Steps towards NNLO QCD calculations: collinear factorization at $\mathcal{O}(\alpha_S^2)$

M. Grazzini*[a]

[a]Institute for Theoretical Physics, ETH-Hönggerberg
CH 8093 Zürich, Switzerland

I consider the singular behaviour of tree-level QCD amplitudes when the momenta of three partons become simultaneously parallel and I discuss the universal factorization formula that controls the singularities of the multiparton matrix elements in this collinear limit.

1. INTRODUCTION

The properties of QCD in the soft and collinear limits play an important role in doing higher order calculations. The singularities arising in these limits prevent a straightforward integration of matrix elements to obtain physical cross sections. However, as far as NLO calculations are concerned, the universality property of these singularities provided a method to solve these problems. The singular behaviour at this order is well known and is given in terms of $\mathcal{O}(\alpha_S)$ factorization formulae for tree level [1] and one loop [2] amplitudes in the soft and collinear limits.

As a first step towards NNLO calculations it would be of great importance to extend this knowledge at $\mathcal{O}(\alpha_S^2)$. At this order one has to consider two loop corrections, one loop corrections with real emission and double real emission. The singularities in two loop corrections have been discussed in Ref. [3]. The properties of one-loop amplitudes in the soft and collinear limits have been recently obtained at all orders in ϵ in Ref. [4]. The singular behaviour of tree-level amplitudes has been studied in Ref. [5,6]. Here I will concentrate on the collinear behaviour.

2. NOTATION AND KINEMATICS

We consider a generic scattering process involving final-state QCD partons with momenta $p_1, p_2, \ldots$ Non-QCD partons $(\gamma^*, Z^0, W^\pm, \ldots)$, carrying a total momentum Q, are always understood. The corresponding tree-level matrix element is denoted by

$$\mathcal{M}_{a_1,a_2,\ldots}^{c_1,c_2,\ldots;s_1,s_2,\cdots}(p_1,p_2,\ldots) \ , \tag{1}$$

where $\{c_i\}$, $\{s_i\}$ and $\{a_i\}$ are respectively colour, spin and flavour indices. The matrix element squared summed over final-state colours and spins will be denoted by $|\mathcal{M}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)|^2$. If the sum over the spin polarizations of the parton a_1 is not carried out, we define the following 'spin-polarization tensor'

$$\mathcal{T}_{a_1,\ldots}^{s_1 s_1'}(p_1,\ldots) \equiv \sum_{\text{spins}\neq s_1,s_1'} \sum_{\text{colours}} \mathcal{M}_{a_1,a_2,\ldots}^{c_1,c_2,\ldots;s_1,s_2,\ldots} \times \left[\mathcal{M}_{a_1,a_2,\ldots}^{c_1,c_2,\ldots;s_1',s_2,\ldots}\right]^\dagger \ . \tag{2}$$

In the evaluation of the matrix element, we use conventional dimensional regularization ($d = 4-2\epsilon$ space-time dimensions, two helicity states for massless quarks and $d-2$ helicity states for gluons).

The relevant collinear limit at $\mathcal{O}(\alpha_S)$ is approached when the momenta of two partons, say p_1 and p_2, become parallel. This limit is defined by setting:

$$\begin{aligned} p_1^\mu &= z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p\cdot n} \\ p_2^\mu &= (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p\cdot n} \end{aligned} \tag{3}$$

and sending $k_\perp \to 0$. In Eq. (3) the light-like ($p^2=0$) vector p^μ denotes the collinear direction, while n^μ is an auxiliary light-like vector, which is necessary to specify the transverse component $k_\perp$ ($k_\perp^2 < 0$) ($k_\perp \cdot p = k_\perp \cdot n = 0$) or, equivalently,

*Work supported by the Swiss National Foundation.

how the collinear direction is approached. In the small-$k_\perp$ limit (i.e. neglecting terms that are less singular than $1/k_\perp^2$), the square of the matrix element in Eq. (1) fulfils the following factorization formula

$$|\mathcal{M}_{a_1,a_2,\dots}(p_1,p_2,\dots)|^2 \simeq \frac{2}{s_{12}}\,4\pi\mu^{2\epsilon}\alpha_S \times \mathcal{T}^{ss'}_{a,\dots}(p,\dots)\,\hat{P}^{ss'}_{a_1a_2}(z,k_\perp;\epsilon)\ , \qquad (4)$$

where μ is the dimensional-regularization scale. The spin-polarization tensor $\mathcal{T}^{ss'}_{a,\dots}(p,\dots)$ is obtained by replacing the partons a_1 and a_2 on the right-hand side of Eq. (2) with a single parton denoted by a. This parton carries the quantum numbers of the pair a_1+a_2 in the collinear limit.

