Positivity Constraints for Parton Distributions in QCD

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The status of positivity constraints in QCD and their stability against evolution in leading and next-to-leading orders, with the particular emphasis on the role of quantum anomalies, is discussed.

The positivity of density matrix is just the positivity of its eigenvalues. The positivity in QCD is pronounced in the framework of factorization, containing the nonperturbative ingredient – parton distributions. They may be considered (at leading order) as a density matrices of partons in hadrons. The obvious reason for positivity is the positivity of physical cross-sections [1]. The special role are played by partonic subprocesses with a maximal analyzing power. They are emerging due to the helicity and angular momentum conservation and represented, say, by the Drell-Yan process, where helicities of quark and antiquark should be the opposite, or scalar particle production in gluon-gluon fusion, requiring their helicities to be equal. The Higgs particle was suggested as a possible candidate for Gedankenexperiment [1]. At the same time, the production of the MSSM Higgs at RHIC polarized collider should occur at $x \sim 0.3$ where polarization of gluons is supposed to be maximal, so that the resulting asymmetry ~ 0.1 might provide an extra constraint for the Higgs signal extraction. This interesting numerical coincidence is unfortunately undermined by the poor statistics.

The observed effects do not exhaust the possible constraints. The non-diagonal (in helicity) elements of density matrix are also constrained, the well-known example being provided by the case of Soffer inequality [2] for quark transversity distribution:

$$|h_1(x)| \leq q_+(x) = \frac{1}{2}[q(x) + \Delta q(x)]. \tag{1}$$

The similar inequality [3] in the case of gluons is relating the contribution of twist 2 ($G_+(x) = (G(x) + \Delta G(x)/2)$, 3 ($\Delta G_T(x)$) and 4($G_L(x)$)):

$$|\Delta G_T(x)| \leq \sqrt{1/2 G_+(x) G_L(x)}\ . \tag{2}$$

It is most instructive to use this relation to estimate G_L (being the interesting new ingredient of nucleon structure [4]) from below:

$$G_L(x) \geq 2[\Delta G_T(x)]^2/G_+(x). \tag{3}$$

This bound is analogous to well-known condition established long time ago by Doncel and de Rafael [5], written in the form

$$|A_2| \leq \sqrt{R}\ , \tag{4}$$

where A_2 is the usual transverse asymmetry and $R = \sigma_L/\sigma_T$ is the standard ratio in DIS. Moreover, the simple derivation similar to the gluonic case is resulting in stronger inequality for DIS, [6]

$$|A_2| \leq \sqrt{R(1+A_1)/2}\ , \tag{5}$$

which is especially useful at small x (small A_1) and for neutron target (negative A_1). The bound for the twist-2 gluonic transversity [7], corresponding to the flip of the helicity by two units, is even simpler:

$$|\Delta(x)| \leq G_+(x). \tag{6}$$

The stability of positivity constraints against QCD evolution at leading order is provided by the kinetic interpretation of the latter [8,9]. One may consider the generic evolution equation as a gain-loss equation, for which

i) all the gain terms are positive, while

ii) the loss terms are local in all the quantum numbers, i.e., proportional to the distribution with the given values of momentum fraction, helicity etc. To prove the given constraint, one should construct the quantities, being positive at reference scale and check the validity of i) and ii). For Soffer inequality such quantities are just

$$
\begin{aligned}
Q_+(x) &= q_+(x) + h_1(x),\\
Q_-(x) &= q_+(x) - h_1(x).
\end{aligned} \tag{7}
$$

Their evolution is governed by the kernels,

$$
\begin{aligned}
P^Q_{++}(z) &\equiv \frac{P^{(0)}_{qq}(z) + P^{(0)}_h(z)}{2}\\
&= \frac{C_F}{2}[\frac{(1+z)^2}{(1-z)_+} + 3\delta(1-z)],\\
P^Q_{+-}(z) &\equiv \frac{P^{(0)}_{qq}(z) - P^{(0)}_h(z)}{2}\\
&= \frac{C_F}{2}(1-z).
\end{aligned} \tag{8}
$$

The regular (gain) parts of them are positive, while the singular (loss) term is appearing only in the diagonal kernel. The crucial property is the equality of the singular terms in the evolution kernels P_{qq} and P_h (note that the first is related to the current conservation), and the fact that the first is larger, expressed by the expression for P_{+-}. The similar equation for the evolution of positivity bound for twist 2 gluonic transversity (6) is just

$$
\begin{aligned}
P^G_{+-}(z) &\equiv \frac{P^{++}_{GG}(z) - P_\Delta(z)}{2}\\
&= \frac{C_2(G)}{2} \cdot \frac{(1-z)(1+z)^2}{z},
\end{aligned} \tag{9}
$$

so that quantities $G_{++}(x) = G_+(x) + \Delta(x), G_{+-}(x) = G_+(x) - \Delta(x)$ remain positive while evolving. Here, again the singular terms are equal. Note that for P^{++}_{GG} it is proportional to the beta-function, which is related to the appearance of the gluonic axial anomaly. Positivity naturally explain, why the same term should appear in the transversity evolution. As soon as the equal (gain) expressions for the quark and gluon transversity are due to supersymmetry [10], the positivity is relating (non-singlet) current conservation, (singlet) axial anomaly and supersymmetry!

