$\mathcal{N}=1$ supersymmetric constraints for evolution kernels

A.V. Belitsky [a*] and D. Müller [a]

[a]Institut für Theoretische Physik, Universität Regensburg,
D-93040 Regensburg, Germany

We provide a complete set of supersymmetric constraints for the anomalous dimensions of the conformal twist-two operators to all orders of perturbation theory. Employing them we derive new relations between the exclusive evolution kernels and apply them in QCD in order to get definite predictions at leading order and beyond.

In the early days of QCD, Dokshitzer had found a simple relation between the partonic splitting functions [1] once all colour factors are identified. About a decade after, six leading order (LO) constraints in the chiral-even sector and a LO relation in the chiral-odd sector have been derived making use of the conformal invariance of the classical $\mathcal{N}=1$ super Yang-Mills theory [2]. These equations provide a good testing ground for available QCD results and, e.g. the Dokshitzer relation had served as a consistency check for the non-trivial two-loop calculation of the DGLAP kernels in the unpolarized [3] and polarized [4] cases as well as for transversity [5]. However, not all of the remaining constraints are fulfilled at NLO [6] and, thus, deserves further study.

To start with, we define the so-called conformal composite operators in the $\mathcal{N}=1$ super Yang-Mills theory in the chiral-even sector:

$$\begin{Bmatrix} {}^{Q}\mathcal{O}^{V} \\ {}^{Q}\mathcal{O}^{A} \end{Bmatrix}_{jl} = \frac{1}{2}\bar\psi^a_+ (i\partial_+)^l \begin{Bmatrix} \gamma_+ \\ \gamma_+\gamma_5 \end{Bmatrix} C_j^{3/2}\left(\frac{\overleftrightarrow{\mathcal{D}}_+}{\partial_+}\right)\psi^a_+, \quad (1)$$

$$\begin{Bmatrix} {}^{G}\mathcal{O}^{V} \\ {}^{G}\mathcal{O}^{A} \end{Bmatrix}_{jl} = G^{a\perp}_{+\mu}(i\partial_+)^{l-1} \begin{Bmatrix} g^\perp_{\mu\nu} \\ i\epsilon^\perp_{\mu\nu} \end{Bmatrix} C_{j-1}^{5/2}\left(\frac{\overleftrightarrow{\mathcal{D}}_+}{\partial_+}\right) G^{a\perp}_{\nu+},$$

$$\begin{Bmatrix} \mathcal{V} \\ \mathcal{U} \end{Bmatrix}_{jl} = \frac{(j+2)(j+3)}{(j+1)} \times G^{a\perp}_{+\mu}(i\partial_+)^l P_j^{(2,1)}\left(\frac{\overleftrightarrow{\mathcal{D}}_+}{\partial_+}\right)\gamma^\perp_\mu \begin{Bmatrix} 1 \\ \gamma_5 \end{Bmatrix}\psi^a_+,$$

where $\epsilon^\perp_{\mu\nu} \equiv \epsilon_{\mu\nu-+}$, $\partial = \overrightarrow{\partial} + \overleftarrow{\partial}$ and $\overleftrightarrow{\mathcal{D}} = \overrightarrow{\mathcal{D}} - \overleftarrow{\mathcal{D}}$. These operators transform covariantly under supersymmetric transformations and form the irreducible Wess-Zumino chiral supermultiplet (here $\sigma_j = [1-(-1)^j]/2$)

