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Perturbative and Non-perturbative Lattice Calculations for the Study of Parton Distributions

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Introduction

- Lattice QCD provides non-perturbative techniques which allow computations of the moments of structure functions from first principles, without model assumptions
- The results that have been produced using lattice computations are by now rather numerous, and include the calculation of the lowest moments of many structure functions of the quarks:
  - the unpolarized structure functions
    $\rightarrow$ measure the unpolarized distribution q
  - the spin-dependent g_1 and g_2 structure functions
    $\rightarrow$ measure the helicity distribution Δq
  - the h_1 transversity structure function
    $\rightarrow$ measure the transversity distribution δq
- Some higher-twist matrix elements have also been studied
- Lattice perturbation theory is essential for the renormalization of the relevant operators
- Perturbative renormalization factors have been calculated for the lowest three moments of all structure functions, in the case of Wilson and also of overlap fermions

- Lattice:
  - discretization of a continuum field theory
  - non-perturbative regularization

    the lattice spacing a is an UV cutoff: $\Lambda \sim 1/a$
  - not tied to any specific approximation method
  - allows calculations from first principles , without model assumptions
- Non-perturbative investigations of QCD, long-distance physics
- Strong coupling: quark confinement (Wilson, 1974)
- Weak coupling: perturbation theory , renormalization
- In this talk we will shortly discuss:
  - Structure functions on the lattice
  - Renormalization of operators
  - Wilson fermions
  - Overlap fermions
  - Improvement
  - Computer codes for analytic calculations
  - Results

Structure Functions on the Lattice

- It is not possible to compute the complete structure functions directly on the lattice (which is set up in Euclidean space)
- The reason is that the structure functions describe the physics close to the light cone, which is essentially a feature of Minkowski space

  $\Rightarrow$ this is lost when going to Euclidean space (Monte Carlo simulations)
- However, on a Euclidean lattice it is possible to compute the moments of the structure functions, via the operator product expansion :

$$\int_0^1 x^n \, \mathcal{F}_{(i)}(x, Q^2) \sim C_{(i)}^{(n)}(Q^2/\mu^2) \cdot \langle h | O^{(n)}(\mu) | h \rangle$$

- `Wilson coefficients`:

  short-distance physics $\Rightarrow$ perturbation theory (continuum)
- `matrix elements`:

  long-distance physics $\Rightarrow$ Monte Carlo simulations

  $\rightarrow$ lattice renormalization

- Operators of twist two (twist: dimension minus spin) dominate the operator product expansion
- Moments of the unpolarized quark distributions:

$$\langle x^n \rangle \sim \langle\, h \,|\, \overline{\psi}\, \gamma_{\{\mu} D_{\mu_1} \cdots D_{\mu_n\}}\, \psi \,|\, h \,\rangle$$

- Moments of the spin-dependent structure function g_1:

$$\langle (\Delta x)^n \rangle \sim \langle\, h \,|\, \overline{\psi}\, \gamma_5 \gamma_{\{\mu} D_{\mu_1} \cdots D_{\mu_n\}}\, \psi \,|\, h \,\rangle$$

- Moments of the spin-dependent structure function g_2:

$$\langle x^n \rangle_{g_2} \sim \langle\, h \,|\, \overline{\psi}\, \gamma_5 \gamma_{[\mu} D_{\{\mu_1} D_{\mu_2]} \cdots D_{\mu_n\}}\, \psi \,|\, h \,\rangle$$

- Moments of the transversity structure function h_1:

$$\langle (\delta x)^n \rangle \sim \langle\, h \,|\, \overline{\psi}\, \gamma_5 \sigma_{\mu\{\mu_1} D_{\mu_2} \cdots D_{\mu_n\}}\, \psi \,|\, h \,\rangle$$

- The moments of the unpolarized gluon distribution

$$\langle x^n \rangle_g \sim \langle\, h \,|\, \sum_\rho \mathrm{Tr}\, (F_\mu{}^\rho D_{\mu_2} \cdots D_{\mu_n} F_{\rho\mu_1}) \,|\, h \,\rangle$$

have been so far quite hard to compute on the lattice

Renormalization of Operators

- Once the various matrix elements are simulated using Monte Carlo algorithms, they need to be renormalized from the lattice to a continuum scheme

  $\Rightarrow$ computation of the renormalization factors of the operators

- To obtain physical continuum matrix elements from Monte Carlo simulations, one needs to compute 1-loop matrix elements of the relevant operators on the lattice as well as in the continuum

- Lattice operators: chosen so that at tree level (for $p \ll \pi/a$) have the same matrix elements as the original continuum operators

$$\langle O_i^{\rm lat} \rangle = \sum_j \left(\delta_{ij} + \frac{g_0^2}{16\pi^2} \Big(-\gamma_{ij} \log a^2 p^2 + R_{ij}^{\rm lat} \Big) \right) \cdot \langle O_j^{\rm tree} \rangle$$

$$\langle O_i^{\rm cont} \rangle = \sum_j \left(\delta_{ij} + \frac{g^2_{\overline{\rm MS}}}{16\pi^2} \Big(-\gamma_{ij} \log \frac{p^2}{\mu^2} + R_{ij}^{\rm cont} \Big) \right) \cdot \langle O_j^{\rm tree} \rangle$$

  where the continuum is in this case the $\overline{\rm MS}$ scheme

- In general $R_{ij}^{\rm lat} \neq R_{ij}^{\rm cont}$

  $\Rightarrow$ the 1-loop renormalization factors on the lattice and in the continuum are different

