Hadronic light-by-light scattering contribution to the muon $g-2$

or

Loops and Legs with Hadrons

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Contents

M. Knecht, A.N., PRD 65, 073034 (2002); hep-ph/0111058

M. Knecht, A.N., M. Perrottet, E. de Rafael, PRL 88, 071802 (2002); hep-ph/0111059

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1. Introduction

1. Muon $g-2$: • Important test of Standard Model
   • Sensitive to New Physics

   Need to understand hadronic contributions: vacuum polarization; light-by-light scattering

2. General approach by the Barcelona-Marseille group to this and similar problems: Reduce model dependence by implementing correct matching with QCD at low and high energies for the relevant Green's functions:

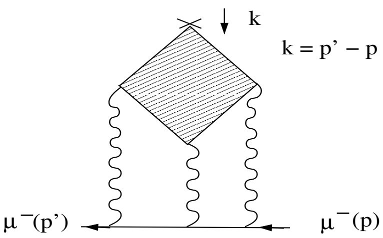
   $$\boxed{\begin{array}{c}\text{Chiral Perturbation Theory (ChPT) at low } Q^2 \\ \updownarrow \\ \text{Resonance region} \\ \updownarrow \\ \text{Operator Product Expansion (OPE) at high } Q^2\end{array}}$$

   Framework of large-N_C QCD: infinite set of narrow resonances; in practice, sufficient to take a few resonance states in each channel to reproduce leading behavior in ChPT and OPE

3. Motivations for our study

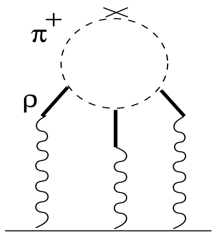
   - Troubled history of theoretical estimates for hadronic light-by-light scattering since 1976
   - Pion-exchange contribution: use new ansatz for pion-photon-photon transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ (Knecht + A.N., 2001) with correct high-energy behavior
   - Wanted to push analytical calculation further

Hadronic light-by-light scattering: present status

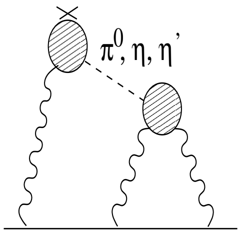


Contribution to $a_\mu \times 10^{10}$

Hayakawa, Kinoshita, (Sanda): +9.0 (1.5)
Bijnens, Pallante, Prades: +8.3 (3.2)

$=$  $+ \;---$

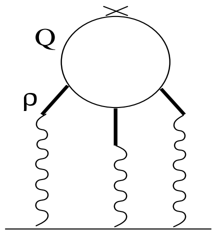
-0.5 (0.8)
-1.9 (1.3)
Undressed: -4.5

$+$  $+ \;---$

$$\boxed{\begin{array}{ll} & +8.3\ (0.6) \\ & +8.5\ (1.3) \\ \text{Knecht, A.N. :} & +8.3\ (1.2) \end{array}}$$

$+$ Exchange of higher resonances $(f_0, a_1, \ldots)$

+0.174 (only a_1)
-0.4 (0.3) $(f_0 + a_1)$

$+$  $+ \;---$

+1.0 (1.1)
+2.1 (0.3)
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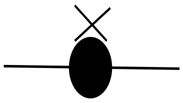
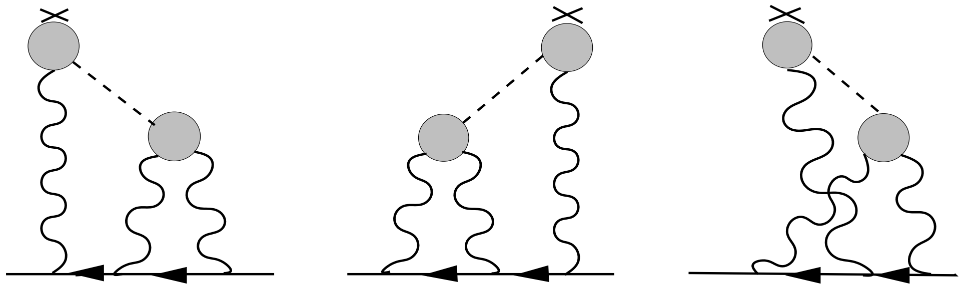
$\Uparrow$

corresponds to local counterterm $\sim \bar\psi \sigma^{\mu\nu} \psi F_{\mu\nu}$
in Effective Field Theory approach (de Rafael, 1994)



