Potential NRQCD and Heavy Quarkonium Spectrum in NNNLO

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TOPICS DISCUSSED

- Effective theory description of the nonrelativistic heavy quark-antiquark system:
QCD $\longrightarrow$ NRQCD $\longrightarrow$pNRQCD.

- Asymptotic expansion of Feynman integrals and dimensional regularization

- pNRQCD Hamiltonian in NNNLO.

- Retardation (ultrasoft) effects.

- Heavy quarkonium spectrum at $\mathcal{O}(\alpha_s^5 m_q)$.

- Phenomenology of $t\bar{t}$ threshold production in e^+e^- annihilation.

WHY NR REGIME OF QCD IS SO INTERESTING?

- Highly non-trivial multiscale dynamics.

- “Clean” on both theoretical and experimental side in contrast to *e.g.* BFKL.

- First principle QCD predictions.

- High order approximation is available.

- Reach phenomenology: heavy quarks masses, strong coupling constant, Higgs (?)

NONRELATIVISTIC EFFECTIVE THEORY

• Near the threshold the heavy quarks are non-relativistic $\longrightarrow$ two small parameters : the quark velocity v and the strong coupling constant α_s.

• The expansion in v may be performed directly in the Lagrangian of QCD that leads us to the the concept of the effective field theory.

Characteristic scales (regions) of the threshold problem:

Hard $\longrightarrow$ energy and momentum scale like m_q,

Soft $\longrightarrow$ energy and momentum scale like vm_q,

Potential $\longrightarrow$ energy scales like v^2m_q, momentum scales like vm_q,

Ultrasoft (gluons) $\longrightarrow$ energy and momentum scale like v^2m_q.

QCD $\rightarrow$ NRQCD

$\uparrow$

Hard quarks and gluons integrated out

(Caswell, Lepage)

$$
\begin{aligned}
\mathcal{L}_{\text{NRQCD}} = \psi^\dagger \left(iD^0 + \frac{\boldsymbol{D}^2}{2m_q} \right) \psi + \tfrac{1}{8m_q^3}\, \psi^\dagger \boldsymbol{D}^4 \psi \\
- \tfrac{c_1\, g_s}{2m_q}\, \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{B} \psi + \tfrac{c_2\, g_s}{8m_q^2}\, \psi^\dagger \left(\boldsymbol{D} \cdot \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{D} \right) \psi \\
+ \tfrac{c_3\, i g_s}{8m_q^2}\, \psi^\dagger \boldsymbol{\sigma} \cdot \left(\boldsymbol{D} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{D} \right) \psi + \ldots \\
+ (\psi \rightarrow \chi) \\
+ \textstyle\sum_\Gamma \tfrac{c_4}{m_q^2} (\psi^\dagger \Gamma \chi)(\chi^\dagger \Gamma \psi) + \ldots + \mathcal{L}_{\text{light}}
\end{aligned}
$$

- The contributions from harder scales result in the Wilson coefficients c_i, leading to an expansion in α_s, as well as in the higher-dimensional operators of the nonrelativistic Hamiltonian, corresponding to an expansion in v.

NRQCD $\rightarrow$ pNRQCD

$\uparrow$

Soft quarks, soft and potential gluons
integrated out

(Pineda, Soto)

$$\mathcal{L}_{\text{PNRQCD}} = \psi^\dagger \left(i\partial^0 + \frac{\boldsymbol{\partial}^2}{2m_q} - g_s \boldsymbol{r}\boldsymbol{E}(t,0) \right) \psi$$

$$+ \; (\psi \rightarrow \chi)$$

$$+ \int d\boldsymbol{r}' \left[\psi^\dagger \psi\right](\boldsymbol{r}+\boldsymbol{r}')\, V(\boldsymbol{r}') \left[\chi^\dagger \chi\right](\boldsymbol{r}) \; + \ldots + \mathcal{L}_{\text{light}}$$

- The dynamics of the quarks is governed by the effective, nonrelativistic Schrödinger equation and by their multipole interaction to the ultrasoft gluons.

