Numerical evaluation of master integrals from differential equations.

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- Introduction
- Master differential equations
- 4-th order Runge-Kutta method
- Results: Sunrise, ...
- Perspectives

Introduction

- Integration by part identities
  and Master Integrals selection
  F.V. Tkachov (1981),
  K.G. Chetyrkin e F.V. Tkachov (1981)

- Master Integrals calculation:
  - analytically,
  - numerically via integral representation,
  - numerically via difference equations,
  - numerically via differential equations.

2-loop self-mass topologies:

O.V. Tarasov (1997),
M. Caffo, H. Czyż, S. Laporta and E. Remiddi (1998).

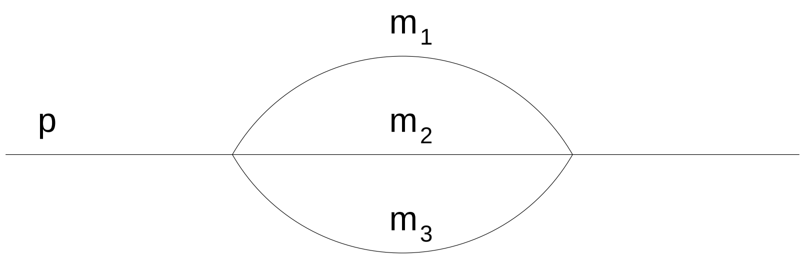


Figure 1: Sunrise: 4 master integrals

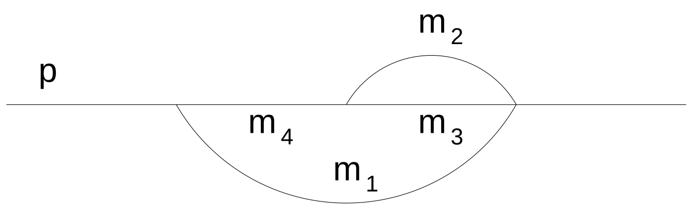


Figure 2: 4-denominators: 1 master integral

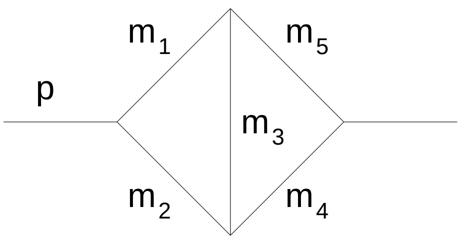


Figure 3: 5-denominators: 1 master integral

Master differential equations

A.V. Kotikov (1991),
E. Remiddi (1997),
M. Caffo, H. Czyż, S. Laporta and E. Remiddi (1998),
....

Obtained by derivation of internal masses m_i or external invariants s_e and use of IPI

$$K_k(m_i^2, s_e)\,\frac{\partial}{\partial s_j}\;F_k(n, m_i^2, s_e) =$$

$$\sum_l M_{k,l}(n, m_i^2, s_e)\;\;F_l(n, m_i^2, s_e)\;\;+T_k(n, m_i^2, s_e)$$

$K_k(m_i^2, s_e)$ and $M_{k,l}(n, m_i^2, s_e)$ are polynomials, $T_k(n, m_i^2, s_e)$ are polynomials times lower order MI

$$K_k(m_i^2, s_j) = 0$$

provides 'special' values (analytic calculations)

4-th order Runge-Kutta method

Given the first-order differential equation

$$\frac{\partial y(x)}{\partial x} = f(x, y)$$

Euler method: starting from a point x_n, where the solution y_n is known, it advances the solution from x_n to $x_{n+1} = x_n + \Delta$

$$y_{n+1} = y_n + \Delta f(x_n, y_n) + \mathcal{O}(\Delta^2)$$

4-th order Runge-Kutta formula:

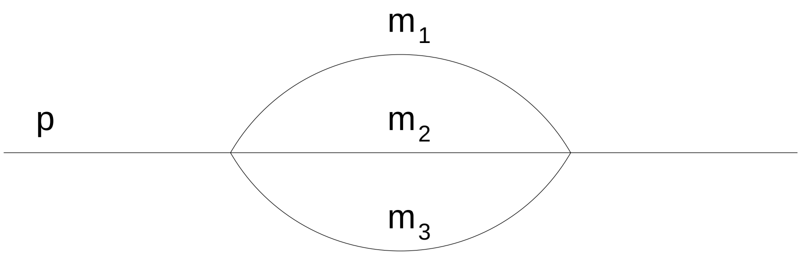
$$\begin{aligned}
k_1 &= \Delta f(x_n, y_n), \; k_2 = \Delta f(x_n + \frac{\Delta}{2}, y_n + \frac{k_1}{2}), \\
k_3 &= \Delta f(x_n + \frac{\Delta}{2}, y_n + \frac{k_2}{2}), \\
k_4 &= \Delta f(x_n + \Delta, y_n + k_3), \\
y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + \mathcal{O}(\Delta^5)
\end{aligned}$$

Robust approach!!!