2. Pion-pole contribution



Blob: Pion-photon-photon transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$

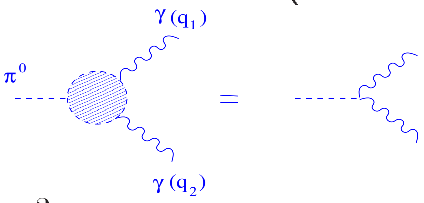
$i \int d^4x e^{iq\cdot x} \langle \Omega | T\{j_\mu(x) j_\nu(0)\} | \pi^0(p) \rangle = \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha p^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q^2, (p-q)^2)$

$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} = & -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \\
& \times \frac{1}{[(p+q_1)^2 - m^2][(p-q_2)^2 - m^2]} \\
& \times \left[\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, (q_1+q_2)^2)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2, 0)}{q_2^2 - M_\pi^2}\, T_1(q_1, q_2; p) \right. \\
& \left. + \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0)}{(q_1+q_2)^2 - M_\pi^2}\, T_2(q_1, q_2; p) \right]
\end{aligned}
$$

- $T_1(q_1, q_2; p), T_2(q_1, q_2; p)$: polynomial of 6th-degree in the momenta
- 5 independent variables in two-loop integrals:
  $p \cdot q_1, p \cdot q_2, q_1^2, q_2^2$, and $q_1 \cdot q_2$ (Note: $p^2 = m^2$)
- Difficult to proceed for general form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$
- Consider here a certain class of form factors
  (a) Correct high-energy behavior (Knecht + A.N., 2001)
  (b) Have already been used in literature (VMD) with results comparable to those from more complicated models (ENJL, HLS, χQM)

$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ form factors

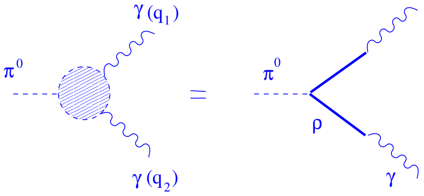
- Point-like vertex (Wess-Zumino-Witten WZW; Adler-Bell-Jackiw anomaly)



$\Rightarrow \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \text{const}$

$q_1^2, q_2^2 \to 0 \;\widehat{=}\;$ long wavelengths $\Rightarrow$ photons cannot resolve internal structure of vertex $\Rightarrow$ all form factors go to a constant

- Vector meson dominance (VMD)



$$\Rightarrow \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \sim \frac{1}{(q_1^2 - M_\rho^2)} \frac{1}{(q_2^2 - M_\rho^2)}$$

- More sophisticated form factors: framework of large-N_C QCD ('t Hooft, 1974)

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)|_{N_C\to\infty} = \sum_{ij} \frac{c_{ij}(q_1^2, q_2^2)}{(q_1^2 - M_{V_i}^2)(q_2^2 - M_{V_j}^2)}$$

with constraints on $c_{ij}(q_1^2, q_2^2)$ from Long Distances (WZW term):

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = -\frac{N_C}{12\pi^2 F_\pi}$$

and Short Distances (OPE):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma^*}(\lambda^2 q^2, (p-\lambda q)^2) = \frac{2}{3}\frac{F_\pi}{q^2}\left\{\frac{1}{\lambda^2} + \frac{1}{\lambda^3}\frac{q\cdot p}{q^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)\right\}$$

Constraints from OPE not fulfilled by VMD form factor !

$$\mathcal{F}^{LMD}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3}\frac{q_1^2 + q_2^2 - c_V}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}$$

$$\mathcal{F}^{LMD+V}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3}\frac{q_1^2 q_2^2 (q_1^2 + q_2^2) + h_1 (q_1^2 + q_2^2)^2 + h_2 q_1^2 q_2^2 + h_5 (q_1^2 + q_2^2) + h_7}{(q_1^2 - M_{V_1}^2)(q_1^2 - M_{V_2}^2)(q_2^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2)}$$

LMD+V ansatz: unknown parameters h_1, h_2, h_5
Determined phenomenologically: CLEO data for $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, 0)$ [$\sim 1/Q^2$ for $Q^2 \to \infty$], Decay $\pi^0 \to e^+e^-$

Two-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$

For the above class of form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ we can perform all angular integrations in two-loop integrals analytically !