- Regular expansion in α_s and v.

pNRQCD FEYNMAN RULES

Coupling to ultrasoft gluon field

$$g_s \boldsymbol{r} \cdot \boldsymbol{E} + \ldots$$

Singlet and octet propagators

$$(\mathcal{H}^{s,o} - E)\, G^{s,o}(\boldsymbol{r}\,, \boldsymbol{r}', E) = \delta(\boldsymbol{r} - \boldsymbol{r}')\,,$$

$$\begin{aligned} G^s(\boldsymbol{r}, \boldsymbol{r}', E) &= \sum_{n=1}^{\infty} \frac{\psi_n^{s*}(\boldsymbol{r})\psi_n^s(\boldsymbol{r}')}{E_n - E} + \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{\psi_{\boldsymbol{k}}^{s*}(\boldsymbol{r})\psi_{\boldsymbol{k}}^s(\boldsymbol{r}')}{\boldsymbol{k}^2/m_q - E}\,, \\ G^o(\boldsymbol{r}, \boldsymbol{r}', E) &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{\psi_{\boldsymbol{k}}^{o*}(\boldsymbol{r})\psi_{\boldsymbol{k}}^o(\boldsymbol{r}')}{\boldsymbol{k}^2/m_q - E} \end{aligned}$$

Effective Hamiltonian

$$\mathcal{H}^{s,o} = -\frac{\Delta \boldsymbol{r}}{m_q} + V_C^{s,o}(r) + \delta\mathcal{H}^{s,o}\,,$$

$$\begin{aligned} V_C^s(r) &= -C_F \frac{\alpha_s}{r}\,, \\ V_C^o(r) &= \left(\frac{C_A}{2} - C_F\right)\frac{\alpha_s}{r} \end{aligned}$$

Time-independent perturbation theory

$$\delta G^s(\boldsymbol{r}, \boldsymbol{r}', E) = -\int \mathrm{d}^3\boldsymbol{r}'' G_C^s(\boldsymbol{r}, \boldsymbol{r}'', E)\delta\mathcal{H}^s(r'')G_C^s(\boldsymbol{r}'', \boldsymbol{r}', E) + \ldots$$

Perturbative calculations within the effective theory

Expansion of the Lagrangian $\rightarrow$
Particular subspace of the total phase space $\rightarrow$
Restrictions on the virtual momenta.

Scale separation $\longrightarrow$ spurious divergences.

"Old" technique:	Explicit momentum cut-off + gluon mass
"Modern" technique:	Dimensional regularization (Pineda, Soto; Czarnecki,Melnikov, Yelkhovsky)

Threshold expansion of Feynman diagram

(Beneke, Smirnov)

One can expand a Feynman diagram in v BEFORE integration over virtual momentum as follows

- Consider various regions of the loop momenta and expand, in every region, the integrand in Taylor series with respect to the parameters that are there considered small;
- Integrate the expanded integrand over the whole integration domain of the loop momenta;
- Put to zero any scaleless integral.

After the expansion, the effective theory vertices/propagators are recovered

The Lagrangian of the effective theory can be employed for a perturbative calculation without explicit restrictions on the virtual momenta if dimensional regularization is used and the formal expressions derived from the Feynman rules of the effective theory are understood in the sense of the threshold expansion (Kniehl,Penin,Smirnov,Steinhauser).

Formulation of effective theory with two crucial virtues: the absence of additional regulator scales and the automatic matching of the contributions from different scales.

In the threshold expansion the contribution of a given region is in one-to-one correspondence to an effective theory quantity:

hard region	$\longrightarrow$	Wilson coefficients,
soft region, potential gluons	$\longrightarrow$	corrections to the potential,
potential quarks	$\longrightarrow$	time-independent perturbation theory,
ultrasoft region	$\longrightarrow$	emission/absorption of dynamical gluons by the bound state.

No ambiguity in the effective theory calculations, *e.g.* with the expansion of the quark propagator

$$S(k) = \frac{1}{k_0 - \boldsymbol{k}^2/(2m_q) + i\varepsilon},$$

in $1/m_q$.

Threshold expansion prescription:

to expand if k is soft;

not to expand if k is potential*.

* Expansion is allowed if there is no Coulomb pinches.

