A Numerical Method for NNLO Calculations

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Radcor/Loops & Legs

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Outline

1. Introduction

2. The Method in General
   (sector decomposition)
   T.Binoth, G.H., Nucl. Phys. B **585** (2000) 741

3. From Loops to Legs

4. Examples/Results

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Introduction

QCD
(massless partons $\to$ infrared singularities)

$2 \to 2$ or $1 \to 3$ scattering processes at NNLO:

- enormous progress in calculation of two-loop amplitudes
- single unresolved limits of one-loop amplitudes understood
- double unresolved limits of tree amplitudes:
  – soft, collinear limits of ME's: known
  – phase space integrals: unknown (except special cases)
- level of complication similar to virtual double-box

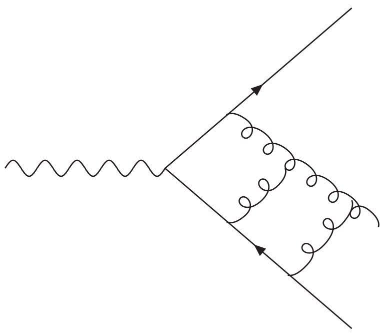
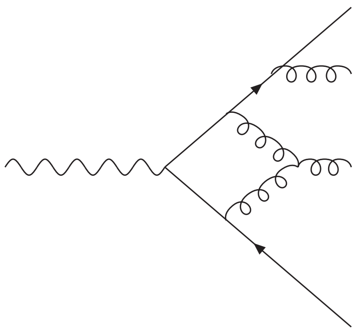
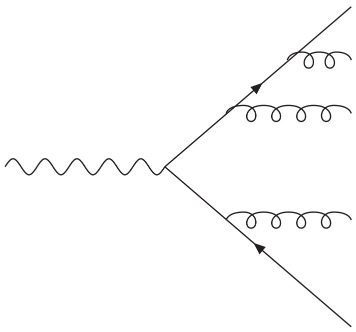
Weak interactions
(massive particles $\to$ multi-scales)

- loop integrals with arbitrary masses known analytically only up to 1-loop, N legs by scalar reduction expressions extremely lengthy for $N \geq 4$

QCD+Weak interactions
Infrared poles **and** multi-scales

- analytical approaches very limited but need to isolate IR divergences

$e^+e^- \to$ 3 jets at NNLO:

	loops	partons in final state
	2	3
	1	4
	0	5

The Method

"sector decomposition" (K.Hepp, M.Roth)

- based on dimensional regularization
- disentangles overlapping soft and collinear regions in parameter space (works also for UV divergences)
- originally developed for multi-loop integrals
- now extended to more general parameter integrals

Four building blocks:

I, II : isolate singular contributions in parameter space by dividing it into sectors with non-overlapping singular regions

III : construct a Laurent series for an integral G: $G = \sum_m C_m \, \epsilon^{-m}$ where C_m are finite functions of integration variables

IV : integrate the functions C_m (analytically if possible, otherwise numerically)

1. Loop Integrals

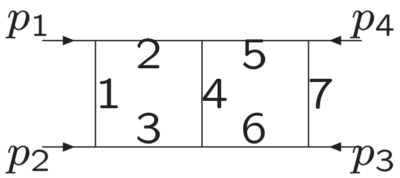
scalar graph G: (method works also for tensor integrals and/or propagator powers $\neq 1$)

N propagators
L loops
D dimensions

$$
\begin{aligned}
G &= \frac{\Gamma(N)}{(i\pi^{\frac{D}{2}})^L} \int d^N x \, \delta(1 - \sum_{l=1}^{N} x_l) \int d^D k_1 \dots d^D k_L \\
&\quad \left[\sum_{j,l=1}^{L} k_j \cdot k_l \, M_{jl} - 2 \sum_{j=1}^{L} k_j \cdot Q_j + J \right]^{-N} \\
&= (-1)^N \Gamma(N - LD/2) \int_0^\infty d^N x \, \delta(1 - \sum_{l=1}^{N} x_l) \\
&\quad \cdot \frac{\mathcal{U}^{N-(L+1)D/2}}{\mathcal{F}^{N-LD/2}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}(\vec{x}) &= \det(M) \left[\sum_{j,l=1}^{L} Q_j \cdot Q_l \, M_{jl}^{-1} - J \right] \\
\mathcal{U}(\vec{x}) &= \det(M)
\end{aligned}
$$

