}{90} - 8\zeta(3)^2\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)^2 S_1(n)\frac{1}{n} &= -\frac{\pi^6}{432} + 2\zeta(3)^2\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)S_1(n)^3\frac{1}{n} &= \frac{269\pi^6}{22680}\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)\psi''(n+1)\frac{1}{n} &= \frac{61\pi^6}{22680} - 2\zeta(3)^2\,, \\
\sum_{n=1}^{\infty} \psi'''(n+1)\frac{1}{n} &= -\pi^2\zeta(3) + 12\zeta(5)\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)S_1(n)^2\frac{1}{n} &= \frac{\pi^2\zeta(3)}{3}\,, \\
\sum_{n=1}^{\infty} \psi''(n+1)S_1(n)\frac{1}{n} &= -\frac{2\pi^2\zeta(3)}{3} + 7\zeta(5)\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)^2\frac{1}{n} &= \frac{5\pi^2\zeta(3)}{6} - 9\zeta(5)\,, \\
\sum_{n=1}^{\infty} \psi''(n+1)\frac{1}{n} &= -\frac{\pi^4}{180}\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)S_1(n)\frac{1}{n} &= \frac{7\pi^4}{360}\,, \\
\sum_{n=1}^{\infty} \psi'(n+1)\frac{1}{n} &= \zeta(3)\,.
\end{aligned}
$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^6} = H(0,0,0,0,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1)\frac{z^n}{n^5} = H(0,0,0,0,1,1;z),$$
$$\sum_{n=1}^{\infty} S_2(n-1)\frac{z^n}{n^4} = H(0,0,0,1,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1)^2\frac{z^n}{n^4} = H(0,0,0,1,0,1;z) + 2H(0,0,0,1,1,1;z),$$
$$\sum_{n=1}^{\infty} S_3(n-1)\frac{z^n}{n^3} = H(0,0,1,0,0,1;z),$$
$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)^3\frac{z^n}{n^3} = {}& H(0,0,1,0,0,1;z) + 3H(0,0,1,0,1,1;z) + 3H(0,0,1,1,0,1;z)\\ &+6H(0,0,1,1,1,1;z),\end{aligned}$$
$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{z^n}{n^3} = {}& H(0,0,1,0,0,1;z) + H(0,0,1,0,1,1;z)\\ &+H(0,0,1,1,0,1;z),\end{aligned}$$
$$\sum_{n=1}^{\infty} S_{12}(n-1)\frac{z^n}{n^3} = H(0,0,1,0,0,1;z) + H(0,0,1,1,0,1;z),$$
$$\sum_{n=1}^{\infty} S_4(n-1)\frac{z^n}{n^2} = H(0,1,0,0,0,1;z),$$
$$\sum_{n=1}^{\infty} S_{13}(n-1)\frac{z^n}{n^2} = H(0,1,0,0,0,1;z) + H(0,1,1,0,0,1;z),$$
$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1)\frac{z^n}{n^2} = {}& H(0,1,0,0,0,1;z) + H(0,1,0,0,1,1;z)\\ &+H(0,1,1,0,0,1;z),\end{aligned}$$
$$\sum_{n=1}^{\infty} S_2(n-1)^2\frac{z^n}{n^2} = H(0,1,0,0,0,1;z) + 2H(0,1,0,1,0,1;z),$$
$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1)\frac{z^n}{n^2} = {}& H(0,1,0,0,0,1;z) + H(0,1,0,0,1,1;z)\\ &+H(0,1,0,1,0,1;z) + 2H(0,1,1,0,0,1;z)\\ &+H(0,1,1,0,1,1;z) + 2H(0,1,1,1,0,1;z),\end{aligned}$$
$$\begin{aligned}\sum_{n=1}^{\infty} S_1(n-1)^2S_2(n-1)\frac{z^n}{n^2} = {}& H(0,1,0,0,0,1;z) + 2H(0,1,0,0,1,1;z)\\ &+2H(0,1,0,1,0,1;z) + 2H(0,1,0,1,1,1;z)\\ &+2H(0,1,1,0,0,1;z) + 2H(0,1,1,0,1,1;z) + 2H(0,1,1,1,0,1;z),\end{aligned}$$