Method of Gegenbauer polynomials

(Baker, Johnson, Willey 1964, 1967; Rosner 1967; Levine + Roskies 1974; Levine, Remiddi, Roskies 1979)

Key observation:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3}\left[f(q_1^2) - \sum_{M_{V_i}} \frac{1}{q_2^2 - M_{V_i}^2} g_{M_{V_i}}(q_1^2)\right]$$

$\Rightarrow$ can cancel all dependences on $q_1 \cdot q_2$ in numerators in form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, (q_1+q_2)^2)$ in loop integrals

Propagators in Euclidian space:

$$\frac{1}{(K-L)^2 + M^2} = \frac{Z_{KL}^M}{|K||L|} \sum_{n=0}^{\infty} (Z_{KL}^M)^n \, C_n(\hat{K} \cdot \hat{L})$$

$$Z_{KL}^M = \frac{K^2 + L^2 + M^2 - \sqrt{(K^2 + L^2 + M^2)^2 - 4K^2L^2}}{2|K||L|}$$

Use orthogonality conditions of the Gegenbauer polynomials:

$$\begin{aligned}\int d\Omega(\hat{K})\, C_n(\hat{Q}_1 \cdot \hat{K})\, C_m(\hat{K} \cdot \hat{Q}_2) &= 2\pi^2 \frac{\delta_{nm}}{n+1} C_n(\hat{Q}_1 \cdot \hat{Q}_2) \\ \int d\Omega(\hat{K})\, C_n(\hat{Q} \cdot \hat{K})\, C_m(\hat{K} \cdot \hat{Q}) &= 2\pi^2 \delta_{nm}\end{aligned}$$

$\hat{Q}_1 \cdot \hat{K} =$ Cosine of angle between the four-dim. vectors Q_1 and K