- The anomalous dimensions are the same in both cases, and thus only a finite renormalization connects the two schemes

- The connection between the original lattice numbers and the final continuum physical results is given by

$$\langle O_i^{\rm cont}\rangle = \sum_j \left(\delta_{ij} - \frac{g_0^2}{16\pi^2}\left(R_{ij}^{\rm lat} - R_{ij}^{\rm cont}\right)\right)\cdot\langle O_j^{\rm lat}\rangle$$

- The differences $\Delta R_{ij} = R_{ij}^{\rm lat} - R_{ij}^{\rm cont}$ enter then in the renormalization factors

$$Z_{ij}(a\mu, g_0) = \delta_{ij} - \frac{g_0^2}{16\pi^2}\Delta R_{ij}(a\mu)$$

  which are the main object of the computations of renormalization constants on the lattice

- While $R^{\rm lat}$ and $R^{\rm cont}$ depend on the state, ΔR is independent of the state and depends only on $a\mu$

- Using these Z's a matrix element obtained from Monte Carlo simulations can be renormalized to a continuum scheme

- The mixing of operators under renormalization has features rather different than in the continuum
- Since Lorentz and (for Wilson fermions) chiral symmetry are broken on the lattice (which is invariant under the hypercubic group) , the mixing of operators under renormalization is more complicated than in the continuum theory

  $\rightarrow$ operators multiplicatively renormalizable in the continuum may mix with other operators when they are put on the lattice

  $\rightarrow$ one can even have the case of mixing coefficients which are power divergent $\sim \frac{1}{a^n}$
- 1-loop renormalization factors are now known for the lowest three moments of all structure functions, and they have been computed using Wilson fermions and also overlap fermions (which do no break chiral symmetry)
- For many operators these renormalization factors are known also in the $O(a)$ improved theory
- Perturbative renormalization factors have been calculated also for some higher-twist matrix elements (4-quark operators)
- Non-perturbative renormalization factors have been instead much more difficult to compute for structure functions: it is hard to find a plateau for the extraction of the numerical signal

  Exception: Schrödinger Functional (but much more demanding)

WILSON FERMIONS

- Continuum $\rightarrow$ lattice:

$$\int d^4x \rightarrow a^4 \sum_x$$

where $x_\mu = a \cdot n_\mu$ defines the lattice sites

- One possible discretization of the (Euclidean) QCD action

$$S^f = \int d^4x \left[\overline{\psi}(x) \left(\not{D} + m_f \right) \psi(x) + \frac{1}{2} \mathrm{Tr} \left[F_{\mu\nu}(x) F_{\mu\nu}(x) \right] \right]$$

is the WILSON ACTION :

$$S_W^f = a^4 \sum_x \Bigg[-\frac{1}{2a} \sum_\mu \Big[\overline{\psi}(x)(r-\gamma_\mu) U_\mu(x) \psi(x+\hat{\mu}) + \overline{\psi}(x+\hat{\mu})(r+\gamma_\mu) U_\mu^\dagger(x) \psi(x) \Big] + \overline{\psi}(x) \left(m_f + \frac{4r}{a} \right) \psi(x) \Bigg] - \frac{1}{g_0^2} \sum_{x,\mu\nu} \Big[\mathrm{Tr} \left[U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x) \right] - N_c \Big]$$

where the massless Dirac kernel is (in short notation)

$$D_W = \frac{1}{2} \left[\gamma_\mu (\nabla_\mu^\star + \nabla_\mu) - a r \nabla_\mu^\star \nabla_\mu \right]$$

- This is a gauge-invariant regularization:
  the lattice theory has exact gauge invariance at any finite a
- The Wilson term (proportional to r) breaks chiral symmetry , but is needed in order to get rid of the extra fermions (doublers)
  $\Rightarrow$ additive mass renormalization (even for vanishing bare mass)
  $\Rightarrow$ heavy pion: $m_\pi \sim 500\ MeV$
- One needs then to perform extrapolations to the chiral limit
  $\Rightarrow$ source of systematic errors, to be controlled
- Recent formulations of chiral fermions on the lattice:
  Ginsparg-Wilson (overlap, domain-wall, fixed-point action)
  $\Rightarrow$ maintain exact chiral symmetry also at finite lattice spacing
  $\Rightarrow$ no need for extrapolations to the chiral limit
- Lattice perturbation theory is more complicated than the continuum: more vertices, more diagrams, expressions containing a huge number of terms, more complicated integrals
- Wilson quark propagator:

$$S^{ab}(k,m_f) = \delta^{ab}\, a\, \frac{-\mathrm{i}\sum_\mu \gamma_\mu \sin ak_\mu + am_f + 2r\sum_\mu \sin^2 \frac{ak_\mu}{2}}{\sum_\mu \sin^2 ak_\mu + \left[2r\sum_\mu \sin^2 \frac{ak_\mu}{2} + am_f\right]^2}$$

$$\stackrel{a\to 0}{\longrightarrow} \delta^{ab}\, \frac{-\mathrm{i}\sum_\mu \gamma_\mu k_\mu + m_f}{\sum_\mu k_\mu^2 + m_f^2}$$