The relations of positivity and anomalies are starting to be even more profound, when one is passing to NLO effects. The contribution of the axial anomaly to the spin-dependent parton distributions may be accommodated by performing the finite renormalization transformation of each quark density, putting all the short distance contribution to the coefficient function [11]:

$$
\begin{aligned}
\Delta q_i(x,Q^2) &= \Delta \tilde{q}_i(x,Q^2) + \\
K(x) \otimes \Delta G(x,Q^2), &\quad K(x) = \frac{\alpha_s}{\pi}(1-x),
\end{aligned} \tag{10}
$$

where the convolution is defined as

$$A(x) \otimes B(x) = \int_0^1 dy \int_0^1 dz\ \delta(x-yz)A(y)B(z).$$

As soon as the densities $\tilde{q}$ are related to the matrix elements of the quark axial current and enter to the observable structure function g_1, they should definitely obey the positivity constraints:

$$|\Delta \tilde{q}_i(x,Q^2)| \leq \tilde{q}_i(x,Q^2). \tag{11}$$

At the same time, the transformation (10) is leading to the larger (in the case of positive gluon polarization) polarized density, which, generally speaking, may violate the positivity constraint. One should however note, that in order to keep all the short-distance contributions in the coefficient function, the similar finite transformation, related (albeit indirectly) to the trace anomaly should be performed also for spin-averaged distributions [4,12]:

$$q_i(x,Q^2) = \tilde{q}_i(x,Q^2) + K(x) \otimes G(x,Q^2), \tag{12}$$

with the *same* function K. The universality of this function, allows to write the transformations for separate helicities:

$$q^i_\pm(x,Q^2) = \tilde{q}^i_\pm(x,Q^2) + K(x) \otimes G_\pm(x,Q^2), \tag{13}$$

so that positivity of all the physical densities and function K guarantees the positivity of the transformed densities. Physically this means that the simultaneous redefinition of the observed polarized and unpolarized cross-sections by, say, jet momentum cutoff, is preserving the positivity. At the same time, the transformation (12) is violating the momentum conservation (note at this point that momentum conservation therefore requires the absorption of some short-distance contribution to the parton density rather than to the coefficient function), so that partonic picture is again violated. The resulting situation is the complicated interplay of the anomalies and positivity. The anomalies are known to appear in pairs, so that one may preserve one of the classical symmetries, violating the other. The removal of the axial anomaly from the spin-dependent quark distribution is actually the rather unusual choice of the chiral symmetry to be preserved, while the gauge one is violated (at large distances, described by the parton distribution, but not in the full factorized cross-section, of course). The reason is that quark spin is starting to be conserved, allowing for a contact with the low energy description. At the same time, the violation of gauge symmetry is not leading to the observable consequences, as the first moment of (singlet) quark distribution is anyway infinite. The consideration of non-local generalization of axial anomaly [11], resulting in (10), changes the situation. It forces, due to positivity, to make the choice in another anomaly pair (which for local anomalies was independent) in favor of dilatation invariance, violating the translational one. The latter, in turn, is resulting in observable, in principle, consequences for parton distribution like momentum non-conservation. Finally, one meets a sort of the uncertainty principle, when it is, generally speaking, impossible to preserve simultaneously all three ingredients of partonic description, namely, conserved quark spin, conserved moment and positivity. Such a role of the anomalies is not surprising, as they are remnants of the quantum theory divergences, which, in principle, may spoil the standard proofs of positivity. This is especially pronounced for QCD because of the confinement property, when intermediate states [2,13,14] providing the positivity, are colored and therefore non-physical. One may combine the two sorts of positivity constraints (for diagonal and non-diagonal elements of density matrices), by considering the unphysical [13], but color-neutral currents, so that one is not confined to the observed asymmetries, while using the color neutral intermediate states. The successful proof of the Soffer inequality [13] is leading to an assumption [14] that the use of colored states is just the formal recipes, substituting the use of corresponding auxiliary currents.

However, the further theoretical and experimental studies of the inequalities for parton distribution is highly desirable, especially of the non-diagonal ones, which are sensitive to the details of underlying field theory. The uncertainty principle due to the interplay of the anomalies of positivity is of special interest. Unlike in the case of the standard uncertainty principle, we cannot at the moment state, that one of the three ingredients (positivity, momentum and spin conservation) must be violated, but rather that it may be violated. The theoretical and experimental check of such a violation seems very interesting.

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