SUNRISE with arbitrary masses

M. Caffo, H. Czyż, S. Laporta and E. Remiddi (1998),

M. Caffo, H. Czyż and E. Remiddi (2002).



$$A(n, m_1^2, m_2^2, m_3^2, p^2, -\alpha_1, -\alpha_2, -\alpha_3, \beta_1, \beta_2) = \mu^{8-2n}$$

$$\int \frac{d^n k_1}{(2\pi)^{n-2}} \int \frac{d^n k_2}{(2\pi)^{n-2}} \frac{(p \cdot k_1)^{\beta_1} (p \cdot k_2)^{\beta_2}}{(k_1^2 + m_1^2)^{\alpha_1} (k_2^2 + m_2^2)^{\alpha_2} ((p - k_1 - k_2)^2 + m_3^2)^{\alpha_3}} \ .$$

The 4 Master Integrals are

$$\begin{aligned}
F_0(n, m_1^2, m_2^2, m_3^2, p^2) &= A(n, m_1^2, m_2^2, m_3^2, p^2, -1, -1, -1, 0, 0) \ , \\
F_1(n, m_1^2, m_2^2, m_3^2, p^2) &= A(n, m_1^2, m_2^2, m_3^2, p^2, -2, -1, -1, 0, 0) \ , \\
F_2(n, m_1^2, m_2^2, m_3^2, p^2) &= A(n, m_1^2, m_2^2, m_3^2, p^2, -1, -2, -1, 0, 0) \ , \\
F_3(n, m_1^2, m_2^2, m_3^2, p^2) &= A(n, m_1^2, m_2^2, m_3^2, p^2, -1, -1, -2, 0, 0) \ .
\end{aligned}$$

$$F_i(n, m_1^2, m_2^2, m_3^2, p^2) = -\frac{\partial}{\partial m_i^2} F_0(n, m_1^2, m_2^2, m_3^2, p^2)\,, \quad i = 1, 2, 3.$$

IPI are up to $\Sigma(\alpha_i - 1) = 2$ and $\Sigma\beta_j = 2$.

The proper scale for numerical discussion comes out to be $\mu = m_1 + m_2 + m_3$.

Lower order diagrams are known functions, like the 1-loop vacuum graph

$$T(n, m^2) = \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{k^2 + m^2} = \frac{m^{n-2} C(n)}{(n-2)(n-4)}\,,$$

$$\begin{aligned} C(n) &= \left(2\sqrt{\pi}\right)^{(4-n)} \Gamma\left(3 - \frac{n}{2}\right) \\ &= 1 - (n-4)\left[\log(2\sqrt{\pi}) - \frac{1}{2}\gamma_E\right] + \mathcal{O}\left((n-4)^2\right)\,, \end{aligned}$$

Differential equations for arbitrary n:

$$p^2\frac{\partial}{\partial p^2}F_0(n,m_1^2,m_2^2,m_3^2,p^2) = (n-3)F_0(n,m_1^2,m_2^2,m_3^2,p^2)$$
$$+\,m_1^2F_1(n,m_1^2,m_2^2,m_3^2,p^2) + m_2^2F_2(n,m_1^2,m_2^2,m_3^2,p^2) + m_3^2F_3(n,m_1^2,m_2^2,m_3^2,p^2)\ .$$

$$\begin{aligned}
p^2\frac{\partial}{\partial p^2}F_i(n,m_1^2,m_2^2,m_3^2,p^2) &= \frac{n-4}{2}F_i(n,m_1^2,m_2^2,m_3^2,p^2) \\
&+ \frac{n-3}{D(m_1^2,m_2^2,m_3^2,p^2)}\Big[\ P_{i,0}(m_1^2,m_2^2,m_3^2,p^2)\frac{3n-8}{2}F_0(n,m_1^2,m_2^2,m_3^2,p^2) \\
&\qquad + P_{i,1}(m_1^2,m_2^2,m_3^2,p^2)F_1(n,m_1^2,m_2^2,m_3^2,p^2) \\
&\qquad + P_{i,2}(m_1^2,m_2^2,m_3^2,p^2)F_2(n,m_1^2,m_2^2,m_3^2,p^2) \\
&\qquad + P_{i,3}(m_1^2,m_2^2,m_3^2,p^2)F_3(n,m_1^2,m_2^2,m_3^2,p^2)\ \Big] \\
&+ \frac{(n-2)^2}{D(m_1^2,m_2^2,m_3^2,p^2)}\Big[\ Q_{i,1}(m_1^2,m_2^2,m_3^2,p^2)T(n,m_2^2)T(n,m_3^2) \\
&\qquad + Q_{i,2}(m_1^2,m_2^2,m_3^2,p^2)T(n,m_1^2)T(n,m_3^2) \\
&\qquad + Q_{i,3}(m_1^2,m_2^2,m_3^2,p^2)T(n,m_1^2)T(n,m_2^2)\ \Big]\ ,
\end{aligned}$$