$$\begin{aligned} &\delta^Q \mathcal{S}^1_{jl} = \sigma_j\, \bar\zeta \mathcal{V}_{j-1l}, \qquad \delta^Q \mathcal{S}^2_{jl} = \sigma_j\, \bar\zeta \mathcal{V}_{jl}, \quad (2)\\ &\delta^Q \mathcal{P}^1_{jl} = \sigma_{j+1}\, \bar\zeta \mathcal{U}_{j-1l}, \quad \delta^Q \mathcal{P}^2_{jl} = \sigma_{j+1}\, \bar\zeta \mathcal{U}_{jl},\\ &\delta^Q \mathcal{V}_{j-1l-1} = -\gamma_-\zeta\left\{\mathcal{S}^1_{jl} + \mathcal{S}^2_{j-1l}\right\} \\ &\qquad\qquad -\gamma_-\gamma_5\zeta\left\{\mathcal{P}^1_{jl} + \mathcal{P}^2_{j-1l}\right\}, \end{aligned}$$

where we have introduced particular combinations of the boson operators

$$\begin{Bmatrix} \mathcal{S}^1 \\ \mathcal{P}^1 \end{Bmatrix}_{jl} = \frac{6}{j}\,{}^{G}\mathcal{O}^{\Gamma}_{jl} + {}^{Q}\mathcal{O}^{\Gamma}_{jl}, \quad (3)$$

$$\begin{Bmatrix} \mathcal{S}^2 \\ \mathcal{P}^2 \end{Bmatrix}_{jl} = \frac{6}{j+1}\,{}^{G}\mathcal{O}^{\Gamma}_{jl} - \frac{j+3}{j+1}\,{}^{Q}\mathcal{O}^{\Gamma}_{jl},$$

with $\Gamma = V(A)$ standing for the $\mathcal{S}$ ($\mathcal{P}$) operator.

Similarly, we introduce for the chiral-odd sector the following composite boson and fermion operators, respectively,

$${}^{Q}\mathcal{O}^{T}_{\mu;jl} = \frac{1}{2}\bar\psi^a_+(i\partial_+)^l\gamma_+\gamma^\perp_\mu C_j^{3/2}\left(\frac{\overleftrightarrow{\mathcal{D}}_+}{\partial_+}\right)\psi^a_+, \quad (4)$$

$${}^{G}\mathcal{O}^{T}_{\mu\nu;jl} = G^{a\perp}_{+\rho}(i\partial_+)^{l-1}\tau^\perp_{\mu\nu;\rho\sigma}C_{j-1}^{5/2}\left(\frac{\overleftrightarrow{\mathcal{D}}_+}{\partial_+}\right)G^{a\perp}_{\sigma+},$$

$$\Theta_{\mu\nu;jl} = (j+2)(j+3) \times \tau^\perp_{\mu\nu;\rho\sigma}\gamma^\perp_\rho G^{a\perp}_{\sigma+}(i\partial_+)^l P_j^{(2,1)}\left(\frac{\overleftrightarrow{\mathcal{D}}_+}{\partial_+}\right)\psi^a_+,$$

where $\tau^\perp_{\mu\nu;\rho\sigma} \equiv \frac{1}{2}(g^\perp_{\mu\rho}g^\perp_{\nu\sigma} + g^\perp_{\mu\sigma}g^\perp_{\nu\rho} - g^\perp_{\mu\nu}g^\perp_{\rho\sigma})$. They transform under $\mathcal{N}=1$ supertransformations as

$$\delta^Q\,{}^{Q}\mathcal{O}^{T}_{\mu;jl} \quad (5)$$

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$$= \frac{\sigma_j}{2j+3}\left\{\bar\zeta\gamma_\nu^\perp\Theta_{\mu\nu;j-1,l} - \bar\zeta\gamma_\nu^\perp\Theta_{\mu\nu;jl}\right\},$$

$$\delta^Q\, {}^G\mathcal{O}^T_{\mu\nu;jl}$$

$$= \frac{1}{6}\frac{\sigma_j}{2j+3}\left\{(j+3)\ \bar\zeta\Theta_{\mu\nu;j-1,l} + j\ \bar\zeta\Theta_{\mu\nu;jl}\right\},$$