Example massless double box:



$$G = -\Gamma(3+2\epsilon)\int\limits_0^\infty \prod_{l=1}^{7} dx_l\,\delta(1-\sum_{l=1}^{7} x_l)\frac{\mathcal{U}^{1+3\epsilon}}{\mathcal{F}^{3+2\epsilon}}$$

$$\begin{aligned}\mathcal{U} &= (x_1+x_2+x_3)(x_5+x_6+x_7)\\ &\quad +x_4(x_1+x_2+x_3+x_5+x_6+x_7)\end{aligned}$$

$$\begin{aligned}\mathcal{F} &= (-s)\{x_2x_3(x_4+x_5+x_6+x_7)\\ &\quad +x_5x_6(x_1+x_2+x_3+x_4)\\ &\quad +x_2x_4x_6+x_3x_4x_5\}\\ &\quad +(-t)\,x_1x_4x_7\end{aligned}$$

Necessary condition for IR divergence:
$\mathcal{F} = 0$
singularities are at $x_i = 0 \quad (i = 1, \ldots, N)$
singular regions are in general overlapping

Method to disentangle overlapping IR divergences:
Iterated sector decomposition

Block I: Generation of primary sectors

- split integration domain into N parts:

$$\int_0^\infty d^N x \;=\; \sum_{l=1}^{N} \int_0^\infty d^N x \prod_{\substack{j=1\\ j\neq l}}^{N} \theta(x_l - x_j)\theta(x_l)$$

$$\Rightarrow \quad G \;=\; \sum_{l=1}^{N} G_l$$

- map $N-1$ integration domains to [0,1] by

$$x_j = \begin{cases} x_l t_j & j < l \\ x_l & j = l \\ x_l t_{j-1} \quad , & j > l \end{cases}$$

- x_l factorizes: $\mathcal{U}(\vec{x}) \to \mathcal{U}_l(\vec{t})\, x_l^{L}$
  $\mathcal{F}(\vec{x}) \to \mathcal{F}_l(\vec{t})\, x_l^{L+1}$

  scaling behaviour:

  $$x_l^{N-1} x_l^{L(N-(L+1)D/2)} x_l^{-(L+1)(N-L\,D/2)} = x_l^{-1}$$

- eliminate x_l in each G_l by using δ-function

  $$\int dx_l/x_l \;\delta(1 - x_l(1 + \sum_{j=1}^{N-1} t_j)) = 1$$

Result:

- N integrals G_l over $N-1$ Feynman parameters t
- integration domains from 0 to 1
- singularities only for $t_k = 0$

$$G_l = \int_0^1 d^{N-1}t \, \frac{\mathcal{U}_l(\vec{t})^{N-(L+1)D/2}}{\mathcal{F}_l(\vec{t})^{N-LD/2}}$$

Block II: Iterated sector decomposition

– repeat the following steps for each G_l:

II.1: Determine $\mathcal{S} = \{t_{\alpha_1}, \ldots, t_{\alpha_r}\}$, such that $\mathcal{U}_l$, resp. $\mathcal{F}_l$, vanish if the parameters of $\mathcal{S}$ are set to zero.

II.2: Decompose the corresponding r-cube into r sectors

$$\prod_{j=1}^{r} \theta(1 \geq t_{\alpha_j} \geq 0) = \sum_{k=1}^{r} \prod_{\substack{j=1 \\ j \neq k}}^{r} \theta(t_{\alpha_k} \geq t_{\alpha_j} \geq 0)$$

II.3: Remap the variables to the unit cube in each new subsector:

$$t_{\alpha_j} \to \begin{cases} t_{\alpha_k} t_{\alpha_j} & j \neq k \\ t_{\alpha_k} & j = k \end{cases}$$

t_{α_k} factorizes from at least one of the functions $\mathcal{U}_l$, $\mathcal{F}_l$

Repeat steps II.1. - II.3. until $\min\,(\mathcal{U},\mathcal{F}) \neq 0$

$$\begin{aligned}\mathcal{U}_{lk} &= 1 + u(\vec{t}\,)\\ \mathcal{F}_{lk} &= -s_0 + \sum_\beta(-s_\beta)f_\beta(\vec{t}\,)\end{aligned}$$

Block III: Extraction of the poles

- integrand in each subsector integral G_{lk} :

$$G_{lk} = \int_0^1 \prod_{j=1}^{N-1} dt_j \, t_j^{(A_j+B_j\epsilon)} \, \mathcal{I}(t_j,\epsilon)$$

If $A_j < 0$: subtraction

typically $A_j = -1$:

$$\int_0^1 dt_j \, t_j^{-1+B_j\epsilon} \, \mathcal{I}(t_j,\epsilon) =$$

$$\frac{\mathcal{I}(t_j=0,\epsilon)}{B_j\epsilon} + \int_0^1 dt_j \, t_j^{B_j\epsilon} \, \frac{\mathcal{I}(t_j,\epsilon)-\mathcal{I}(0,\epsilon)}{t_j}$$