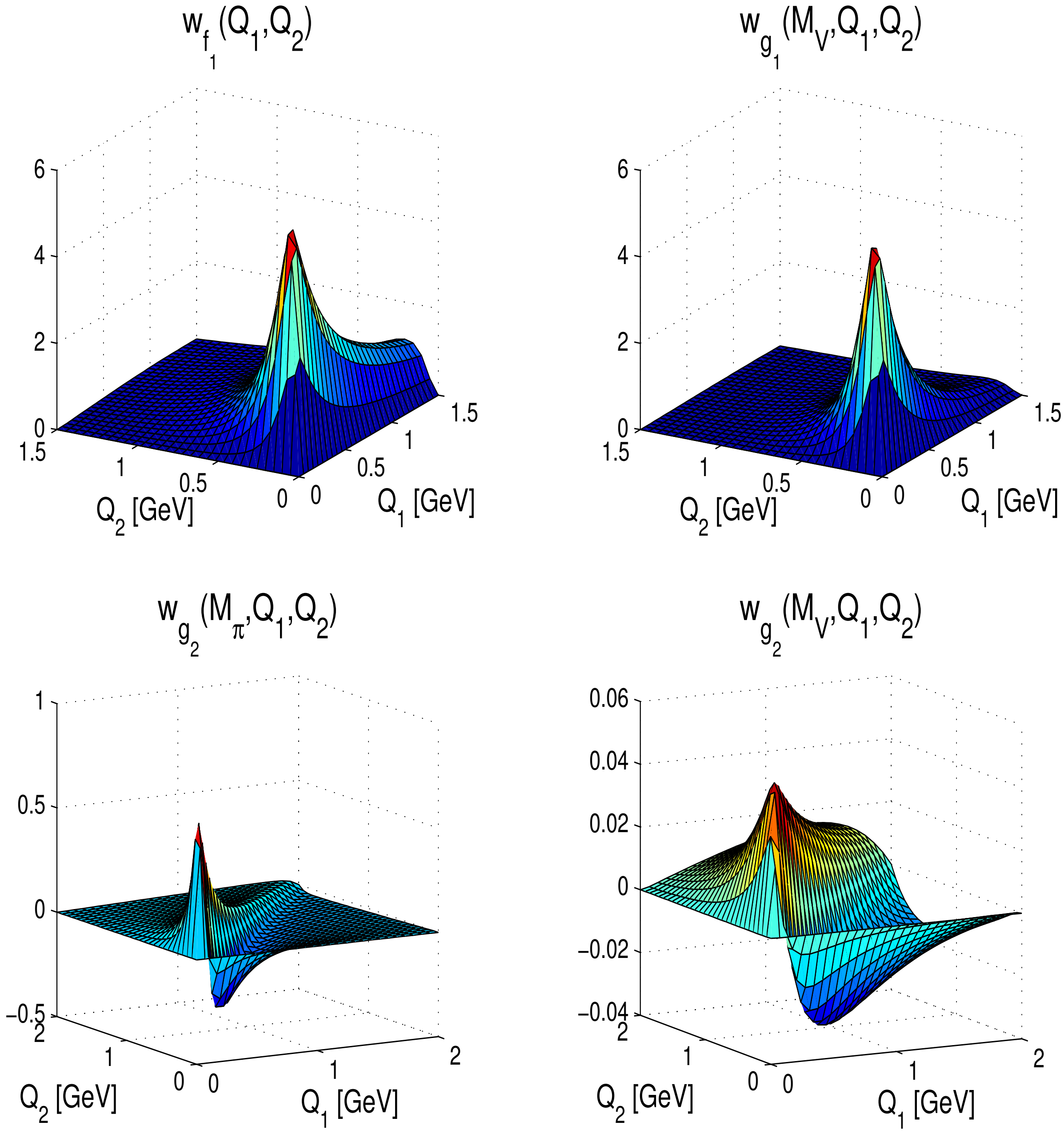
One obtains (schematically):

$$\boxed{a_\mu^{\mathrm{LbyL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\ \tilde{f}_i(Q_1, Q_2)}$$

- Integration over Euclidian momenta Q_i
- Universal (for above class of form factors) weight functions w_i (rational functions, square roots and logarithms)
- Dependence on form factors resides in $\tilde{f}_i$
- Can separate generic features from model dependence !
  Not visible in analytical result for $a_\mu^{\mathrm{LbyL};\pi^0}$ (as series expansion) for VMD form factor by Blokland, Czarnecki, Melnikov (PRL 2002)

  Note that there is an intrinsic uncertainty in the form factor of $10 - 30\ \%$, furthermore VMD form factor has wrong high-energy behavior !

Weight functions



- Region around $Q_i \sim 0.5$ GeV dominates $\rightarrow$ Better control of error in $a_\mu^{\mathrm{LbyL};\pi^0}$ that results from model dependence in $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$
- Log-square divergence $\sim \mathcal{C} \ln^2 \Lambda$ with cutoff Λ for constant WZW form factor due to tail in w_{f_1} (or $\mathcal{C} \ln^2 M_V$ for $M_V \rightarrow \infty$ for VMD and LMD form factors)

Numerical results

$$a_\mu^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{\mathrm{LbyL};\pi^0(1)} + a_\mu^{\mathrm{LbyL};\pi^0(2)}\right]$$

Form factor	$a_\mu^{\mathrm{LbyL};\pi^0(1)}$	$a_\mu^{\mathrm{LbyL};\pi^0(2)}$	$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{10}$
WZW	0.095	0.0020	12.2
VMD	0.044	0.0013	5.6
LMD	0.057	0.0014	7.3
LMD+V ($h_2 = -10$ GeV^2)	0.049	0.0013	6.3
LMD+V ($h_2 = 0$ GeV^2)	0.045	0.0013	5.8
LMD+V ($h_2 = 10$ GeV^2)	0.041	0.0013	5.3