$$\begin{aligned}
D(m_1^2,m_2^2,m_3^2,p^2) &= \left(p^2+(m_1+m_2+m_3)^2\right)\ \left(p^2+(m_1+m_2-m_3)^2\right) \\
&\quad \left(p^2+(m_1-m_2+m_3)^2\right)\ \left(p^2+(m_1-m_2-m_3)^2\right)
\end{aligned}$$

$P_{i,j}(m_1^2, m_2^2, m_3^2, p^2)$ and $Q_{i,j}(m_1^2, m_2^2, m_3^2, p^2)$ are polynomials of p^2 and of the masses

For $n \simeq 4$ the master integrals are expanded

$$F_i(n, m_1^2, m_2^2, m_3^2, p^2) = C^2(n)\Big\{\frac{1}{(n-4)^2} F_i^{(-2)}(m_1^2, m_2^2, m_3^2, p^2)$$
$$+\frac{1}{(n-4)} F_i^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) + F_i^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + \mathcal{O}(n-4)\Big\} \ .$$

for $i = 0, 1, 2, 3,$

The coefficients of the poles in $n - 4$ are

$$\begin{aligned} F_0^{(-2)}(m_1^2, m_2^2, m_3^2, p^2) &= -\frac{1}{8}(m_1^2 + m_2^2 + m_3^2) \ , \\ F_0^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) &= \frac{1}{8}\Big\{\frac{p^2}{4} + \frac{3}{2}(m_1^2 + m_2^2 + m_3^2) \\ &\quad - \Big[m_1^2 \log(\frac{m_1^2}{\mu^2}) + m_2^2 \log(\frac{m_2^2}{\mu^2}) + m_3^2 \log(\frac{m_3^2}{\mu^2})\Big]\Big\} \ , \end{aligned}$$

$$\begin{aligned} F_i^{(-2)}(m_1^2, m_2^2, m_3^2, p^2) &= \frac{1}{8} \ , \\ F_i^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) &= -\frac{1}{16} + \frac{1}{8}\log(\frac{m_i^2}{\mu^2}) \ , \ i = 1, 2, 3. \end{aligned}$$

The finite parts satisfy the differential eq.s:

$$p^2 \frac{\partial}{\partial p^2} F_0^{(0)}(m_1^2, m_2^2, m_3^2, p^2) \; = \; F_0^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + F_0^{(-1)}(m_1^2, m_2^2, m_3^2, p^2)$$
$$+ \, m_1^2 F_1^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + m_2^2 F_2^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + m_3^2 F_3^{(0)}(m_1^2, m_2^2, m_3^2, p^2) \; ,$$

$$\begin{aligned}
&8D(m_1^2, m_2^2, m_3^2, p^2) p^2 \frac{\partial}{\partial p^2} F_i^{(0)}(m_1^2, m_2^2, m_3^2, p^2) = \\
&\quad 4D(m_1^2, m_2^2, m_3^2, p^2) F_i^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) \\
&\quad + P_{i,0}(m_1^2, m_2^2, m_3^2, p^2) \Big[16 F_0^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + 28 F_0^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) \\
&\qquad\qquad + 12 F_0^{(-2)}(m_1^2, m_2^2, m_3^2, p^2) \Big] \\
&\quad + 8 P_{i,i}(m_1^2, m_2^2, m_3^2, p^2) \Big[F_i^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + F_i^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) \Big] \\
&\quad + 8 P_{i,j}(m_1^2, m_2^2, m_3^2, p^2) \Big[F_j^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + F_j^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) \Big] \\
&\quad + 8 P_{i,k}(m_1^2, m_2^2, m_3^2, p^2) \Big[F_k^{(0)}(m_1^2, m_2^2, m_3^2, p^2) + F_k^{(-1)}(m_1^2, m_2^2, m_3^2, p^2) \Big] \\
&\quad + Q_{i,i}(m_1^2, m_2^2, m_3^2, p^2) \; m_j^2 m_k^2 \Big[\log(m_j^2) + \log(m_k^2) \Big]^2 \\
&\quad + Q_{i,j}(m_1^2, m_2^2, m_3^2, p^2) \; m_i^2 m_k^2 \Big[\log(m_i^2) + \log(m_k^2) \Big]^2 \\
&\quad + Q_{i,k}(m_1^2, m_2^2, m_3^2, p^2) \; m_i^2 m_j^2 \Big[\log(m_i^2) + \log(m_j^2) \Big]^2 \; ,
\end{aligned}$$