Effective Hamiltonian up to NNNLO ($l=0$)

$$\mathcal{H} = (2\pi)^3\delta(\boldsymbol{q})\left(\frac{\boldsymbol{p}^2}{m_q} - \frac{\boldsymbol{p}^4}{4m_q^3}\right) + C_c(\alpha_s)V_C(|\boldsymbol{q}|) + C_{1/m}(\alpha_s)V_{1/m}(|\boldsymbol{q}|)$$
$$+\frac{\pi C_F\alpha_s(\mu)}{m_q^2}\left[C_\delta(\alpha_s) + C_p(\alpha_s)\frac{\boldsymbol{p}^2+\boldsymbol{p}'^2}{2\boldsymbol{q}^2} + C_s(\alpha_s)\boldsymbol{S}^2\right],$$

Static potential

$$C_c(\alpha_s)V_C(|\boldsymbol{q}|) = -\frac{4\pi C_F\alpha_s(|\boldsymbol{q}|)}{\boldsymbol{q}^2}\left[1 + \frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2\right.$$
$$\left. + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3\left(a_3 + 8\pi^2 C_A^3\ln\frac{\mu^2}{\boldsymbol{q}^2}\right) + \ldots\right].$$

$1/m_q$ potential

$$C_{1/m}(\alpha_s)V_{1/m}(|\boldsymbol{q}|) = \frac{\pi^2 C_F\alpha_s^2(|\boldsymbol{q}|)}{m_q|\boldsymbol{q}|}\left\{b_1\right.$$
$$\left. + \frac{\alpha_s(|\boldsymbol{q}|)}{\pi}\left[b_2 - \frac{4}{3}\left(C_A^2 + 2C_AC_F\right)\ln\frac{\mu^2}{\boldsymbol{q}^2}\right] + \ldots\right\}.$$

“Breit” potential

$$\alpha_s(\mu)C_\delta(\alpha_s) = \alpha_s(|\boldsymbol{q}|)\left\{d_0^\delta + \frac{\alpha_s(\mu)}{\pi}\left[d_1^\delta + \frac{4}{3}(C_A - 2C_F)\ln\frac{\mu^2}{\boldsymbol{q}^2}\right] + \ldots\right\},$$
$$\alpha_s(\mu)C_p(\alpha_s) = \alpha_s(|\boldsymbol{q}|)\left[d_0^p + \frac{\alpha_s(\mu)}{\pi}\left(d_1^p - \frac{8}{3}C_A\ln\frac{\mu^2}{\boldsymbol{q}^2}\right) + \ldots\right],$$
$$\alpha_s(\mu)C_s(\alpha_s) = \alpha_s(|\boldsymbol{q}|)\left[d_0^i + \frac{\alpha_s(\mu)}{\pi}d_1^i + \ldots\right],$$
$$\alpha_s(\mu)C_s^a(\alpha_s) = \alpha_s(m_q)\left[d_0^{s,a} + \frac{\alpha_s(m_q)}{\pi}d_1^{s,a} + \ldots\right],$$

Wilson coefficients

$$
\begin{aligned}
a_1 &= \frac{31}{9}C_A - \frac{20}{9}T_F n_l \,, \\
a_2 &= \left[\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right] C_A^2 + \left(\frac{20}{9}T_F n_l\right)^2 \\
&- \left[\frac{1798}{81} + \frac{56}{3}\zeta(3)\right] C_A T_F n_l - \left[\frac{55}{3} - 16\zeta(3)\right] C_F T_F n_l \,,
\end{aligned}
$$

$$
a_3 = 4^3 \times \begin{cases} 98 & \text{if } n_l = 4\,, \\ 60 & \text{if } n_l = 5\,. \end{cases}
$$

(1-loop: Fischler; 2-loop: Peter; Schröder; Kniehl, Penin, Smirnov, Steinhauser; 3-loop Padé: Chishtie, Elias)