- repeat subtraction for each t_j

  $(j = 1 \ldots N-1)$

  $\rightarrow$ poles are isolated

- expand in ϵ up to order ϵ^n :

  $\rightarrow$ Laurent series in each subsector

$$G_{lk} = \sum_{m=-n}^{2L} C_{lk,m}\, \epsilon^{-m}$$

$$G = \sum_{l=1}^{N} G_l = \sum_{l=1}^{N} \sum_{k=1}^{R_l} G_{lk}$$

Block IV:

Calculation of the pole coefficients

- coefficient $C_{lk,m}$ of $1/\epsilon^m$: $(N-1-m)$ dimensional parameter integral

- analytic integration possible for large m or simple functions

- numerical integration no problem for regular function $\mathcal{F}$, limitation only from CPU time

Examples

graph	boxtriangle	massless 2-box	1 mass 2-box
N	6	7	7
sectors	84	282	197
graph	massless Cbox	1 mass Cbox$_a$	1 mass Cbox$_b$
N	7	7	7
sectors	384	240	225
graph	2mass 2-box, adj.	2m 2-box, diag.	rk 1 tensor 2-box
N	7	7	7
sectors	192	126	908

graph	3L 3point mercedes	3L 3point ladder	3L 3point crossed
N	8	9	9
sectors	968	2319	1806

graph	3L 2point mercedes	3L 2point ladder	4L 2point UV subdiv
N	8	8	$7+\epsilon$
sectors	8	8	7

graph	3L planar box		
N	10		
sectors	8650		

planar double box, two adjacent external legs off-shell
$(m_3^2 \neq 0, m_4^2 \neq 0)$:

$$DB_{2mass,a} = \Gamma^2(1+\epsilon)\,(-m_4^2)^{-2\epsilon}\frac{1}{s^2 t}\{\,\frac{1}{4\epsilon^4} - \frac{1}{\epsilon^3}[\,\frac{1}{2}\log(s/m_4^2) + \log(t/m_4^2) - \frac{1}{2}\log(m_3^2/m_4^2)]\} + \mathcal{O}(\frac{1}{\epsilon^2})$$

Numerically e.g. at point
$(-s, -t, -u, -m_3^2, -m_4^2) = (2/3, 2/3, 2/3, 1, 1)$:

$$DB_{2mass,a}(2/3, 2/3, 2/3, 1, 1) = \Gamma^2(1+\epsilon)(\,-\frac{0.8437}{\epsilon^4} - \frac{2.0524}{\epsilon^3} - \frac{10.52}{\epsilon^2} - \frac{48.62}{\epsilon} - 140.77)$$

2. General multi-parameter integrals

isolate poles from a function $\mathcal{G}$ without specific loop integral properties:

1) not necessarily $\delta(1-\sum_{l=1}^{N} x_l)$ constraint
2) no fixed scaling properties like for
   $\mathcal{F}$ and $\mathcal{U}$
3) poles not only at $x_i = 0$

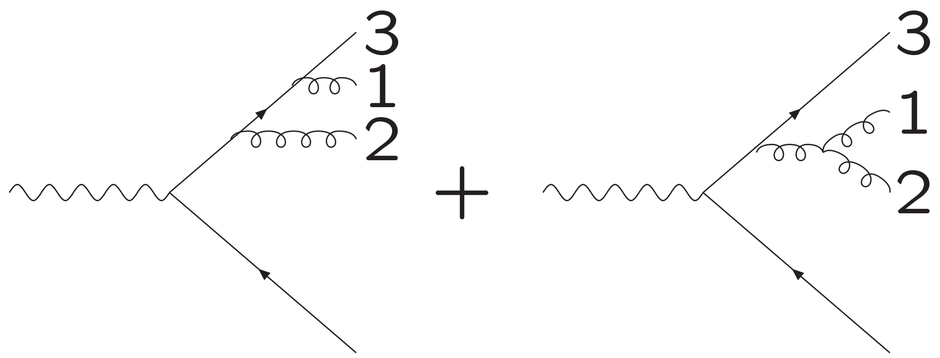
sector decomposition straightforward
only requirement for numerical integration:
no (integrable) singularities within the integration domain
$\Rightarrow$ perform variable transformations such that all integrals are from 0 to 1 with singularities at zero before calling the decomposition subroutine