$$\sum_{n=1}^{\infty} S_1(n-1)^4 \frac{z^n}{n^2} = H(0,1,0,0,0,1;z) + 4H(0,1,0,0,1,1;z) + 6H(0,1,0,1,0,1;z)$$
$$+12H(0,1,0,1,1,1;z) + 4H(0,1,1,0,0,1;z) + 12H(0,1,1,0,1,1;z)$$
$$+12H(0,1,1,1,0,1;z) + 24H(0,1,1,1,1,1;z),$$
$$\sum_{n=1}^{\infty} S_{112}(n-1) \frac{z^n}{n^2} = H(0,1,0,0,0,1;z) + H(0,1,0,1,0,1;z)$$
$$+H(0,1,1,0,0,1;z) + H(0,1,1,1,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1) S_4(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + H(1,0,0,0,1,1;z)$$
$$+H(1,1,0,0,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1) S_{13}(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + H(1,0,0,0,1,1;z)$$
$$+H(1,0,1,0,0,1;z) + 2H(1,1,0,0,0,1;z)$$
$$+H(1,1,0,0,1,1;z) + 2H(1,1,1,0,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1)^2 S_3(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + 2H(1,0,0,0,1,1;z)$$
$$+H(1,0,0,1,0,1;z) + 2H(1,0,0,1,1,1;z) + H(1,0,1,0,0,1;z)$$
$$+2H(1,1,0,0,0,1;z) + 2H(1,1,0,0,1,1;z) + 2H(1,1,1,0,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1) S_2(n-1)^2 \frac{z^n}{n} = H(1,0,0,0,0,1;z) + H(1,0,0,0,1,1;z)$$
$$+2H(1,0,0,1,0,1;z) + 2H(1,0,1,0,0,1;z) + 2H(1,0,1,0,1,1;z)$$
$$+2H(1,0,1,1,0,1;z) + H(1,1,0,0,0,1;z) + 2H(1,1,0,1,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1)^2 S_{12}(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + 2H(1,0,0,0,1,1;z)$$
$$+2H(1,0,0,1,0,1;z) + 2H(1,0,0,1,1,1;z) + 3H(1,0,1,0,0,1;z)$$
$$+2H(1,0,1,0,1,1;z) + 3H(1,0,1,1,0,1;z) + 3H(1,1,0,0,0,1;z)$$
$$+4H(1,1,0,0,1,1;z) + 4H(1,1,0,1,0,1;z) + 2H(1,1,0,1,1,1;z)$$
$$+6H(1,1,1,0,0,1;z) + 4H(1,1,1,0,1,1;z) + 6H(1,1,1,1,0,1;z),$$
$$\sum_{n=1}^{\infty} S_1(n-1)^3 S_2(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + 3H(1,0,0,0,1,1;z)$$
$$+4H(1,0,0,1,0,1;z) + 6H(1,0,0,1,1,1;z) + 4H(1,0,1,0,0,1;z)$$
$$+6H(1,0,1,0,1,1;z) + 6H(1,0,1,1,0,1;z) + 6H(1,0,1,1,1,1;z)$$
$$+3H(1,1,0,0,0,1;z) + 6H(1,1,0,0,1,1;z) + 6H(1,1,0,1,0,1;z)$$
$$+6H(1,1,0,1,1,1;z) + 6H(1,1,1,0,0,1;z)$$
$$+6H(1,1,1,0,1,1;z) + 6H(1,1,1,1,0,1;z),$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_1(n-1)^5 \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + 5H(1,0,0,0,1,1;z) + 10H(1,0,0,1,0,1;z) \\
& +20H(1,0,0,1,1,1;z) + 10H(1,0,1,0,0,1;z) + 30H(1,0,1,0,1,1;z) \\
& +30H(1,0,1,1,0,1;z) + 60H(1,0,1,1,1,1;z) + 5H(1,1,0,0,0,1;z) \\
& +20H(1,1,0,0,1,1;z) + 30H(1,1,0,1,0,1;z) + 60H(1,1,0,1,1,1;z) \\
& +20H(1,1,1,0,0,1;z) + 60H(1,1,1,0,1,1;z) \\
& +60H(1,1,1,1,0,1;z) + 120H(1,1,1,1,1,1;z),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_1(n-1) S_{112}(n-1) \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + H(1,0,0,0,1,1;z) \\
& +H(1,0,0,1,0,1;z) + 2H(1,0,1,0,0,1;z) + H(1,0,1,0,1,1;z) \\
& +2H(1,0,1,1,0,1;z) + 2H(1,1,0,0,0,1;z) + H(1,1,0,0,1,1;z) \\
& +2H(1,1,0,1,0,1;z) + 3H(1,1,1,0,0,1;z) \\
& +H(1,1,1,0,1,1;z) + 3H(1,1,1,1,0,1;z),
\end{aligned}
$$

$$
\sum_{n=1}^{\infty} S_5(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z),
$$

$$
\sum_{n=1}^{\infty} S_{14}(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + H(1,1,0,0,0,1;z),
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_2(n-1) S_3(n-1) \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + H(1,0,0,1,0,1;z) \\
& +H(1,0,1,0,0,1;z),
\end{aligned}
$$