- Gluon propagator (in covariant gauge):

$$G^{ab}_{\mu\nu}(k) = \delta^{ab}\,\frac{1}{\frac{4}{a^2}\sum_\lambda \sin^2\frac{ak_\lambda}{2}}\left[\delta_{\mu\nu} - (1-\alpha)\frac{\sin\frac{ak_\mu}{2}\,\sin\frac{ak_\nu}{2}}{\sum_\lambda \sin^2\frac{ak_\lambda}{2}}\right]$$

- Gauge invariance is imposed directly on the lattice action

  $\Rightarrow$ group elements U_μ instead of the algebra elements A_μ

  $\Rightarrow$ ∞ number of vertices (example: quark - quark - n gluons)

- Only a finite number of vertices is needed at any given order in g_0

- Quark-quark-gluon (old "continuum") vertex:

$$(V_1^a)^{bc}_\mu(p_1,p_2) = -g_0(t^a)^{bc} \times\left(\mathrm{i}\gamma_\mu\cos\frac{a(p_1+p_2)_\mu}{2} + r\sin\frac{a(p_1+p_2)_\mu}{2}\right) \stackrel{a\to 0}{\longrightarrow} -g_0(t^a)^{bc}\,\mathrm{i}\gamma_\mu$$

  (p_1 and p_2 are the momenta of the quarks)

- Quark-quark-gluon-gluon ("irrelevant") vertex:

$$(V_2^{ab})^{cd}_{\mu_1\mu_2}(p_1,p_2) = -\frac{1}{2}ag_0^2\,\delta_{\mu_1\mu_2}\left(\frac{1}{N_c}\delta^{ab} + d^{abe}t^e\right)^{cd} \times\left(-\mathrm{i}\gamma_\mu\sin\frac{a(p_1+p_2)_\mu}{2} + r\cos\frac{a(p_1+p_2)_\mu}{2}\right)$$

  which is zero in the continuum limit, but still gives non-vanishing contributions to Feynman diagrams (divergent loops, tadpoles)

- To derive the gluon vertices in the Wilson action, one expands the U_μ's in the plaquette in terms of the A_μ's
- The result is an infinite number of vertices, of which only a finite number is needed at any given order in g_0
- 3-gluon vertex:

$$W^{abc}_{\mu\nu\rho}(k,p,q) = \mathrm{i}g_0\, f^{abc}\, \frac{2}{a}\Bigg\{ \delta_{\mu\nu}\, \sin\frac{a(k-p)_\rho}{2}\, \cos\frac{aq_\mu}{2}$$

$$+ \delta_{\nu\rho}\, \sin\frac{a(p-q)_\mu}{2}\, \cos\frac{ak_\nu}{2} + \delta_{\rho\mu}\, \sin\frac{a(q-k)_\nu}{2}\, \cos\frac{ap_\rho}{2}\Bigg\}$$

$$\stackrel{a\to 0}{\longrightarrow} \mathrm{i}g_0\, f^{abc}\left\{\delta_{\mu\nu}\,(k-p)_\rho + \delta_{\nu\rho}\,(p-q)_\mu + \delta_{\rho\mu}\,(q-k)_\nu\right\}$$

  (where $k+p+q=0$, all momenta entering, assigned counterclockwise)

- The 4-gluon vertex is too complicated to be reported here
- The gauge measure at order g_0^2 gives a $1/a^2$ mass counterterm
- Using a Faddeev-Popov procedure one gets the Feynman rules for the ghost propagator , the ghost-ghost-gluon vertex , . . .
- The effective ghost-gauge field interaction is no longer linear in the gauge potential A_μ

  $\Rightarrow$ also in this sector new vertices appear that have no analogous in the continuum, like the ghost-ghost-gluon-gluon vertex

Overlap Fermions

- A Dirac operator D satisfying the Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = a\,\frac{1}{\rho}\,D\gamma_5 D$$

defines fermions with exact chiral symmetry, with no doublers and which also possess flavor symmetry, locality, unitarity and gauge-invariance

  - Locality is not meant here to be strict locality, but in the larger sense that the strength of the interaction decays exponentially with the distance

- Lüscher has shown that a fermion obeying the Ginsparg-Wilson relation possesses an exact chiral symmetry of the general form

$$\begin{aligned}\delta\psi &= \epsilon\cdot\gamma_5\left(1-\frac{c}{\rho}\,aD\right)\psi\\ \delta\bar{\psi} &= \epsilon\cdot\bar{\psi}\left(1-\frac{1-c}{\rho}\,aD\right)\gamma_5\end{aligned}$$

also for non-zero lattice spacing

Recent numerical work shows the very good accuracy with which chiral symmetry is attained

- Solutions of the Ginsparg-Wilson relation: overlap, domain-wall, fixed-point action ("classically perfect")