Example from $e^+e^- \rightarrow$ 2 jets at NNLO

IR singular regions:

1. three particles collinear
2. two pairs of particles collinear
3. two particles collinear, one soft
4. two particles soft
5. soft $q\bar{q}$ pair

example collinear decay $q \rightarrow g_1 g_2 q_3$



$$d\Phi(p_1, p_2, p_3, p_4) \;\rightarrow\; d\Phi^{(2)}(\tilde{p}_{123}, \tilde{p}_4) \times d\Phi^{\mathrm{tc}}$$

$$|M|^2 \;\rightarrow\; |M_B|^2 \times \langle P_{g_1 g_2 q_3} \rangle / s_{123}^2$$

triple collinear phase space factor:

$$\int d\Phi^{\rm tc} = \frac{\tilde{s}^{2-2\epsilon}}{(4\pi)^d} \frac{1}{\Gamma^2(1-\epsilon)}$$
$$\int dz_1\, dz_2\, dz_3\, \delta(1-\sum_{i=1}^{3} z_i)\, [z_1 z_2 z_3]^{-\epsilon}$$
$$\int_0^1 dy\, [y(1-y)]^{1-2\epsilon} \int_0^1 du\, [u(1-u)]^{-\epsilon}$$
$$\frac{\Gamma(1-\epsilon)}{\sqrt{\pi}\Gamma(\frac{1}{2}-\epsilon)} \int_0^{\pi} d\theta (\sin\theta)^{-2\epsilon}$$

$$\begin{aligned}
\tilde{s} &= 2\, \tilde{p}_{123} \tilde{p}_4 \\
\tilde{p}_{123} &= p_1 + p_2 + p_3 - \frac{y}{1-y} p_4 \\
\tilde{p}_4 &= \frac{1}{1-y} p_4 \\
y &= \frac{s_{123}}{s_{1234}} \\
z_i &= \frac{p_i \tilde{p}_4}{\tilde{p}_{123} \tilde{p}_4} \quad (i = 1, 2, 3)\,, \quad \sum_{i=1}^{3} z_i = 1
\end{aligned}$$

In above PS parametrization:

$$
\begin{aligned}
s_{123} &= \tilde{s} y \\
s_{13} &= \tilde{s} y u (1 - z_2) \\
s_{23} &= \tilde{s} y u \frac{z_1 z_2}{(1 - z_2)} (1 + \nu^2 - 2\nu \cos\theta) \\
s_{12} &= \tilde{s} y u \frac{z_2 z_3}{(1 - z_2)} (1 + \lambda^2 + 2\lambda \cos\theta) \\
s_{i4} &= \tilde{s} (1 - y) z_i \quad (i = 1, 2, 3)
\end{aligned}
$$

$$
\nu^2 = \frac{z_3}{z_1 z_2} \frac{(1 - u)}{u} \quad , \quad \lambda^2 = \frac{z_1^2}{z_3^2} \nu^2
$$

$$
\langle P_{g_1 g_2 q_3}^{Abel} \rangle / s_{123}^2 = C_F^2 \{ \frac{A(z)}{s_{13} s_{23}}
+ \frac{B_1(z)}{s_{13} s_{123}} + \frac{B_2(z)}{s_{23} s_{123}}
+ C \left(\frac{s_{23}}{s_{13} s_{123}^2} + \frac{s_{13}}{s_{23} s_{123}^2} \right) + \frac{D}{s_{123}^2} \}
$$

y-integration always factorizes immediately

θ-integral:

$$\frac{\Gamma(1-\epsilon)}{\sqrt{\pi}\Gamma(\frac{1}{2}-\epsilon)}\int_0^\pi d\theta(\sin\theta)^{-2\epsilon} \cdot[1+\nu^2 \pm 2\nu\cos\theta]^{-1}$$

$$\begin{aligned} &= \theta(1-\nu^2)\ {}_2F_1(1,1+\epsilon,1-\epsilon,\nu^2) \\ &+ \theta(\nu^2-1)\frac{1}{\nu^2}\ {}_2F_1(1,1+\epsilon,1-\epsilon,1/\nu^2) \end{aligned}$$

split into two parts at $\nu^2 = 1$,

remap to integrals from 0 to 1,

use integral representation of ${}_2F_1$

typical integral:

$$
\begin{aligned}
&J^{Abel} = \\
&B_\epsilon \int dz_1\, dz_2\, dz_3\, \delta(1 - \sum z_i)\, [z_1 z_2 z_3]^{-2\epsilon} \tilde{A}(z) \\
&\int_0^1 dv\, v^{-1-\epsilon} (1-v)^{-2\epsilon} (z_1 z_2 + z_3 v)^{2\epsilon} \\
&\int_0^1 dt\, t^{-1-2\epsilon} (1-t)^{\epsilon} (1-tv)^{\epsilon}
\end{aligned}
$$

$$
\tilde{A}(z) = \frac{1+z_3^2}{z_1 z_2} - \epsilon \frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon(1+\epsilon)
$$

$$
B_\epsilon = \frac{1}{(4\pi)^d} \frac{\Gamma(2-2\epsilon)}{\Gamma(2-4\epsilon)\Gamma(1+\epsilon)\Gamma(1-\epsilon)}
$$