From these diff. eq.s analytic expressions were completed for the 1st term in the expansions around the points (special points):

- $p^2 = 0$
- $p^2 = \infty$
- $p^2 = -(m_1 + m_2 + m_3)^2$, threshold
- $p^2 = -(m_1 + m_2 - m_3)^2$, pseudo-thresholds

M. Caffo, H. Czyż and E. Remiddi (2000), (2001), (2002).

Most laborious part of the calculation, which sometimes requires external knowledge
$\rightarrow$ weakest aspect of the method

A FORTRAN program using the fourth-order Runge-Kutta formula is implemented to obtain numerical results, starting from the points indicated.

When starting from $p^2 = \infty$ or threshold, subtracted differential equations are used.

Due to the particular starting points also the first term in the expansion is necessary.

Fast and rather precise execution
(on an Intel Pentium III of 1 GHz):

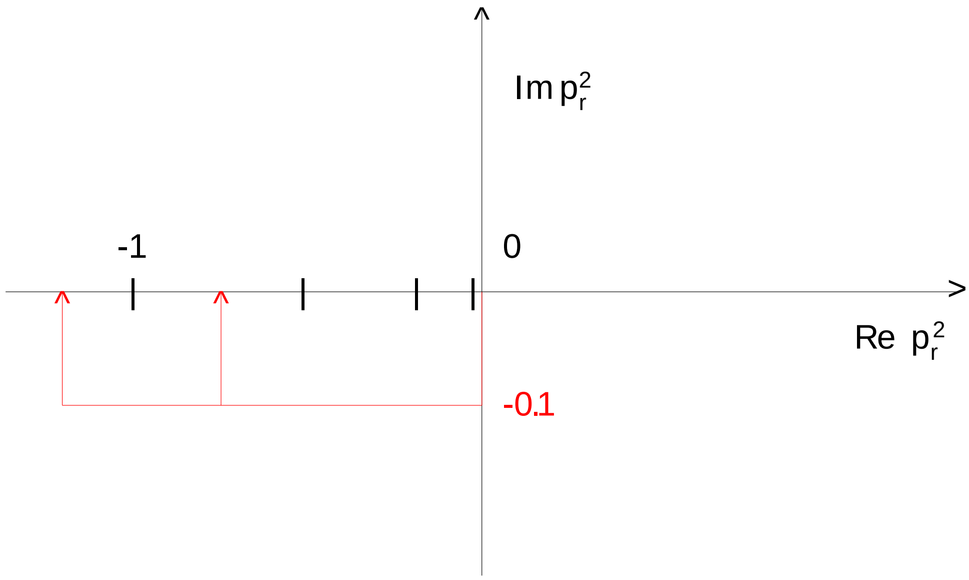
7 digits $\rightarrow$ fraction of a sec — 10 sec CPU
11 digits $\rightarrow$ few tens of sec — one hour CPU.

For numerical convenience dimensionless quantities are better:

$$m_{i,r} \equiv \frac{m_i}{m_1 + m_2 + m_3} \quad , \quad p_r^2 \equiv \frac{p^2}{(m_1 + m_2 + m_3)^2} \ ,$$

$$F_{0,r}^{(0)} \equiv \frac{F_0^{(0)}}{(m_1 + m_2 + m_3)^2} \ ;$$

The path followed is in the complex p_r^2 plane:



Consistency between different paths.