The kernel $\hat{P}_{a_1a_2}$ in Eq. (4) is the d-dimensional Altarelli–Parisi (AP) splitting function [7]. It depends on the momentum fraction z but also on the transverse momentum $k_\perp$ and on the helicity of the parton a in the matrix element $\mathcal{M}^{c,\dots;s,\dots}_{a,\dots}(p,\dots)$. More precisely, $\hat{P}_{a_1a_2}$ is in general a matrix acting on the spin indices s,s' of the parton a in the spin-polarization tensor $\mathcal{T}^{ss'}_{a,\dots}(p,\dots)$. Because of these *spin correlations*, the spin-average square of the matrix element $\mathcal{M}^{c,\dots;s,\dots}_{a,\dots}(p,\dots)$ cannot be simply factorized on the right-hand side of Eq. (4).

3. COLLINEAR FACTORIZATION AT $\mathcal{O}(\alpha_S^2)$

In the following we are interested in the collinear limit at $\mathcal{O}(\alpha_S^2)$. At this order there are two different collinear limits to be considered [5]. The first is when *two pairs* of parton momenta become independently parallel. In this case the factorization formula is obtained by a simple iteration of Eq. (4). In the second case three parton momenta can simultaneously become parallel. Denoting these momenta by p_1,p_2 and p_3, their most general parametrization is

$$p_i^\mu = x_i p^\mu + k_{\perp i}^\mu - \frac{k_{\perp i}^2}{x_i}\frac{n^\mu}{2p\cdot n}\ , \qquad i=1,2,3\ , \quad (5)$$

where the notations are the same as in Eq. (3). Note that no constraint is imposed on the longitudinal and transverse variables x_i and $k_{\perp i}$.

It can be shown [6] that in the triple-collinear limit ($k_{\perp i}\to 0$) the matrix element squared still fulfils a factorization formula analogous to Eq. (4), namely

$$|\mathcal{M}_{a_1,a_2,a_3,\dots}(p_1,p_2,p_3,\dots)|^2 \simeq \frac{4}{s_{123}^2}(4\pi\mu^{2\epsilon}\alpha_S)^2 \times \mathcal{T}^{ss'}_{a,\dots}(xp,\dots)\,\hat{P}^{ss'}_{a_1a_2a_3}\ . \qquad (6)$$

As in Eq. (4), the spin-polarization tensor $\mathcal{T}^{ss'}_{a,\dots}(xp,\dots)$ is obtained by replacing the partons a_1, a_2 and a_3 with a single parent parton, whose flavour a is determined by flavour conservation in the splitting process $a\to a_1+a_2+a_3$.

The three-parton splitting functions $\hat{P}_{a_1a_2a_3}$ generalize the AP splitting functions in Eq. (4). The spin correlations produced by the collinear splitting are taken into account in a universal way, i.e. independently of the specific matrix element on the right-hand side of Eq. (6). Besides depending on the spin of the parent parton, the functions $\hat{P}_{a_1a_2a_3}$ depend on the momenta p_1,p_2,p_3. However, due to their invariance under longitudinal boosts along the collinear direction, the splitting functions can depend in a non-trivial way only on the sub-energy ratios s_{ij}/s_{123} and on the following longitudinal and transverse variables:

$$z_i = \frac{x_i}{x} \qquad \widetilde{k}_i^\mu = k_{\perp i}^\mu - \frac{x_i}{x}\sum_{j=1}^3 k_{\perp j}^\mu\ , \qquad (7)$$

where $x=\sum_{i=1}^3 x_i$.