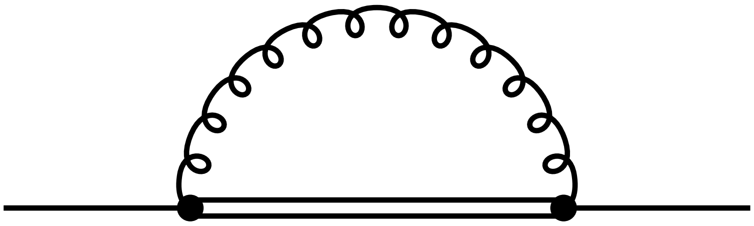
$$
\begin{aligned}
b_1 &= -C_A + \frac{C_F}{2} \,, \\
b_2 &= -\left(\frac{101}{36} + \frac{4}{3}\ln 2\right) C_A^2 + \left(\frac{65}{18} - \frac{8}{3}\ln 2\right) C_A C_F \\
&+ \frac{49}{36} C_A T_F n_l - \frac{2}{9} C_F T_F n_l \,.
\end{aligned}
$$

(1-loop Gupta, Radford; 2-loop Kniehl, Penin, Smirnov, Steinhauser)

$$
\begin{aligned}
d_1^\delta &= \left(2 - \frac{17}{6}\ln\frac{\boldsymbol{q}^2}{m_q^2}\right) C_A + \left(2 - \frac{1}{3}\ln\frac{\boldsymbol{q}^2}{m_q^2}\right) C_F - \left(\frac{64}{15} + 4\ln 2 - i2\pi\right) T \\
d_1^s &= \left(\frac{22}{27} + \frac{7}{6}\ln\frac{\boldsymbol{q}^2}{m_q^2}\right) C_A - \frac{2}{3}C_F - \frac{20}{27}T_F n_l + (2 - 2\ln 2 + i\pi)T_F \,, \\
d_1^p &= -\frac{31}{9}C_A + \frac{20}{9}T_F n_l \,.
\end{aligned}
$$

(Manohar, Stewart; Kniehl, Penin, Smirnov, Steinhauser).

Radiation and absorption of ultrasoft gluons in NNNLO



Instanteneous contribution (Kniehl, Penin; Kniehl, Penin, Smirnov, Steinhauser)

$$\delta^{\rm us}\mathcal{H} = \left(\frac{5}{18} - \frac{\ln 2}{3} + \frac{1}{6}\ln\frac{\mu^6}{\left(E_1^C\right)^2 \boldsymbol{q}^4}\right)\frac{C_A^3 C_F \alpha_s^4}{\boldsymbol{q}^2}$$
$$+ \left(\left(\frac{13}{9} + \frac{2}{3}\ln\frac{\mu^4}{\left(E_1^C\right)^2 \boldsymbol{q}^2}\right) C_A^2 + \left(-\frac{4}{9} + \frac{4}{3}\ln\frac{\mu^4}{\left(E_1^C\right)^2 \boldsymbol{q}^2}\right) C_A C_F\right)\frac{\pi C_F \alpha_s^2}{m_q|\boldsymbol{q}|}$$
$$- \left(\frac{20}{9} - \frac{8}{3}\ln 2 + \frac{4}{3}\ln\frac{\mu^2}{\left(E_1^C\right)^2}\right)\left((C_A - 2C_F) + 2C_A\frac{\boldsymbol{p}^2 + \boldsymbol{p}'^2}{2\boldsymbol{q}^2}\right)\frac{C_F\alpha_s}{m_q^2}\Bigg] ,$$

Retarded contribution is given by QCD analog of QED "Bethe logarithm" (Kniehl, Penin)

$$L_n^E = \frac{1}{\alpha_s^2 C_F^2 E_n^C}\int_0^\infty \frac{d^3k}{(2\pi)^3}|\langle \mathbf{r}\rangle_{\mathbf{k}n}|^2\left(E_n^C - k^2/m_q\right)^3 \ln\frac{E_1^C}{E_n^C - k^2/m_q} .$$

Heavy Quarkonium Spectrum at $\mathcal{O}(\alpha_s^5 m_q)$

Perturbative ground state energy

$$E_1^{\text{p.t.}} = E_1^C + \delta E_1^{(1)} + \delta E_1^{(2)} + \delta E_1^{(3)} + \dots\,,$$

$\delta E_1^{(1,2)}$ are known since 1998 (Pineda, Yndurain)

The $\mathcal{O}\left(\alpha_s^3\right)$ corrections to the energy levels arise from several sources:

(i) matrix elements of the N^{3}LO operators of the effective Hamiltonian between Coulomb wave functions;

(ii) higher iterations of the NLO and NNLO operators of the effective Hamiltonian in time-independent perturbation theory;

(iii) matrix elements of the N^{3}LO instantaneous operators generated by the emission and absorption of ultrasoft gluons;

(iv) the retarded ultrasoft contribution.