ERROR ESTIMATION:

$$\Delta = \frac{L}{N} \quad = \; step \; ,$$

$$\Delta^5 = error \; in \; a \; step \; of \; 4th \; ord. \; RK \; ,$$

$$\epsilon_{RK} = N\Delta^5 = \frac{L^5}{N^4} \; ,$$

$$\epsilon_{RK} = \frac{V(N) - V(N/10)}{V(N)} \; , (effect. \; est.)$$

Cumulative rounding error:

$$\epsilon_{cre} = \sqrt{N} \times 10^{-15} \; , (double \; precision)$$

$$\epsilon_{tot} = \epsilon_{RK} + \epsilon_{cre} \; .$$

COMPARISONS:

[1] *F.A. Berends, M. Böhm, M. Buza and R. Scharf (1994),*

[2] *G. Passarino (2001)*

Finite and μ-independent combination:

$$\begin{aligned} T_{123N}(p^2, m_1^2, m_2^2, m_3^2) = -16[& + F_0^{(0)}(m_1^2, m_2^2, m_3^2, p^2) \\ & - F_0^{(0)}(m_1^2, 0, m_3^2, p^2) \\ & - F_0^{(0)}(0, m_2^2, m_3^2, p^2) \\ & + F_0^{(0)}(0, 0, m_3^2, p^2)] \ , \end{aligned}$$

Numerically in the $m_i \to 0$ limit

or analytically,

Excellent agreement! $\to$ 12 digits

Other calculations exist:

S. Bauberger, F.A. Berends, M. Böhm and M. Buza (1995),

A. Ghinculov and J.J. van der Bij (1995),

..

Perspectives

- To complete the 2-loop self-mass for arbitrary masses
- Easy extension to more loops
- Easy extension to more legs
- Comforting support to analytic calculations

p_r^2	Re $F_{0,r}^{(0)}$	Im $F_{0,r}^{(0)}$	Re $F_1^{(0)}$	Im $F_1^{(0)}$
-1000.	-113.349786296(3)	96.922241476(2)	-0.4811355157(1)	3.255217395(1)
-30.	-0.230629539580(2)	2.31604333072(1)	2.86686026427(1)	2.46075767262(1)
-15.	0.044413679180(8)	0.964565210432(6)	3.47247970401(2)	2.23096593908(2)
-1.5	-0.2536003902785(5)	0.0032295545873(1)	5.03180993338(1)	0.564979990606(8)

Table 1: The benchmark values of $F_{0,r}^{(0)}$ and $F_1^{(0)}$ for masses $m_1 = 1$, $m_2 = 9$, $m_3 = 200$ and $\mu = m_1 + m_2 + m_3$.

p_r^2	Re $F_2^{(0)}$	Im $F_2^{(0)}$	Re $F_3^{(0)}$	Im $F_3^{(0)}$
-1000.	-1.4689531571(1)	2.393163661(1)	-0.81238274708(3)	1.17931522036(6)
-30.	0.880149099634(4)	1.624323639603(8)	0.1097269441157(5)	0.496397637240(4)
-15.	1.26943734837(1)	1.42094925579(1)	0.195266696001(4)	0.366097572324(2)
-1.5	2.066516140875(4)	0.239828506599(7)	0.2185459121886(5)	0.0195952353591(3)

Table 2: The benchmark values of $F_2^{(0)}$ and $F_3^{(0)}$ for masses $m_1 = 1$, $m_2 = 9$, $m_3 = 200$ and $\mu = m_1 + m_2 + m_3$.

p_r^2	$F_{0,r}^{(0)}$	$F_1^{(0)}$	$F_2^{(0)}$	$F_3^{(0)}$
-1	-0.279454902855371	4.83093725524177	1.89253423642110	0.185424043556224
-0.99	-0.2798928396415(5)	4.80520291678(1)	1.884686031133(5)	0.1846128537367(4)
$p_{ps3,r}^2$	-0.280281633048667	4.78848044412341	1.87869903011051	0.183940826371101
-0.9	-0.2836780811878(5)	4.68669935479(1)	1.836380342548(8)	0.1786103121374(3)
$p_{ps2,r}^2$	-0.286233415451605	4.62888926299134	1.80974413459626	0.174898795219002
-0.825	-0.2866587055221(5)	4.620058473554(8)	1.805561498336(6)	0.1742958493924(4)
$p_{ps1,r}^2$	-0.286906928933491	4.61498769104262	1.80314761916276	0.173945546345814
-0.8	-0.2876221116285(5)	4.60069487253(1)	1.796297920597(6)	0.1729424293568(4)
-0.1	-0.3101507246241(3)	4.261426874519(4)	1.619837556072(1)	0.1432933391084(2)
0	-0.312816604092084	4.22788075922252	1.60134292154365	0.139821925866842
1.0	-0.3340476235037(6)	3.970691522343(6)	1.455289136414(2)	0.1103513887593(2)
30.0	0.323213333716(3)	2.33632892937(2)	0.425519391740(4)	-0.178170159306(2)
1000.	115.539777092(3)	-0.8050335305(2)	-1.7888035846(2)	-1.11980894482(3)

Table 3: The benchmark values of $F_{0,r}^{(0)}$, $F_1^{(0)}$, $F_2^{(0)}$ and $F_3^{(0)}$ for masses $m_1 = 1$, $m_2 = 9$, $m_3 = 200$ and $\mu = m_1 + m_2 + m_3$.