- One of the possible solutions of the Ginsparg-Wilson relation has been found by Neuberger starting from the overlap formalism
- In the massless case the overlap-Dirac operator is

$$D_N = \frac{1}{a}\rho\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right], \qquad X = D_W - \frac{1}{a}\rho$$

  where D_W is the usual Wilson-Dirac operator:

$$D_W = \frac{1}{2}\Big[\gamma_\mu(\nabla^\star_\mu + \nabla_\mu) - ar\nabla^\star_\mu\nabla_\mu\Big]$$

$$\nabla_\mu\psi(x) = \frac{1}{a}\Big[U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\Big]$$

- In the range $0 < \rho < 2r$ the right spectrum of massless fermions is obtained
- Since additive mass renormalization is forbidden, one avoids altogether a source of systematic errors that is always present with Wilson fermions
- Many 1-loop calculations with overlap fermions have been already carried out, including:
  - the relation between the Λ parameter in the lattice scheme defined by the overlap operator and in the $\overline{\rm MS}$ scheme
  - the renormalization factors of the quark bilinears $\bar{\psi}\Gamma\psi$, of the lowest moments of all structure functions, of the $\Delta S = 2$ and $\Delta S = 1$ effective weak Hamiltonians

- The massless quark propagator in the overlap is given by

$$S^{ab}(k) = \delta^{ab}\left(\frac{-\mathrm{i}\sum_\mu \gamma_\mu \sin ak_\mu}{2\rho\,(\omega(k)+b(k))} + \frac{a}{2\rho}\right)$$

where

$$\omega(k) = \frac{1}{a}\sqrt{\sum_\mu \sin^2 ak_\mu + \left(2r\sum_\mu \sin^2\frac{ak_\mu}{2} - \rho\right)^2}$$

$$b(k) = \frac{1}{a}\left(2r\sum_\mu \sin^2\frac{ak_\mu}{2} - \rho\right)$$

- The interaction vertices needed for 1-loop calculations can be entirely given in terms of the vertices of the QED Wilson action

$$W_{1\mu}(p_1,p_2) = -g_0\left(\mathrm{i}\gamma_\mu \cos\frac{a(p_1+p_2)_\mu}{2} + r\sin\frac{a(p_1+p_2)_\mu}{2}\right)$$

$$W_{2\mu}(p_1,p_2) = -\frac{1}{2}ag_0^2\left(-\mathrm{i}\gamma_\mu \sin\frac{a(p_1+p_2)_\mu}{2} + r\cos\frac{a(p_1+p_2)_\mu}{2}\right)$$

(where p_1 and p_2 are the quark momenta flowing in and out of the vertices) and of the quantity

$$X_0(p) = \frac{1}{a}\left(\mathrm{i}\sum_\mu \gamma_\mu \sin ap_\mu + 2r\sum_\mu \sin^2\frac{ap_\mu}{2} - \rho\right)$$

- The quark-quark-gluon vertex in the overlap theory has then the expression

$$(V_1^a)_\mu^{bc}(p_1,p_2) = (t^a)^{bc}\cdot\rho\,\frac{1}{\omega(p_1)+\omega(p_2)}\Bigg[W_{1\mu}(p_1,p_2) - \frac{1}{\omega(p_1)\omega(p_2)}X_0(p_2)W_{1\mu}^\dagger(p_1,p_2)X_0(p_1)\Bigg]$$
$$\stackrel{a\to 0}{\longrightarrow} -g_0(t^a)^{bc}\,\mathrm{i}\gamma_\mu$$

- The quark-quark-gluon-gluon vertex in the overlap theory is

$$(V_2^{ab})_{\mu\nu}^{cd}(p_1,p_2) = \delta_{\mu\nu}\left(\frac{1}{N_c}\delta^{ab}+d^{abe}t^e\right)^{cd}\cdot\Bigg\{\rho\,\frac{1}{\omega(p_1)+\omega(p_2)} \times\Bigg[W_{2\mu}(p_1,p_2)-\frac{1}{\omega(p_1)\omega(p_2)}X_0(p_2)W_{2\mu}^\dagger(p_1,p_2)X_0(p_1)\Bigg]$$
$$+\frac{1}{2}\rho\,\frac{1}{\omega(p_1)+\omega(p_2)}\frac{1}{\omega(p_1)+\omega(k)}\frac{1}{\omega(k)+\omega(p_2)} \times\Bigg[X_0(p_2)W_{1\mu}^\dagger(p_2,k)W_{1\nu}(k,p_1) + W_{1\mu}(p_2,k)X_0^\dagger(k)W_{1\nu}(k,p_1) + W_{1\mu}(p_2,k)W_{1\nu}^\dagger(k,p_1)X_0(p_1) - \frac{\omega(p_1)+\omega(k)+\omega(p_2)}{\omega(p_1)\omega(k)\omega(p_2)}X_0(p_2)W_{1\mu}^\dagger(p_2,k)X_0(k)W_{1\nu}^\dagger(k,p_1)X_0(p_1)\Bigg]\Bigg\}$$
$$\stackrel{a\to 0}{\longrightarrow} 0$$