$$\delta^Q\,\Theta_{\mu\nu;jl} = -6\left\{\tau^\perp_{\mu\nu;\rho\sigma}\gamma_-\zeta + i\epsilon^\perp_{\mu\nu;\rho\sigma}\gamma_-\gamma_5\zeta\right\}$$
$$\times\left\{{}^G\mathcal{O}^T_{\mu\nu;j+1,l+1} + {}^G\mathcal{O}^T_{\mu\nu;j,l+1}\right\}$$
$$-\frac{1}{2}\tau^\perp_{\mu\nu;\rho\sigma}\gamma^\perp_\rho\left\{g^\perp_{\lambda\sigma}\gamma_-\zeta + i\epsilon^\perp_{\lambda\sigma}\gamma_-\gamma_5\zeta\right\}$$
$$\times\left\{(j+1)\ {}^Q\mathcal{O}^T_{\lambda;j+1,l+1} - (j+3)\ {}^Q\mathcal{O}^T_{\lambda;j,l+1}\right\}.$$

Here $\epsilon^\perp_{\mu\nu;\rho\sigma} \equiv \frac{1}{2}(\epsilon^\perp_{\mu\rho}g^\perp_{\nu\sigma} + \epsilon^\perp_{\nu\rho}g^\perp_{\mu\sigma})$.

In order to derive the desired constraints for the anomalous dimensions, we use the commutator relation between the generators of dilatation $\mathcal{D}$ and supersymmetric transformation $\mathcal{Q}$:

$$[\mathcal{Q},\mathcal{D}]_- = \frac{i}{2}\mathcal{Q}. \tag{6}$$

The infinitesimal dilatation, $\mathcal{D}$, is equivalent to a change in the scale, and, therefore, it is governed by the renormalization group equation

$$\mu\frac{d}{d\mu}[\boldsymbol{\mathcal{O}}_{jl}] = -\sum_{k=0}^{j}\boldsymbol{\gamma}^{\mathcal{O}}_{jk}[\boldsymbol{\mathcal{O}}_{kl}], \tag{7}$$

with anomalous dimensions $\boldsymbol{\gamma}^{\mathcal{O}} = \{\lambda_{jk}, \text{for}\,\mathcal{O} = \mathcal{U},\mathcal{V};\ \omega_{jk}, \text{for}\,\mathcal{O} = \Theta\}$. Applying now the commutator (6) on the Green functions with conformal operators, $\langle[\boldsymbol{\mathcal{O}}_{jl}]\prod_i\phi(x_i)\rangle$, and making use of Eqs. (2) and (7, we derive the following constraints for eight anomalous dimensions in the chiral-even sector

$$\begin{aligned}
{}^{11}\gamma^{\mathcal{S}}_{2n+1,2m+1} &= {}^{22}\gamma^{\mathcal{P}}_{2n,2m} = \lambda_{2n,2m},\\
{}^{12}\gamma^{\mathcal{S}}_{2n+1,2m+1} &= {}^{21}\gamma^{\mathcal{P}}_{2n,2m+2} = \lambda_{2n,2m+1},\\
{}^{21}\gamma^{\mathcal{S}}_{2n+1,2m+1} &= {}^{12}\gamma^{\mathcal{P}}_{2n+2,2m} = \lambda_{2n+1,2m},\\
{}^{22}\gamma^{\mathcal{S}}_{2n+1,2m+1} &= {}^{11}\gamma^{\mathcal{P}}_{2n+2,2m+2} = \lambda_{2n+1,2m+1}.
\end{aligned} \tag{8}$$

Since we did not employ the conformal symmetry, which is obviously broken, we obtain the true constraints where both the diagonal (conformal) and the non-diagonal (anomalous) matrix elements are involved. Note that the restriction to the forward case provides six relations, however, two of them necessarily involve the non-diagonal elements of the anomalous dimension matrix of the conformal operators [7]. Omitting this non-diagonal part inevitably lead to the violation of these constraints beyond LO [6].