Result

Sum of all integrals contributing to $P^{Abel}_{g_1 g_2 q_3}$:

$$I^{Abel} = \int d\Phi^{\mathrm{tc}} \langle P^{Abel}_{g_1 g_2 q_3} \rangle / s^2_{123}$$

$$= \frac{\tilde{s}^{-2\epsilon}}{(4\pi)^d} \frac{C_F^2}{\Gamma^2(1-\epsilon)} \sum_{i=1}^{4} \frac{P_i}{\epsilon^i}$$

$$P_4 = 4 \pm 2 \times 10^{-6}$$

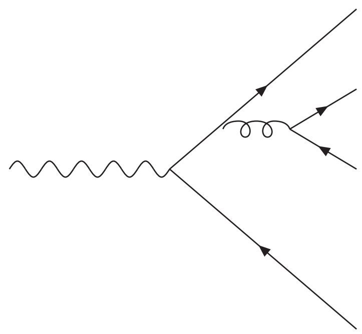
$$P_3 = 13.99999 \pm 4.8 \times 10^{-5}$$

$$P_2 = 9.88098 \pm 0.0016$$

$$P_1 = -39.889 \pm 0.0127$$

$$P_0 = -118.329 \pm 0.0733$$

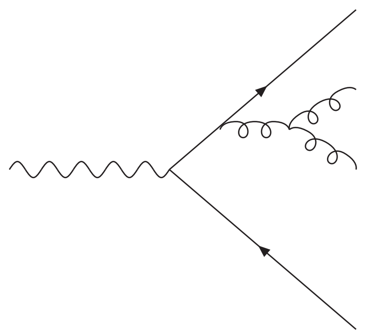
splitting $q \to \bar{q}_1 q_2 q_3$



$$
\begin{aligned}
I^{(id)}_{\bar{q}_1 q_2 q_3} &= \int d\Phi^{\mathrm{tc}} \langle P^{(id)}_{\bar{q}_1 q_2 q_3} \rangle / s^2_{123} \\
&= \frac{\tilde{s}^{-2\epsilon}}{(4\pi)^d} \frac{C_F}{\Gamma^2(1-\epsilon)} \left(C_F - \frac{C_A}{2} \right) \sum_{i=1}^{4} \frac{P_i}{\epsilon^i}
\end{aligned}
$$

$$
\begin{aligned}
P_4 &= 0 \\
P_3 &= 0 \\
P_2 &= -4 \times 10^{-5} \pm 5.5 \times 10^{-5} = 0 \\
P_1 &= -0.7202 \pm 0.00127 \\
P_0 &= -6.062 \pm 0.014
\end{aligned}
$$

most complicated denominator:



$$\langle P^{\text{nonAb}}_{g_1 g_2 q_3} \rangle / s_{123}^2 \sim \frac{A'(z)}{s_{12}^2} \quad + \text{less complicated}$$

θ-integral:

$$\begin{aligned} I &= \frac{\Gamma(1-\epsilon)}{\sqrt{\pi}\Gamma(\frac{1}{2}-\epsilon)} \int_0^\pi d\theta (\sin\theta)^{-2\epsilon} \\ & \quad [1+\lambda^2+2\lambda\cos\theta]^{-2} = \\ & \quad \theta(1-\lambda^2)\ {}_2F_1(2, 2+\epsilon, 1-\epsilon, \lambda^2) \\ &+ \theta(\lambda^2-1)\frac{1}{\lambda^4}\ {}_2F_1(2, 2+\epsilon, 1-\epsilon, 1/\lambda^2) \end{aligned}$$

$\rightarrow$ no problem

Conclusions and Outlook

- constructive algorithm to isolate infrared singularities in multi-loop integrals and phase space integrals
  $\rightarrow$ application to isolate IR singularities in double unresolved parts of NNLO ME's
- analytic integration can be further developed
- numerical checks for analytic calculation of any integral with no singularities within the integration domain
- generalization to functions with thresholds is underway