IMPROVEMENT

- One can implement the improvement on actions and operators, to decrease the errors due to the granularity of the lattice
- It is very expensive to reduce the cut-off effects by "brute force" simply decreasing the lattice spacing a:

$$\text{CPU time} \approx \frac{1}{\text{error}^5}$$

  $\Rightarrow$ reducing by a factor two the discretization errors in this way requires a calculational effort 30 times bigger

  $\Rightarrow$ a better way of reducing them can be had by improving the theory

- A systematic improvement program to reduce the cut-off errors order by order in a was first proposed by Symanzik , and then further developed by Lüscher and Weisz , Sheikholeslami and Wohlert , Heatlie et al. , . . . (on-shell improvement)
- $O(a)$ improvement : removes the systematic error arising from the finiteness of the lattice spacing (contributions of order a):

$$\left\langle p \left| \widehat{\mathcal{O}}_L \right| p' \right\rangle_{MC} = a^d \left[\left\langle p \left| \widehat{\mathcal{O}} \right| p' \right\rangle_{ph} + O(a^2) \right]$$

  $\Rightarrow$ faster convergence to the continuum limit

- For Wilson fermions, this is achieved in on-shell matrix elements by adding a counterterm to the Wilson action

$$\Delta S_I^f = c_{sw} \cdot \mathrm{i} g_0 a^4 \sum_{x,\mu\nu} \frac{r}{4a} \overline{\psi}(x) \sigma_{\mu\nu} F_{\mu\nu}^{\mathrm{clover}}(x) \psi(x)$$

- The improvement coefficient c_{sw} has to be exactly tuned so as to cancel all $O(a)$ contributions coming from the action

- Only with the appropriate value of c_{sw} (for a given lattice spacing) the cancellation of $O(a)$ effects can be achieved

- The "clover-leaf" gluon field is here defined as

$$F_{\mu\nu}^{\mathrm{clover}}(x) = \frac{1}{4} \sum_{\mu\nu=\pm} P_{\mu\nu}(x) = \frac{1}{8 i g_0 a^2} \sum_{\mu\nu=\pm} (U_{\mu\nu}(x) - U_{\mu\nu}^{\dagger}(x))$$

- This new piece of the action means that we have to add to the Wilson quark-quark-gluon interaction vertex

$$(V^a)_{\rho}^{bc} = -g_0 (t^a)^{bc} \left[r \sin \frac{a(k+k')_\rho}{2} + \mathrm{i} \gamma_\rho \cos \frac{a(k+k')_\rho}{2} \right]$$

the SW-improved quark-quark-gluon interaction vertex

$$((V^I)^a)_{\rho}^{bc} = -c_{sw} \cdot g_0 \frac{r}{2} (t^a)^{bc} \cos \frac{a(k-k')_\rho}{2} \sum_{\lambda} \sigma_{\rho\lambda} \sin a(k-k')_\lambda$$

- The fermion propagator and the vertices with an even number of gluons are instead not modified by the improved action

- Neither is the gluon propagator: the first corrections to the pure gauge term in the Wilson action are already of order a^2

- All this is not enough: in addition to improving the action, one also has to improve the various operators
- One must add a basis of higher dimensional (irrelevant) operators with the same quantum numbers ($\dim(\widetilde{O}_i) = \dim(O) + 1$):

$$O \longrightarrow O^{\rm imp} = (1 + b_O\, am)\, O + a \sum_i c_i\, \widetilde{O}_i$$

- Overlap fermions present many advantages compared to Wilson fermions also regarding the Symanzik improvement of operators
- Compared with Wilson fermions, the $O(a)$ improvement for overlap fermions is much simpler:

  1) the action is already improved , and there is no need of new interactions like for Wilson fermions

  2) the renormalization constants for improved and unimproved operators are the same , while with Wilson fermions additional cumbersome calculations are needed to get the Z's for the improved operators

  3) the improved operators are given, to all orders , by

$$O^{\rm imp} = \bar{\psi}\Big(1 - \frac{1}{2\rho} a D_N\Big)\, \widetilde{O}\, \Big(1 - \frac{1}{2\rho} a D_N\Big)\psi$$

  while for Wilson fermions the construction of the improved operator is different in each case

  4) full $O(a)$ improvement is achieved without tuning any coefficients, and is valid to all orders of perturbation theory

- The last point is really a big calculational advantage : improving operators for Wilson fermions means
  - finding out a complete basis of operator counterterms
  - determining for all counterterms the exact values of their coefficients that effectively improve the original operator, order by order in perturbation theory
- This appears even at lowest order to be a highly non-trivial task, as it can be seen for the first moment of unpolarized structure functions, $\bar{\psi}\gamma_{\{\mu}D_{\nu\}}\psi$, where two counterterms are needed:

$$O^{(1)}_{\mu\nu} = -\frac{1}{4}ai\,c_1(g_0^2)\sum_\lambda \bar{\psi}\sigma_{\lambda\{\mu}\Big[D_{\nu\}},D_\lambda\Big]\psi$$
$$O^{(2)}_{\mu\nu} = -\frac{1}{4}a\,c_2(g_0^2)\,\bar{\psi}\Big\{D_\mu,D_\nu\Big\}\psi$$