Numerical result for NNNLO correction to the ground state energy

$$\begin{aligned}\delta E_1^{(3)} &= \alpha_s^3(\mu_s) E_1^C \left[\left(\begin{array}{c} 70.590|_{n_l=4} \\ 56.732|_{n_l=5}\end{array}\right) + 15.297 \ln(\alpha_s(\mu_s))\right.\\ &\qquad \left. + 0.001\, a_3 + \left(\begin{array}{c} 34.229|_{n_l=4} \\ 26.654|_{n_l=5}\end{array}\right)\Bigg|_{\beta_0^3}\right],\end{aligned}$$

Correction to the ground state energy for vanishing β-function (Kniehl, Penin, Smirnov, Steinhauser)

$$\delta E_1^{(3)}\Big|_{\beta(\alpha_s)=0} = -E_1^C \frac{\alpha_s^3}{\pi}\left\{-\frac{a_1 a_2 + a_3}{32\pi^2} + \left[-\frac{C_A C_F}{2} + \left(-\frac{19}{16} + \frac{S(S+1)}{2}\right) C_F^2\right] a_1\right.$$
$$+ \left[-\frac{1}{6} + \ln 2 + L_{\alpha_s}\right]\frac{C_A^3}{6} + \left[-\frac{49}{36} + \frac{4}{3}(\ln 2 + L_{\alpha_s})\right] C_A^2 C_F$$
$$+ \left[\left(-\frac{5}{72} + \frac{10}{3}\ln 2 + \frac{37}{6}L_{\alpha_s}\right) + \left(\frac{85}{54} - \frac{7}{6}L_{\alpha_s}\right) S(S+1)\right] C_A C_F^2$$
$$+ \left[\frac{50}{9} + \frac{8}{3}\ln 2 + 3L_{\alpha_s} - \frac{S(S+1)}{3}\right] C_F^3$$
$$+ \left[-\frac{32}{15} + 2\ln 2 + (1-\ln 2)S(S+1)\right] C_F^2 T_F$$
$$\left. + \frac{49 C_A C_F T_F n_l}{36} + \left[\frac{11}{18n} - \frac{10}{27}S(S+1)\right] C_F^2 T_F n_l + \frac{2}{3} C_F^3 L_1^E\right\},$$