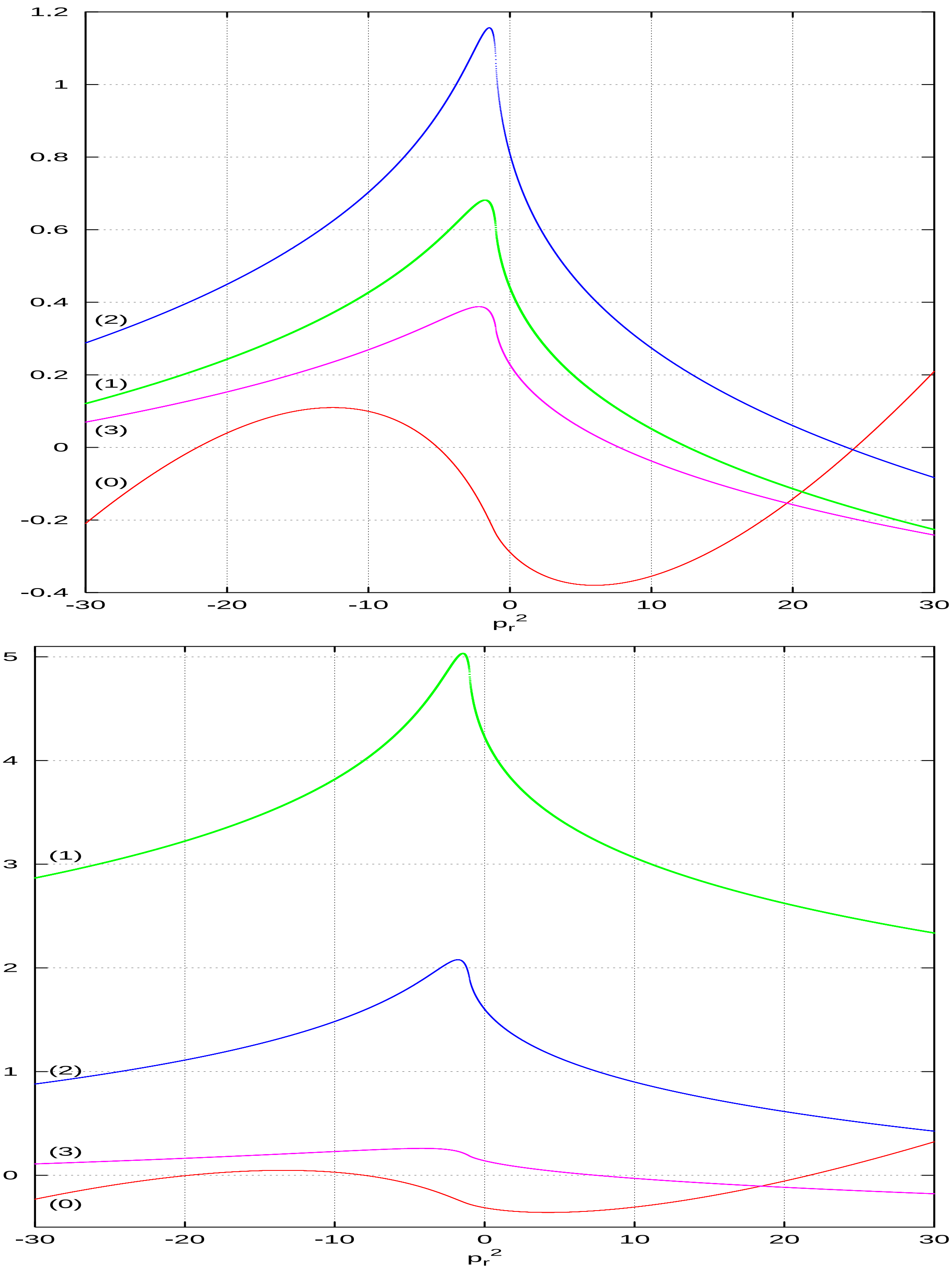


Figure 4: Plots of Re $F_{0,r}^{(0)}$ (labeled as (0)) and Re $F_i^{(0)}$ (labeled as (i)) as a function of p_r^2 above for $m_1 = 2$, $m_2 = 1$ and $m_3 = 4$, below for $m_1 = 1$, $m_2 = 9$ and $m_3 = 200$.