- Only for some particular values of the coefficients

$$c_1(g_0^2) = 1+g_0^2c_1^{(1)}+O(g_0^4)$$
$$c_2(g_0^2) = 1+g_0^2c_2^{(1)}+O(g_0^4)$$

  the $O(a)$ corrections are exactly canceled
- Up to know it has been possible to determine at 1 loop only one of the corresponding coefficients; the other coefficient remains unknown

FORM Codes

- Due to the complexity of the calculations, to the possible great number of diagrams, and above all to the huge amount of terms for each diagram, computer codes have to be used
- To evaluate the Feynman diagrams and obtain the algebraic expressions for the renormalization factors, we have thus developed sets of computer codes written in the symbolic manipulation language FORM
- These codes are able to take as input the Feynman rules for the particular combination of operators, propagators (Wilson or overlap) and vertices (Wilson, and SW-improved, or overlap) appearing in each diagram, to expand them in the lattice spacing a at the appropriate order, to evaluate the gamma algebra on the lattice , and then to work out everything until the final expressions are obtained
- The main difficulty for developing such codes is that the (Euclidean) Lorentz group $O(4)$ breaks down to the hypercubic group W_4

  (hypercubic group: 384 elements, 20 irreps)
- A representation which is irreducible under the (Euclidean) Lorentz group is thus in general reducible under the symmetry group of the lattice

- This gives rise to a whole new series of problems, regarding for example the validity of the Einstein summation convention and the mixing among matrix elements when radiative corrections are considered
- FORM has been developed having in mind the usual continuum calculations
- There are therefore many useful built-in features which are sometimes somewhat of an hindrance when one tries to perform lattice perturbative calculations
- This is for example what FORM would normally do:
  - $\sum_\lambda \gamma_\lambda p_\lambda \longrightarrow \not{p}$ OK
  - $\sum_\lambda \gamma_\lambda p_\lambda \sin k_\lambda \longrightarrow \not{p} \sin k_\lambda$ NO
  - $\sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\lambda \longrightarrow (\gamma \cdot \sin k) \cos^2 k_\lambda$ NO
  - $\sum_{\lambda,\rho} \gamma_\rho \gamma_\lambda \gamma_\rho \sin k_\lambda \cos^2 k_\rho \longrightarrow -2 \sum_\lambda \gamma_\lambda \sin k_\lambda \cos^2 k_\rho$ NO
- The correct answer in the last case is instead

$$-\sum_{\lambda,\rho} \gamma_\lambda \sin k_\lambda \cos^2 k_\rho + 2 \sum_\rho \gamma_\rho \sin k_\rho \cos^2 k_\rho$$

$\Rightarrow$ DEVELOPMENT OF SPECIAL ROUTINES TO DEAL WITH THE GAMMA ALGEBRA ON THE LATTICE

- Integrals: in momentum space, over the first Brillouin zone

$$\sum_{k\in BZ} f(ak,ap) \;\to\; \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4k}{(2\pi)^4}\, f(ak,ap) \;\to\; \int_{-\pi}^{\pi} \frac{d^4k'}{(2\pi)^4}\, a^{-4} f(k',ap)$$

  with the rescaling $k' = ak$

- FORM is also necessary for tackling the huge number of terms arising from the Taylor expansions of the Fourier transforms of operators, propagators, vertices, covariant derivatives, ...

- The moments of the unpolarized structure functions behave like

$$\langle x^n \rangle \quad\sim\quad \langle \overline{\psi}\, \gamma_\mu\, D_{\mu_1} \cdots D_{\mu_n}\, \psi \rangle \quad\sim\quad \frac{1}{a^n}$$

  because $D \sim \dfrac{1}{a}$

  $\Rightarrow$ need to perform a Taylor expansion in a to order n

- Then everything has to be expanded to this order

- This is true also for polarized structure functions

- Example: Wilson quark-quark-gluon vertex to order a^2

$$\begin{aligned}(V^a)^{bc}_\mu(k,ap) \quad &= -g_0\,(t^a)^{bc} \\ &\times\Bigg\{ r\left[\sin\frac{k_\mu}{2} + \frac{1}{2}ap_\mu\,\cos\frac{k_\mu}{2} - \frac{1}{8}a^2p_\mu^2\,\sin\frac{k_\mu}{2}\right] \\ &+\mathrm{i}\gamma_\mu\left[\cos\frac{k_\mu}{2} - \frac{1}{2}ap_\mu\,\sin\frac{k_\mu}{2} - \frac{1}{8}a^2p_\mu^2\,\cos\frac{k_\mu}{2}\right]\Bigg\}\end{aligned}$$