- WZW: $a_\mu^{\mathrm{LbyL};\pi^0(1)}$ with UV-cutoff $\Lambda = 1$ GeV, $a_\mu^{\mathrm{LbyL};\pi^0(2)}$ is UV finite !
- LMD+V: $h_1 = 0$ GeV^2, $h_5 = 6.93$ GeV^4 (from fit to CLEO data)
  h_2 not strongly constraint from $\pi^0 \to e^+e^-$ ($|h_2| < 20$ GeV^2)

- Variation of parameters:

– Fixed h_2: vary $\Delta M_{V_1} = \pm 20$ MeV, $\Delta M_{V_2} = \pm 25$ MeV,
  $\Delta h_5 = \pm 0.5$ GeV^4 $\Rightarrow$ $\Delta a_\mu^{\mathrm{LbyL};\pi^0} = \pm 0.2 \times 10^{-10}$

– Vary h_2 in range $|h_2| < 20$ GeV^2, all other parameters fixed
  $\Rightarrow$ $\Delta a_\mu^{\mathrm{LbyL};\pi^0} = \pm 0.9 \times 10^{-10}$

- It seems more important to fit CLEO data and slope for $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, 0)$, than to correctly reproduce OPE (cf. weight functions peaked at $Q_i \sim 0.5$ GeV)

Our estimate for $a_\mu^{\text{LbyL;PS}}$

π^0 (LMD+V):

$$a_\mu^{\text{LbyL};\pi^0} = +5.8\ (1.0) \times 10^{-10}$$

η, η': use VMD form factor with $M_V = \Lambda_{\text{PS}}$ from fit by CLEO collaboration (Gronberg et al, 1998)

$$\begin{aligned} a_\mu^{\text{LbyL};\eta}|_{VMD} &= +1.3\ (0.1) \times 10^{-10}, \quad \Lambda_\eta = 774 \pm 30 \text{ MeV} \\ a_\mu^{\text{LbyL};\eta'}|_{VMD} &= +1.2\ (0.1) \times 10^{-10}, \quad \Lambda_{\eta'} = 859 \pm 29 \text{ MeV} \end{aligned}$$

The error only reflects the variation in Λ_{PS} from fit !

All contributions from pseudoscalar exchanges together:

$$\begin{aligned} a_\mu^{\text{LbyL;PS}} &\equiv a_\mu^{\text{LbyL};\pi^0}|_{LMD+V} + a_\mu^{\text{LbyL};\eta}|_{VMD} + a_\mu^{\text{LbyL};\eta'}|_{VMD} \\ &= +8.3\ (1.2) \times 10^{-10} \end{aligned}$$

Error includes estimate of model dependence.

3. Effective Field Theory approach

M. Knecht, A.N., M. Perrottet, E. de Rafael (PRL 2002)

Construct effective field theory for $E \ll 1$ GeV with pions, photons and muons

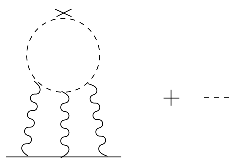
(Based on gauge invariance and chiral symmetry: Gasser, Leutwyler; Ecker et al.; Wess, Zumino; Witten; Savage, Luke, Wise)

Allows, in principle, systematic approach to problem

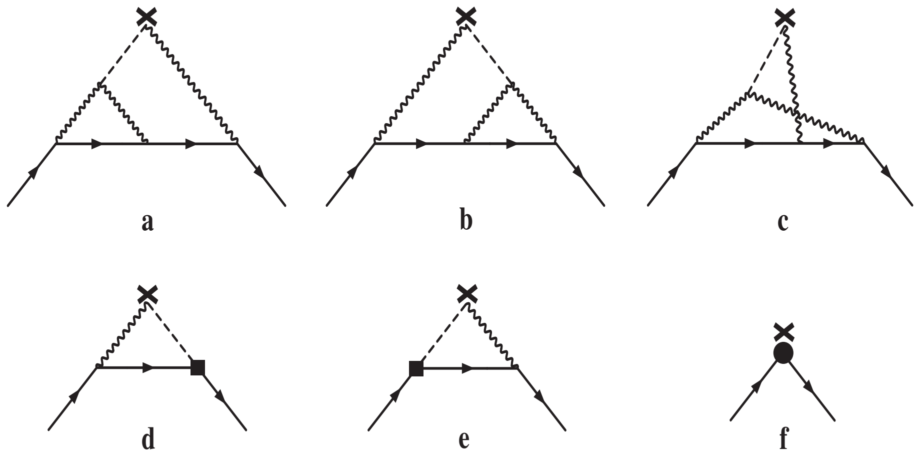
Goal here: Calculate leading log term using renormalization group in effective field theory (Weinberg, 1979)