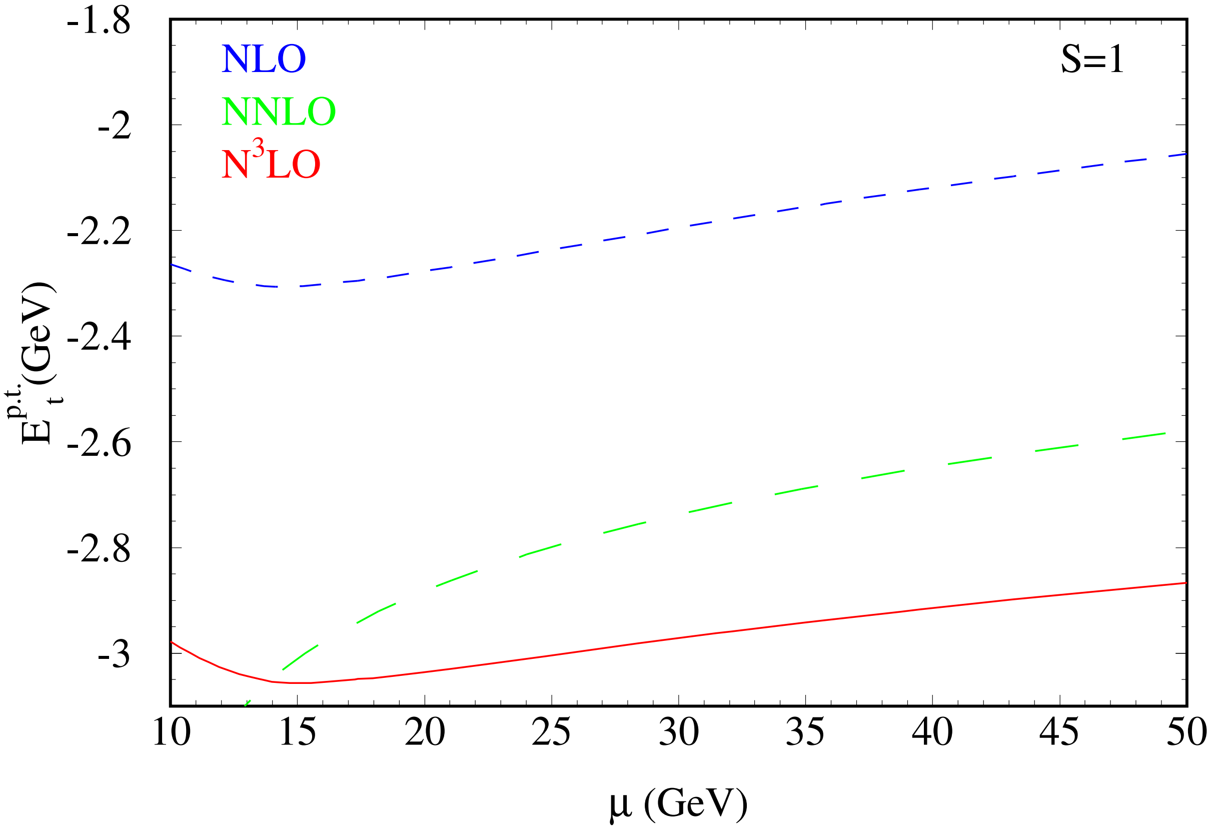
$L_{\alpha_s} = -\ln(C_F\alpha_s)$, $L_1^E = -81.5379$.

Correction to the ground state energy proportional to β-function (Penin, Steinhauser)

$$\delta E_1^{(3)}\Big|_{\beta(\alpha_s)} = E_n^C \left(\frac{\alpha_s}{\pi}\right)^3 \left\{32\beta_0^3 L_\mu^3 + \left[40\beta_0^3 + 12 a_1 \beta_0^2 + 28\beta_1\beta_0\right] L_\mu^2\right.$$
$$+ \left[\left(\frac{16\pi^2}{3} + 64\zeta(3)\right)\beta_0^3 + 10 a_1 \beta_0^2 + \left(40\beta_1 + \frac{a_1^2}{2} + a_2\right.\right.$$
$$\left.\left. + 8\pi^2 C_A C_F + \left(\frac{21\pi^2}{2} - \frac{16\pi^2}{3}S(S+1)\right) C_F^2\right)\beta_0 + 3a_1\beta_1 + 4\beta_2\right] L_\mu$$
$$+ \left(-8 + 4\pi^2 + \frac{2\pi^4}{45} + 64\zeta(3) - 8\pi^2\zeta(3) + 96\zeta(5)\right)\beta_0^3 + \left(\frac{2\pi^2}{3} + 8\zeta(3)\right) a_1\beta_0^2$$
$$+ \left(\left(8 + \frac{7\pi^2}{3} + 16\zeta(3)\right)\beta_1 - \frac{a_1^2}{8} + \frac{3}{4}a_2 + \left(6\pi^2 - \frac{2\pi^4}{3}\right) C_A C_F\right.$$
$$\left.\left. + \left(8\pi^2 - \frac{4\pi^4}{3} + \left(-\frac{4\pi^2}{3} + \frac{4\pi^4}{9}\right) S(S+1)\right) C_F^2\right)\beta_0 + 2a_1\beta_1 + 4\beta_2\right\}.$$

$L_\mu = \ln(\mu/C_F\alpha_s m_q)$.

Top-antitop threshold production



Relatively large top quark width smears out the Coulomb-like resonances below the threshold. Only the ground state pole results in the well-pronounced resonance:

$$E_{\mathrm{res}} = (1.9833 + 0.007\,\frac{m_t - 174.3\ \mathrm{GeV}}{174.3\ \mathrm{GeV}} \pm 0.0009) \times m_t\,,$$

The top quark pole mass with 80 MeV accuracy!