- Other example: Wilson quark propagator to order a

$$S^{ab}(k+aq, am_f) = \delta^{ab} \left\{ \frac{-\mathrm{i} \sum_\mu \gamma_\mu \sin k_\mu + 2r \sum_\mu \sin^2 \frac{k_\mu}{2}}{\sum_\mu \sin^2 k_\mu + \left[2r \sum_\mu \sin^2 \frac{k_\mu}{2} \right]^2} \right.$$

$$+ a \cdot \left[\frac{-\mathrm{i} \sum_\mu \gamma_\mu q_\mu \cos k_\mu + r \sum_\mu q_\mu \sin k_\mu + m_f}{\sum_\mu \sin^2 k_\mu + \left[2r \sum_\mu \sin^2 \frac{k_\mu}{2} \right]^2} \right.$$

$$- \left(-\mathrm{i} \sum_\rho \gamma_\rho \sin k_\rho + 2r \sum_\rho \sin^2 \frac{k_\rho}{2} \right)$$

$$\left. \left. \times \frac{2 \sum_\mu q_\mu \sin k_\mu \cos k_\mu + 4r \sum_\mu \sin^2 \frac{k_\mu}{2} \left(r \sum_\nu q_\nu \sin k_\nu + m_f \right)}{\left\{ \sum_\mu \sin^2 k_\mu + \left[2r \sum_\mu \sin^2 \frac{k_\mu}{2} \right]^2 \right\}^2} \right] \right\}$$

- Consequence of all this: huge number of terms (at least in the initial stages of the manipulations)

  $\Rightarrow$ also the gamma algebra becomes cumbersome to do by hand

- All this also means a limitation on the number of moments of structure functions one can compute
  (apart from the limitation coming from operator mixings: when an operator has more than four indices, two of them have to be equal, and this opens the door to mixings, often with power-divergent coefficients)

RESULTS

- We have calculated the renormalization of several operators which measure the lowest moments of DIS structure functions , in the $\overline{\rm MS}$ scheme
- In some cases for a given moment we have computed two operators ((a) and (b)), which belong to two different representations of the discrete Euclidean Lorentz group
- In the operators (a) all Lorentz indices are distinct
- Operator mixing for overlap fermions is simpler than for Wilson fermions: chiral symmetry prohibits any mixings among operators of different chirality
- Mixing coefficients which are power-divergent like a^{-n} in the continuum limit can be eliminated from the start from overlap calculations if the corresponding operators belong to multiplets with the wrong chirality
- In the Wilson case:
  - the operators measuring the moments of the g_2 structure function have additional mixings with wrong-chirality operators with power-divergent coefficients
  - the Z's of the corresponding moments of q and Δq are not constrained to be equal anymore

moment	overlap		Wilson
	$\rho = 1.0$	$\rho = 1.9$	
$\langle x \rangle_q^{(a)}$	1.41213	1.21841	0.98920
$\langle x \rangle_{\Delta q}^{(a)}$	1.41213	1.21841	0.99709
$\langle x \rangle_q^{(b)}$	1.40847	1.21309	0.97837
$\langle x \rangle_{\Delta q}^{(b)}$	1.40847	1.21309	0.99859
$\langle x^2 \rangle_q$	1.51968	1.32436	1.09763
$\langle x^2 \rangle_{\Delta q}$	1.51968	1.32436	1.10231
$\langle x^3 \rangle_q^{(a)}$	1.61872	1.42279	1.19722
$\langle x^3 \rangle_{\Delta q}^{(a)}$	1.61872	1.42279	1.20040
$\langle x^3 \rangle_q^{(b)}$	1.63737	1.44159	1.21534
$\langle x^3 \rangle_{\Delta q}^{(b)}$	1.63737	1.44159	1.21944
$\langle x \rangle_{g_2}$	1.34794	1.18456	mixing
$\langle x^2 \rangle_{g_2}$	1.47816	1.30997	mixing
$\langle x^3 \rangle_{g_2}$	1.58943	1.41900	mixing
$\langle 1 \rangle_{\delta q}$	1.27252	1.08648	0.85631
$\langle x \rangle_{\delta q}$	1.41153	1.21851	0.99559
$\langle x^2 \rangle_{\delta q}$	1.51865	1.32355	1.10021

Renormalization constants of the multiplicatively renormalized operators measuring the lowest moments of various structure functions, for $a = 1.0\ fm$, in the $\overline{\rm MS}$ scheme

- For reasons of computer power, the structure functions have been studied mostly in the quenched approximation , and only recently the first full QCD computations have been feasible
- The average of an observable O is given by

$$\langle O \rangle = Z^{-1} \int \mathcal{D}U \mathcal{D}\overline{\psi} \mathcal{D}\psi \; O(U, \overline{\psi}, \psi) \; \mathrm{e}^{-S_g[U] - \overline{\psi}(\not{D}[U]+m)\psi}$$

with

$$Z = \int \mathcal{D}U \mathcal{D}\overline{\psi} \mathcal{D}\psi \; \mathrm{e}^{-S_g[U] - \overline{\psi}(\not{D}[U]+m)\psi}$$

- On the lattice the Grassmann variables are integrated out, and one has then

$$Z = \int \mathcal{D}U \;\; \det(\not{D}[U] + m) \; \mathrm{e}^{-S_g[U]}$$

or, equivalently, an effective action

$$S_{\mathrm{eff}}[U] = S_g[U] - \ln \det(\not{D}[U] + m) = S_g[U] - \mathrm{Tr} \ln(\not{D}[U] + m)$$

- Quenching amounts to doing simulations with $\det(\not{D}[U] + m) = 1$
- This means that there are no sea quarks in the calculations: the internal quark loops are neglected
- Although it looks quite drastic, it is not a bad approximation for many physical quantities, and apparently (as it has been understood recently) also for structure functions