Analogously, using Eq. (5) we find two relations in the chiral-odd sector ($v_n \equiv n+2$, $w_n \equiv 2n+1$)

$$\begin{aligned}
&\frac{4n+5}{4m+5}\,{}^{QQ}\gamma^T_{2n+1,2m+1}\\
&\quad = \left\{\omega_{2n+1,2m+1} - \omega_{2n,2m+1}\right\},\\
&\frac{4n+5}{4m+5}\,{}^{GG}\gamma^T_{2n+1,2m+1}\\
&\quad = \left\{\frac{w_n}{w_m}\omega_{2n+1,2m+1} + 2\frac{v_n}{w_m}\omega_{2n,2m+1}\right\}.
\end{aligned} \tag{9}$$

In the forward case we find then one equation: ${}^{QQ}\gamma^T_{2n+1,2n+1} = {}^{GG}\gamma^T_{2n+1,2n+1} = \omega_{2n+1,2n+1}$. Note also that there exist two relations between the anomalous dimensions ω_{jk} which read

$$\begin{aligned}
&\omega_{2n+1,2m+1} - \omega_{2n,2m+1} = \omega_{2n,2m} - \omega_{2n+1,2m}\ ,\\
&\frac{w_n}{w_m}\omega_{2n+1,2m+1} + 2\frac{v_n}{w_m}\omega_{2n,2m+1}\\
&\quad = \frac{v_n}{v_m}\omega_{2n,2m} + \frac{1}{2}\frac{w_n}{v_m}\omega_{2n+1,2m}.
\end{aligned} \tag{10}$$

Let us address the question of use of these equations in QCD. Instead of direct calculations of otherwise different quark-gluon anomalous dimensions at LO, when conformal covariance holds true, we can obtain them from the one-loop anomalous dimensions of fermion operators in $\mathcal{N}=1$ super Yang-Mills theory ($\sigma(j) \equiv (-1)^j$):

$$\begin{aligned}
\frac{\lambda_j}{N_c} &= 2\psi(j+1) + 2\psi(j+4) - 4\psi(1)\\
&\quad - \frac{4\sigma(j)}{(j+1)(j+2)(j+3)} - 3,\\
\frac{\omega_j}{N_c} &= 2\psi(j+2) + 2\psi(j+3) - 4\psi(1) + \frac{2\sigma(j)}{j+2} - 3.
\end{aligned} \tag{11}$$

Obviously, at LO the colour structure, arising in QCD, can be restored in a unique way.

The results for the anomalous dimensions (8) imply constraints for the evolution kernels. While for the splitting functions they are known since long ago, we present them now also for the ER-BL evolution kernels. As demonstrated below, these constraints are also useful beyond LO to reconstruct conformal parts of the exclusive kernels.

Since at LO conformal covariance holds true, the ER-BL evolution kernels are given as a single sum over Gegenbauer polynomials [8] and allow to derive the following relations ($\bar{x} \equiv 1-x$):

$$\frac{d}{dy}{}^{QQ}V^i(x,y) + \frac{d}{dx}{}^{GG}V^i(x,y) = -3\,{}^{QG}V^i(x,y),$$
$${}^{GQ}V^i(x,y) = \frac{(\bar{x}x)^2}{\bar{y}y}\,{}^{QG}V^i(y,x), \qquad (12)$$

for $i = \{V, A\}$. There also exist two differential equations relating the parity odd with the parity even sectors. Thus, from the knowledge of the QQ-channel, we find all other kernels by solving six differential equations and requiring their solutions to respect conformal covariance. Thus, we predict unambiguously the results for the mixed LO kernels ${}^{AB}V^i = \theta(y-x)\,{}^{AB}F^i(x,y) - \left\{ {x\to\bar{x} \atop y\to\bar{y}} \right\}$:

$${}^{QG}F^i = 2T_F N_f \frac{x}{y^2} \begin{cases} -1+2x-4\bar{x}y, & \text{for } i=V \\ -1, & \text{for } i=A \end{cases},$$
$${}^{GQ}F^i = C_F \frac{x^2}{y} \begin{cases} -1+2y-4\bar{x}y, & \text{for } i=V \\ 1, & \text{for } i=A \end{cases}, \qquad (13)$$

which, therefore, resolve the confusion about diversity of kernels available in the literature.