Contributions to a_μ^{LbyL}

$\mathcal{O}(p^6)$: charged pion loop (finite, subleading in $1/N_C$)



$\mathcal{O}(p^8)$: pion-pole (leading in $1/N_C$)
Divergent two-loop contribution
Need counterterms: one-loop graphs with insertion of χ (■) (coupling $\overline{\psi}\gamma_\mu\gamma_5\psi\ \partial^\mu\pi^0$) and local term κ (•)



Schematically:

$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} &= H\left(\frac{m}{\mu}\right) + \chi(\mu) J\left(\frac{m}{\mu}\right) + \kappa(\mu) \\
&= h_2 \ln^2\left(\frac{m}{\mu}\right) + h_1 \ln\left(\frac{m}{\mu}\right) + h_0 + \chi(\mu)\left[j_1 \ln\left(\frac{m}{\mu}\right) + j_0\right] + \kappa(\mu)
\end{aligned}
$$

Renormalization scale invariance:

$$
\begin{aligned}
&\Rightarrow \quad \mu \frac{d}{d\mu} a_\mu^{\mathrm{LbyL};\pi^0} = 0 \\
&\Rightarrow \quad h_2 = \frac{1}{2}\gamma_\chi j_1
\end{aligned}
$$

with $\gamma_\chi \equiv \mu \frac{d}{d\mu}\chi(\mu) = N_C$ known from $\pi^0 \to e^+e^-$

To extract $\ln^2$ term $\sim h_2$ in two-loop integral need only $\ln$ term $\sim j_1$ from one-loop calculation

Result:

$$
a_\mu^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\mathcal{C} \ln^2\left(\frac{m}{\mu_0}\right) + \mathcal{O}\left(\ln\left(\frac{m}{\mu_0}\right)\right)\right]
$$

with

$$
\mathcal{C} = +3\left(\frac{N_C}{12\pi}\right)^2 \left(\frac{m}{F_\pi}\right)^2
$$

Coefficient agrees exactly with calculation in full theory with form factors !

$a_\mu^{\text{LbyL;had}}$: Leading and next-to-leading log's

EFT + large-N_C analysis (see also Ramsey-Musolf + Wise, PRL 2002)

$$a_\mu^{\text{LbyL;had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f\left(\frac{m_{\pi^\pm}}{m_\mu}, \frac{m_{K^\pm}}{m_\mu}\right) + \overbrace{N_C \left(\frac{m_\mu^2}{16\pi^2 F_\pi^2}\frac{N_C}{3}\right)}^{\mathcal{C} \quad (universal)} \left[\ln^2 \frac{\mu_0}{m_\mu} + \overbrace{c_1}^{\chi(\mu_0) \; (\pi^0 \to e^+e^-)} \ln \frac{\mu_0}{m_\mu} + c_0\right] + \mathcal{O}\left(\frac{m_\mu^2}{\mu_0^2} \times \text{log's}\right) + \mathcal{O}\left(\frac{m_\mu^4}{\mu_0^4} \times N_C \times \text{log's}\right)\right\}$$

$f(m_{\pi^\pm}/m_\mu, m_{K^\pm}/m_\mu) = -0.038$: charged pion and kaon loop (formally $\mathcal{O}(1)$)
$\mu_0 \sim M_\rho$: hadronic scale, $\ln \frac{M_\rho}{m_\mu} = 1.98, \ln^2 \frac{M_\rho}{m_\mu} = 3.94$