$$
\sum_{n=1}^{\infty} S_{23}(n-1) \frac{z^n}{n} = H(1,0,0,0,0,1;z) + H(1,0,1,0,0,1;z),
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_{12}(n-1) S_2(n-1) \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + 2H(1,0,0,1,0,1;z) \\
& +H(1,0,1,0,0,1;z) + H(1,0,1,1,0,1;z) \\
& +H(1,1,0,0,0,1;z) + 2H(1,1,0,1,0,1;z),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_{113}(n-1) \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + H(1,0,1,0,0,1;z) \\
& +H(1,1,0,0,0,1;z) + H(1,1,1,0,0,1;z),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_{212}(n-1) \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + H(1,0,0,1,0,1;z) \\
& +H(1,0,1,0,0,1;z) + H(1,0,1,1,0,1;z),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} S_{1112}(n-1) \frac{z^n}{n} = {} & H(1,0,0,0,0,1;z) + H(1,0,0,1,0,1;z) + H(1,0,1,0,0,1;z) \\
& +H(1,0,1,1,0,1;z) + H(1,1,0,0,0,1;z) + H(1,1,0,1,0,1;z) \\
& +H(1,1,1,0,0,1;z) + H(1,1,1,1,0,1;z),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{z^n}{n^5} = H(0,0,0,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)\frac{z^n}{n^4} = H(0,0,0,1,1;z),\\
&\sum_{n=1}^{\infty} S_2(n-1)\frac{z^n}{n^3} = H(0,0,1,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)^2\frac{z^n}{n^3} = H(0,0,1,0,1;z) + 2H(0,0,1,1,1;z),\\
&\sum_{n=1}^{\infty} S_3(n-1)\frac{z^n}{n^2} = H(0,1,0,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)^3\frac{z^n}{n^2} = H(0,1,0,0,1;z) + 3H(0,1,0,1,1;z)\\
&\qquad\qquad\qquad +3H(0,1,1,0,1;z) + 6H(0,1,1,1,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)S_2(n-1)\frac{z^n}{n^2} = H(0,1,0,0,1;z) + H(0,1,0,1,1;z)\\
&\qquad\qquad\qquad +H(0,1,1,0,1;z),\\
&\sum_{n=1}^{\infty} S_{12}(n-1)\frac{z^n}{n^2} = H(0,1,0,0,1;z) + H(0,1,1,0,1;z),\\
&\sum_{n=1}^{\infty} S_4(n-1)\frac{z^n}{n} = H(1,0,0,0,1;z),\\
&\sum_{n=1}^{\infty} S_{13}(n-1)\frac{z^n}{n} = H(1,0,0,0,1;z) + H(1,1,0,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)S_3(n-1)\frac{z^n}{n} = H(1,0,0,0,1;z) + H(1,0,0,1,1;z)\\
&\qquad\qquad\qquad +H(1,1,0,0,1;z),\\
&\sum_{n=1}^{\infty} S_2(n-1)^2\frac{z^n}{n} = H(1,0,0,0,1;z) + 2H(1,0,1,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)S_{12}(n-1)\frac{z^n}{n} = H(1,0,0,0,1;z) + H(1,0,0,1,1;z) + H(1,0,1,0,1;z)\\
&\qquad\qquad +2H(1,1,0,0,1;z) + H(1,1,0,1,1;z) + 2H(1,1,1,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)^2S_2(n-1)\frac{z^n}{n} = H(1,0,0,0,1;z) + 2H(1,0,0,1,1;z)\\
&\qquad\qquad\qquad +2H(1,0,1,0,1;z) + 2H(1,0,1,1,1;z) + 2H(1,1,0,0,1;z)\\
&\qquad\qquad\qquad +2H(1,1,0,1,1;z) + 2H(1,1,1,0,1;z),\\
&\sum_{n=1}^{\infty} S_1(n-1)^4\frac{z^n}{n} = H(1,0,0,0,1;z) + 4H(1,0,0,1,1;z) + 6H(1,0,1,0,1;z)
\end{aligned}
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$$
\begin{aligned}
&+12H(1,0,1,1,1;z)+4H(1,1,0,0,1;z)+12H(1,1,0,1,1;z)\\
&+12H(1,1,1,0,1;z)+24H(1,1,1,1,1;z),
\end{aligned}
$$

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\begin{aligned}
\sum_{n=1}^{\infty} S_{112}(n-1)\frac{z^n}{n} &= H(1,0,0,0,1;z)+H(1,0,1,0,1;z)+H(1,1,0,0,1;z)\\
&\quad +H(1,1,1,0,1;z).
\end{aligned}
$$