Now we address a more interesting issue of applying supersymmetric constraints beyond LO. Let us first mention that in the chiral-even sector the NLO forward anomalous dimensions and the NLO non-diagonal entries, derived from conformal constraints and a one-loop calculation of the special conformal anomalies, fulfill all superconstraints (8). Of course, they do not hold in the conventional dimensional regularization (DREG) scheme but rather in a scheme which preserves supersymmetry (DRED). The transformation $\boldsymbol{\gamma}^{\rm DRED} = \boldsymbol{z}\boldsymbol{\gamma}^{\rm DREG}\boldsymbol{z}^{-1} - \beta(g)\frac{\partial}{\partial g}\boldsymbol{z}\cdot\boldsymbol{z}^{-1}$ is driven by rotation

$$\boldsymbol{z}_{jk} = \mathbb{1}\delta_{jk} + \frac{\alpha_s}{2\pi}N_c\left\{\boldsymbol{z}^{\rm D}_j\delta_{jk} + 2\boldsymbol{z}^{\rm ND}_{jk}\theta_{j-2,k}\sigma_{j-k+1}\right\},$$

with the following matrices

$$\boldsymbol{z}^{{\rm D},V}_j = \begin{pmatrix} -\frac{j(j+3)}{2(j+1)(j+2)} & \frac{12}{j(j+2)(j+3)} \\ \frac{j}{6(j+2)} & -\frac{1}{6} \end{pmatrix}, \qquad (14)$$
$$\boldsymbol{z}^{{\rm D},A}_j = \begin{pmatrix} -\frac{j(j+3)}{2(j+1)(j+2)} & \frac{12}{j(j+1)(j+2)} \\ -\frac{j}{3(j+1)(j+2)} & -\frac{1}{6} - \frac{4}{(j+1)(j+2)} \end{pmatrix},$$
$$\boldsymbol{z}^{{\rm ND},V}_{jk} = \boldsymbol{z}^{{\rm ND},A}_{jk} = \begin{pmatrix} 0 & \frac{6(2k+3)}{k(k+1)(k+2)(k+3)} \\ -\frac{(2k+3)}{6(k+1)(k+2)} & -\frac{(2k+3)(j-k)(j+k+3)}{k(k+1)(k+2)(k+3)} \end{pmatrix}.$$

Moreover, from conformal constraints we have found a simple representation for the NLO corrections to the ER-BL kernels, which reads in matrix notation ($\otimes \equiv \int_0^1$)

$$\boldsymbol{V}^{(1)} = -\dot{\boldsymbol{V}}\otimes\left(\boldsymbol{V}^{(0)} + \frac{\beta_0}{2}\mathbb{1}\right) - \Big[\boldsymbol{g}\underset{,}{\otimes}\boldsymbol{V}^{(0)}\Big]_{-} + \boldsymbol{D} + \boldsymbol{G},$$

where the $\boldsymbol{g}$, $\dot{\boldsymbol{V}}$, and $\boldsymbol{V}^{(0)}$ are known [8]. The diagonal kernel $\boldsymbol{G}$ is defined by contributions coming from the two-loop crossed ladder diagram (in the light-cone gauge) and is known in the QQ-channel. Since this diagram does not contain divergent subgraphs, it respects supersymmetry and tree level conformal covariance. From this information, similar to LO case, one can construct the $\boldsymbol{G}$ kernels in all other channels. The remaining diagonal piece, $\boldsymbol{D}$, can be represented as convolutions of simple kernels. Proceeding along this line we reconstruct the whole ER-BL kernel in NLO without explicit two-loop calculations [9].

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