

# Unique Factorization in QFT

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i) Lie algebra of Feynman graphs

→ forest formula

→ renormalization group

→ pre-Lie structure

ii)  $\zeta$ -functions  $\Leftrightarrow$  Dyson-Schwinger equations

$$\zeta(t, s) = \phi_s(1) + \phi_s [B_*^P(\zeta(t))]$$

$$\bar{\zeta}(t) = (1 + t B_*^P)[\bar{\zeta}(t)]$$

iii) gauge symmetry

$$u \otimes v - v \otimes u = 0$$

$\Leftrightarrow$  Ward Id.

$$\text{circle with arrow} * \text{triangle with arrow} = \text{circle with arrow and triangle inside}$$

$$\text{circle with arrow} * \text{line with arrow} = \text{circle with arrow and line inside} + \text{circle with arrow and line outside}$$

$$\text{triangle with arrow} * \text{circle with arrow} = \text{triangle with arrow and circle inside}$$

Then  $\Gamma_1 * \Gamma_2 - \Gamma_2 * \Gamma_1$  is a Lie bracket.

$$\sum_{\substack{i,j,k \\ \text{cyclic}}} [[\Gamma_i, \Gamma_j], \Gamma_k] = 0$$

$$\Rightarrow \Delta[\Gamma] = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

$$S_R^\phi * \phi(\Gamma) = S_R^\phi(\Gamma) + \underbrace{\sum_{\gamma \subset \Gamma} S_R^\phi(\gamma) \phi(\Gamma/\gamma)}_{\overline{R}}$$

counterterm

$$\Delta[\Gamma] = \Gamma \otimes 1 + (\text{id} \otimes B_+^{\Gamma_{ii}}) \Delta[X], \quad \Gamma = B_+^{\Gamma_{ii}}(X)$$

$$\gamma = \text{triangle with arrow} \quad \Gamma = \text{triangle with arrow and circle inside} \quad B_+^{\Gamma_{ii}}(X) \text{ (non)}$$

$$\text{circle with arrow and triangle inside} = \text{triangle with arrow} + g^2 \text{ (diagram with 4 vertices)} + g^4 \text{ (diagram with 6 vertices)} \quad D-S$$

Interlude

Riemann  $\zeta$

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \sum_n \frac{1}{n^s}$$

"Feynman graphs"



$p_x$  prime

$$B_+^P(\tau) = \prod_{p \mid \tau} \frac{1}{p} \cdot \frac{1}{\deg(\tau)+1}$$

algebraic  
structure  
un dered!

$$\bar{g}(t) = 1 + t \sum_p B_+^P(\bar{g}(t))$$

$$= 1 + t \sum_p + t^2 \frac{1}{2} \sum_{p_1, p_2} \frac{1}{p_1 p_2} + \dots$$

Feynman rules:  $\phi_s(1) = 1$   $\phi_s(\tau) = X(\tau)^{-s}$

$$\bar{g}(t, s) = \phi_s(1) + \phi_s(B_+^P(\bar{g}(t)))$$

$$\bar{g}(1, s) = \zeta(s) \quad \text{Riemann zeta.}$$



$$X = 1 + \sum_{\gamma} B_{\gamma}^{\gamma} (X^{\gamma'}) \gamma^{\gamma'}$$

$\gamma \nwarrow$  skeletons

Then  $X = \prod_{\gamma} \frac{1}{1 - \gamma}$

symmetrized  
insertions  
RING  $\mathcal{R}$

$$A * B + B * A = 0$$

$A, B$   
graphs

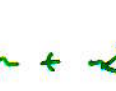
$$\stackrel{?}{\Rightarrow} A = 0 \vee B = 0$$

$\phi_6^3$  yes

$\phi_4^4$  yes

|     |    |
|-----|----|
| QED | no |
| YM  | no |

  $\vee$   = 0

 +  =  \*   
=  \* 

  $\vee$   = 0

Then in the ring  $\mathcal{R}$ , the relation  $\text{circle with dot} = 0$  defines a new ring  $\bar{\mathcal{R}}$  in which the Ward Ids hold.

[ goes through under the Renormalization rules ]