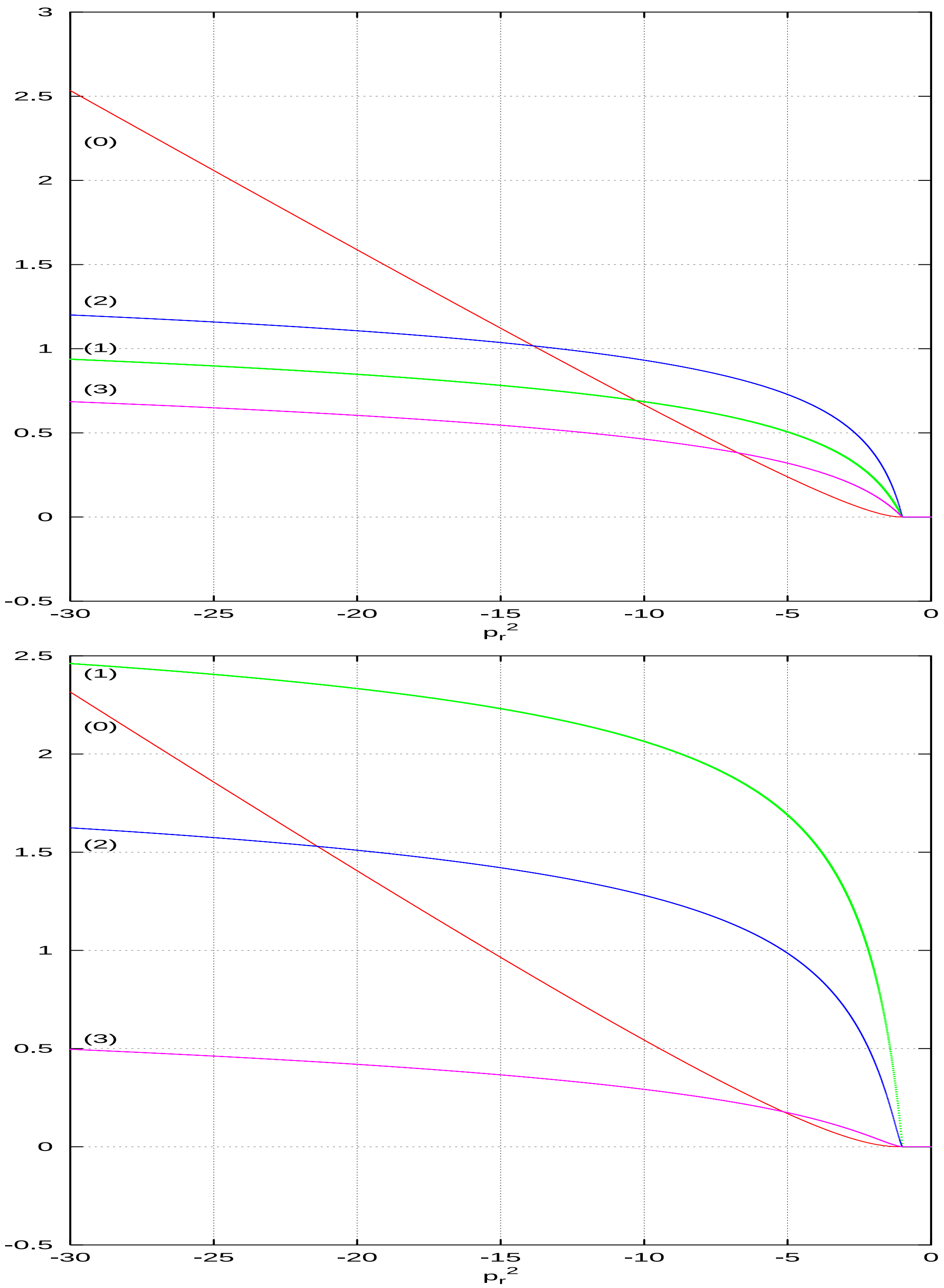


Figure 5: Plots of Im $F_{0,r}^{(0)}$ (labeled as (0)) and Im $F_i^{(0)}$ (labeled as (i)) as a function of p_r^2 above for $m_1 = 2$, $m_2 = 1$ and $m_3 = 4$, below for $m_1 = 1$, $m_2 = 9$ and $m_3 = 200$.