- Structure functions in full Wilson QCD (unquenched) have been the major progress in the past couple of years (LHPC & QCDSF)
- Apparently, after the first studies with dynamical quarks , there is no statistically significant difference between quenched and full QCD results
- Previously it had been conjectured that quenching was the cause of the observed discrepancy of lattice results with experiment
- Evidently this is not the case, at least for the values of quark masses attainable at present
- The largest discrepancies concern unpolarized quark distributions
- The spin-dependent g_1 and g_2 structure functions have also been extensively studied, as well as the axial charge $g_A = \Delta u - \Delta d$
- g_2 with Wilson fermions presents a mixing problem due to the breaking of chirality (which is an artifact of Wilson fermions): there is a mixing with lower-dimensional operators

  $\Rightarrow$ power divergences $\sim \frac{1}{a^n}$ in the continuum limit

  With Ginsparg-Wilson fermions the chiral symmetry is not broken, and this mixing is forbidden
- The h_1 transversity structure function, whose lowest moment is the tensor charge $\delta u - \delta d$, has also been studied on the lattice

- One limitation is given by the difficulty to numerically simulate disconnected diagrams (except through gluon lines)
- The disconnected diagrams are flavor independent

  $\Rightarrow$ they do not contribute to the difference between u and d structure functions (for $SU(2)$ degeneracy)
- This means that quantities like $g_A = \Delta u - \Delta d$ and $\langle x^n \rangle_{u-d}$ do not receive contributions from disconnected diagrams
- Extrapolations to the chiral limit : the linear extrapolations in $m_q \sim m_\pi^2$ could also be an explanation for the discrepancies (while the continuum limit does not seem to be a problem)
- It has been recently suggested that extrapolations of the lattice data which use chiral perturbation theory could solve the discrepancy with experiment
- It looks like the pion cloud of the nucleon is not adequately described by current lattices
- Extrapolation formulae using chiral perturbation theory look like (Thomas et al., Chen, Ji, Arndt, Savage, . . .)

$$\begin{aligned} \langle x^n \rangle_{u-d} \quad = \quad & A_n \left[1 - \frac{(3g_A^2+1)}{(4\pi f_\pi)^2} \, m_\pi^2 \, \ln\left(\frac{m_\pi^2}{m_\pi^2 + \Lambda^2} \right) \right] \\ & + B_n \, (m_\pi r_0)^2 + C_n \, (a/r_0)^2 \end{aligned}$$

  where Λ is a phenomenological cutoff which describes the size of the source generating the pion cloud

- The phenomenological chiral extrapolation formulae seem to resolve the discrepancy with experiment for a few observables
- However, the results by the various groups can be reproduced only with Λ's with too wide a range to have some predictive power
- Moreover, the currently available lattice data do not even discriminate between linear and chiral perturbation theory fits
- So far we are able to reproduce the phenomenological results, but only after we know them
- The problems is that the available pions are not sufficiently light
- We need a smaller pion mass, $m_\pi < 250\ MeV$, in order that the parameters of the chiral expansions can be well determined on the lattice
- For such calculations we need about $8\ Teraflops$ for one year
- The next generation of computers, coming in a couple of years, will be able to perform these calculations
- So, the question is still open
- What is not in doubt: the pion cloud is very important
- For the pion cloud of the proton to be adequately included in the lattice box and properly measured in Monte Carlo simulations, the pion correlation length should be much smaller than the lattice size $\Rightarrow$ larger lattices

- Higher-twist corrections have also been studied
- The lattice results seem to indicate that the twist four $(1/Q^2)$ corrections are in a few particular cases quite small
- The renormalization factors of twist-4 4-quark operators like

$$\sum_a \overline{\psi}\gamma_\mu\gamma_5\, t^a\psi \cdot \overline{\psi}\gamma_\nu\gamma_5\, t^a\psi$$

  have been calculated in perturbation theory
- First moment of the unpolarized pion structure function:

$$\int_0^1 dx\, F_2(x,Q^2)\Big|_{I=2} = 1.67(64)\cdot\alpha_S(Q^2)\cdot\frac{f_\pi^2}{Q^2}$$

- First moment of the unpolarized proton structure function:

$$\int_0^1 dx\, F_2(x,Q^2)\Big|_{\mathbf{27},I=1} = -0.0006(5)\cdot\alpha_S(Q^2)\cdot\frac{m_P^2}{Q^2}$$

- These flavor and isospin combinations avoid mixing with lower-dimensional operators
- These twist-4 results are much smaller than the corresponding twist-2 matrix elements computed on the lattice
- They are also smaller than the phenomenological numbers
- These are not complete and systematic studies, like the ones concerning leading-twist operators
- Not all operators have been considered, because of the mixings

SUMMARY

- The lattice provides techniques which are invaluable for non-perturbative, first-principle, model-independent studies
- A lot of work has been done in the past years for the study of all kinds of structure functions
- Connection to continuum schemes ($\overline{\rm MS}$): renormalization in perturbation theory (Wilson and overlap)
- Simulations (Wilson) : now also in full QCD , but the discrepancies between lattice and experiment remain
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