The method used to derive these results exploits the fact that the interference of Feynman diagrams obtained by squaring the amplitude $\mathcal{M}(p_1,\dots,p_m,\dots)$ is collinearly suppressed when computed in a physical gauge. Thus, in the evaluation of the triple collinear limit we can limit ourselves to consider the diagrams in Fig. 1. Details on the method and on our calculation are given in Ref. [6]. The basic observation is that if we rescale the transverse momenta as $k_{\perp i}\to\lambda k_{\perp i}$ the matrix element squared has the singular behaviour

$$|\mathcal{M}_{a_1,a_2,a_3,\dots}(p_1,p_2,p_3,\dots)|^2 \sim 1/\lambda^4 + \dots \qquad (8)$$

where dots stand for less singular contributions when $\lambda\to 0$. To extract the singular behaviour one has to put on shell the parent parton leg in Fig. 1. Since $p_1+p_2+p_3 = xp+\mathcal{O}(k_\perp)$, in order to do this we have to neglect $\mathcal{O}(k_\perp)$ terms

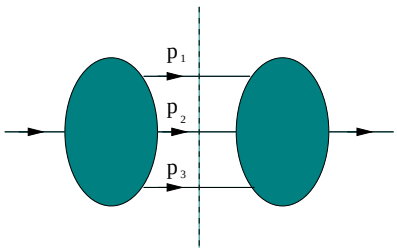

Figure 1. Triple collinear limit: dominant diagrams in a physical gauge.

from the amplitude which may affect in a non-universal way the singular behaviour. But since we are interested in the *most singular* behaviour we can safely put $p_1 + p_2 + p_3 \to xp$ and reconstruct the gauge invariant spin polarization tensor in (6). Thus $\hat{P}_{a_1 a_2 a_3}$ can be computed by evaluating the process-independent diagrams in Fig.1 in the collinear limit. Our explicit results are presented in [6]. We find that when the parent parton is a fermion, spin-correlations are absent. This is analogous to what happens at $\mathcal{O}(\alpha_S)$ and it is a consequence of helicity conservation in the quark vector coupling. On the contrary, in the case in which the parent parton is a gluon spin correlations are highly non trivial.

A check of the calculation is provided by the strong-ordered limit. In this limit the three partons become collinear sequentially and $\hat{P}^{ss'}_{a_1 a_2 a_3}$ factorize in the product of two AP splitting functions. A further non-trivial check of the calculation is provided by supersymmetry. As for AP splitting functions, we find that $\hat{P}_{a_1 a_2 a_3}$ obey a $N = 1$ SUSY identity in the limit $\epsilon \to 0$ and when $C_F = C_A = 2T_R$:

$$\left[\hat{P}_{\bar{q}_1 q_2 q_3} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\right] + \left[\hat{P}_{q_3 g_1 g_2} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right] = P_{g_1 g_2 g_3} + \left[\hat{P}_{g_3 \bar{q}_1 q_2} + 5 \text{ perm.}\right] \quad (9)$$

which expresses the fact that the total quark (actually, gluino) and gluon decay probability are the same.

The $\mathcal{O}(\alpha_S^2)$-collinear limit of tree-level QCD amplitudes has been independently considered by Campbell and Glover [5]. Taking for granted Eq. (6) they neglected spin correlations and computed only the spin-averaged splitting functions by performing the limit of known amplitudes. We have compared our results with those of Ref. [5] and found complete agreement.

4. SUMMARY

I have discussed the three-parton collinear limit of tree-level QCD amplitudes. In this limit the singular behaviour of the matrix element squared is controlled by process-independent splitting functions, which are analogous to the Altarelli–Parisi splitting functions.

These splitting functions are one of the necessary ingredients to extend QCD predictions at higher perturbative orders. In particular, they will be relevant to set up general methods to compute jet cross sections at NNLO. The knowledge of the collinear splitting functions, when combined with a consistent analysis of soft-gluon coherence properties, could also be used to improve the logarithmic accuracy of parton showers available at present for Monte Carlo event generators.

The results presented in this talk have been obtained in collaboration with S. Catani.

REFERENCES

1. M.L. Mangano and S.J. Parke, Phys. Rep. **200** (1991) 301 and references therein.
2. Z. Bern, L. Dixon and D.A. Kosower, Annu. Rev. Nucl. Part. Sci. **46** (1996) 109 and references therein; W.T. Giele and E.W.N. Glover, Phys. Rev. **D46** (1992) 1980; Z. Kunszt, A. Signer and Z. Trócsányi, Nucl. Phys. B420 (1994) 550.
3. S. Catani, Phys. Lett. **B427** (1998) 161.
4. D.A. Kosower and P. Uwer, hep-ph/9903515; Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, hep-ph/9903516
5. J.M. Campbell and E.W.N. Glover, Nucl. Phys. **B527** (1998) 264.
6. S. Catani and M. Grazzini, Phys. Lett. **B446** (1999) 143 and preprint in preparation.
7. G. Altarelli and G. Parisi, Nucl. Phys. **B126** (1977) 298.