m_1	m_2	m_3	p^2	T_{123N}	p^2	T_{123N}
3	3	10	-9	-7.3129877443	9	-6.93244055931
				-7.3129877442(26)		-6.93244055924(50)
2	3	10	-20	-4.1493850173	20	-3.63591843327
				-4.1493850171(18)		-3.63591843320(95)
2	2	10	-25	-2.3353847298	25	-1.9428452190
				-2.3353847298(14)		-1.9428452191(10)
1	2	10	-30	-0.8117674738	30	-0.6306847352
				-0.8117674738(15)		-0.6306847353(12)
1	1	10	-49	-0.3167501084	49	-0.1950338472
				-0.3167501085(24)		-0.1950338472(21)
3	4	15	-50	-7.9471022759	50	-6.8270303849
				-7.9471022760(33)		-6.8270303852(19)
3	4	20	-100	-6.0171476156	100	-4.9485063889
				-6.0171476159(59)		-4.9485063897(42)
3	4	20	-150	-6.3903568683	150	-4.7506023184
				-6.3903568686(73)		-4.7506023184(66)
5	5	25	-150	-14.5339444977	150	-1.21816923644
				-14.5339444982(87)		-1.21816923648(68)
5	5	25	-200	-15.0523063012	200	-11.8790613597
				-15.0523063010(95)		-11.8790613597(83)

Table 4: Comparison for small $|p^2| < (m_1+m_2+m_3)^2$. In each box first entry is multiple series value of Table 1 of [1], second entry is present result (in parenthesis it is given the error on the last digits). The value of p^2 here corresponds to $-p^2$ in [1].

p^2	T_{123N}	p^2	T_{123N}	p^2	T_{123N}
-1	-70.6856984	-25	-70.75620346	-81	-70.92141286
	-70.6856977(39)		-70.75620352(11)		-70.92141291(70)
	-70.686011		-70.756299		-70.921481
1	-70.6798310	25	-70.609519049	81	-70.446146678
	-70.6798305(21)		-70.609519051(97)		-70.446146655(35)
	-70.680106		-70.609231		-70.446044

Table 5: Comparison for small $|p^2| < (m_1+m_2+m_3)^2$ and for $m_1 = 10, m_2 = 20, m_3 = 100$. In each box first entry is multiple series value of [1], second entry is present result (in parenthesis it is given the error on the last digits), third entry is from numerical integration in Table 7 of [2]. The value of p^2 here corresponds to $-p^2$ in [1] and to s in [2].

m_1	m_2	m_3	p^2	Re T_{123N}	Im T_{123N}
2	3	2	-80	0.587432001	-11.262835755
				0.587431990(41)	-11.262835744(24)
3	3	3	-100	-1.28284949	-20.899600723
				-1.28284943(16)	-20.899600741(10)
2	3	4	-100	-0.3286481685	-11.84587606309
				-0.3286481687(16)	-11.84587606302(60)
3	4	4	-150	-1.26795173	-26.491194705
				-1.26795173(21)	-26.491194694(41)
2	4	3	-150	1.5662482672	-11.9689438774
				1.5662482670(14)	-11.9689438778(12)
3	3	4	-150	0.9865824304	-16.10213970663
				0.9865824312(19)	-16.10213970687(77)
3	4	5	-150	-4.7638745(12)	-29.601246304(26)
				-4.7638748416(32)	-29.60124631204(68)
2	3	4	-200	1.6960823345	-6.02417248918
				1.6960823350(17)	-6.02417248906(81)
3	4	4	-200	1.86355967	-20.39852237
				1.86355979(34)	-20.39852240(12)
4	4	4	-250	2.64395201	-27.60904430
				2.64395222(31)	-27.60904439(18)

Table 6: Comparison for p^2 large ($|p^2| > (m_1+m_2+m_3)^2$) and negative. In each box first entry is multiple series value of Table 2 of [1], second entry is present result (in parenthesis it is given the error on the last digits). The value of p^2 here corresponds to $-p^2$ in [1].

m_1	m_2	m_3	p^2	T_{123N}	m_1	m_2	m_3	p^2	T_{123N}
2	2	2	50	-3.728125558	2	4	3	100	-7.1836810855
				-3.728125610(46)					-7.1836810854(54)
3	3	3	100	-8.79126989	3	4	3	120	-12.430997190
				-8.79127000(11)					-12.430997250(30)
3	3	4	150	-7.0830520665	3	4	4	150	-10.85647158
				-7.0830520661(28)					-10.85647158(26)
3	4	3	150	-11.361931056	3	4	4	200	-9.64611359
				-11.361931121(31)					-9.64611360(29)
2	3	4	200	-3.3018636831	4	4	4	250	-13.57188440
				-3.3018636830(21)					-13.57188463(21)

Table 7: Comparison for p^2 large ($|p^2| > (m_1+m_2+m_3)^2$) and positive. In each box first entry is multiple series value of Table 2 of [1], second entry is present result (in parenthesis it is given the error on the last digits). The value of p^2 here corresponds to $-p^2$ in [1].