Problem: π^0-exchange $\to$ cancellation between $\log^2$ and $\log$:

$$\begin{aligned} \left. a_\mu^{\text{LbyL;}\pi^0} \right|_{\text{VMD}} &= \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \quad \left[\ln^2 \frac{M_\rho}{m_\mu} + c_1 \ln \frac{M_\rho}{m_\mu} + c_0\right] \\ &\overset{\text{Fit}}{=} \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C} \quad [3.94 - 3.30 + 1.08] \\ &= \quad [12.3 - 10.3 + 3.4] \times 10^{-10} = 5.4 \times 10^{-10} \end{aligned}$$

cf. Blokland, Czarnecki, Melnikov (PRL 2002), set $m_{\pi^0} = m_\mu$:

$$\Rightarrow \left. a_\mu^{\text{LbyL;}\pi^0} \right|_{\text{VMD}} = (12 - 8.0 + 1.7) \times 10^{-10} = 5.7 \times 10^{-10}$$

Cancellation now also visible in revised analysis by Ramsey-Musolf + Wise !

Sign of hadronic light-by-light scattering and quark-loop

Constituent quark loop (Laporta + Remiddi, 1993):

$$\left. a_\mu^{\text{LbyL}} \right|_{\text{CQM}} \sim + \left(\frac{\alpha}{\pi}\right)^3 N_C \frac{m_\mu^2}{M_Q^2} + \ldots, \qquad M_Q \gg m_\mu$$

M_Q acts as hadronic UV cutoff from EFT point of view
Result not consistent with $\log^2$ behavior in EFT, contributes only to c_0

4. Conclusions and open problems

- Found sign error in $a_\mu^{\mathrm{LbyL};\pi^0}$

  $\rightarrow$ substantial reduction of last year's discrepancy between SM prediction for a_μ and BNL experiment. 2002: Discrepancy back to $2-3\ \sigma$, or maybe not !?

- Pseudoscalar exchange seems under control (at 15% level)

  – Theory input: Long + short distance constraints; analytical 2-dim. integral representation (weight functions $\leftrightarrow$ model for $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$)

  – Experimental input: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2,0)$, slope; $\pi^0 \rightarrow e^+e^-$ (check of models; problem: loose exp. bound)

- Effective field theory approach

  – Systematic; brings some order into different contributions

  – $\log^2 + \log$-terms (large-N_C enhanced): are numerically important and seem to have right size compared to model calculations

Open problems:

- How far can we push EFT approach ?

  EFT approach limited: a_μ appears as local counterterm; Log's do not completely dominate numerically (strong cancellations); Momenta of order $Q_i \sim 0.5 - 1.0$ GeV important $\rightarrow$ beyond ChPT

  What about terms subleading in $1/N_C$? Charged pion loop leading in low-energy expansion !

- Do we really understand the dressed charged pion and quark loop ?

  Is strong suppression before $\rightarrow$ after dressing real or model dependent ? Results of two model calculations differ by up to $\pm 1.4 \times 10^{-10}$

- How do we avoid double counting of hadrons and quarks ?

  What is a constituent quark loop dressed with VMD legs ?

Summary (my conservative estimate): $a_\mu^{\mathrm{LbyL;had}} = +8\ (4) \times 10^{-10}$

Hayakawa + Kinoshita, hep-ph/0112102: $a_\mu^{\mathrm{LbyL;had}} = +8.96\ (1.54) \times 10^{-10}$
Bijnens, Pallante, Prades, hep-ph/0112255: $a_\mu^{\mathrm{LbyL;had}} = +8.3\ (3.2) \times 10^{-10}$

And finally:

Note that the same 4-point function

$$\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3) = \int \mathrm{d}^4x\mathrm{d}^4y\mathrm{d}^4z e^{i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\langle\Omega|\mathrm{T}\left\{j_\mu(x)j_\nu(y)j_\rho(z)j_\sigma(0)\right\}|\Omega\rangle$$

that enters in hadronic light-by-light scattering also appears at higher orders $\mathcal{O}\left(\left(\frac{\alpha}{\pi}\right)^3\right)$ in $g-2$ through corrections to the hadronic vacuum polarization: