Running Coupling BFKL Equation and Deep Inelastic Scattering.

R.S. Thorne[a]

[a]Jesus College and Theoretical Physics, University of Oxford,
Oxford, Oxon, OX1 3DW, U.K.

I examine the form of the solution of the BFKL equation with running coupling relevant for deep inelastic scattering. The evolution of structure functions is precisely determined and well described by an effective coupling $\sim 1/(\beta_0(\ln(Q^2/\Lambda^2) + 3.6(\alpha_s(Q^2)\ln(1/x))^{1/2})$ (until extremely small x). Corrections to the LO equation are relatively small, and the perturbative expansion is stable. Comparison to data via a global fit is very successful.

Small x physics has recently been a particular area of both experimental and theoretical interest. The first data on structure functions at very small values of x (down to $x = 10^{-5}$) obtained by HERA [1,2] have themselves been enough to make this an active topic. However, the fact that the splitting functions and coefficient functions required for the construction of structure functions have expansions containing terms $\alpha_s^n(\mu^2)\ln^{n-1}(1/x)$ has added extra impetus, implying that at small x one may have to account for leading terms in $\ln(1/x)$ at high orders in α_s, rather than just expand naively in powers of α_s.

1. THE RUNNING COUPLING BFKL EQUATION.

The BFKL equation [3] makes it possible to take account of the most leading $\ln(1/x)$ terms at each order in α_s, since it is the four-point gluon Green's function containing all the information about the most important small-x behaviour. At leading order, with fixed α_s, the BFKL equation is scale invariant, and has simple eigenfunction $\sim (k^2)^\gamma$, with eigenvalues $\chi^0(\gamma)$, where $\chi^0(\gamma)$ is the Mellin transformation of the kernel. Transforming back to x and k^2 space the leading small x behaviour is driven by the saddle-point at $\gamma = 1/2$, leading to small-x behaviour of the form $x^{-\lambda}$, where $\lambda = \bar{\alpha}_s\chi^0(1/2) = \bar{\alpha}_s 4\ln(2)$ ($\bar{\alpha}_s = 3\alpha_s/\pi$), i.e. if $\alpha_s \approx 0.2$, $\lambda \sim 0.5$. However, this result is potentially subject to large corrections. In particular the equation predicts large diffusion – within the gluon Green's function the average virtuality $\sim k^2$ but the spread of important values has a width $\propto (\alpha_s \ln(1/x))^{1/2}$. Hence, there is considerable influence from both the infrared and ultraviolet regions of virtuality at small x. If the coupling is allowed to run this will clearly be very important.

Recently the NLO BFKL equation became available [4,5]. Not only does this provide information about the scale-invariant NLO corrections, but the running of the QCD coupling now becomes impossible to ignore – the NLO part of the kernel contains a term $\propto \beta_0\alpha_s^2(\mu^2)\ln(k^2/\mu^2)$, where μ is the renormalization scale, and this is absorbed into the renormalization group improved coupling, $\alpha_s(\mu^2) - \beta_0\alpha_s^2(\mu^2)\ln(k^2/\mu^2) \rightarrow \alpha_s(k^2)$. Having done this, a simple way to consider the effect of the NLO corrections to the BFKL kernel is to consider its action on the LO eigenfunctions $(k^2)^\gamma$. This results in a NLO *eigenvalue* of the form $\bar{\alpha}_s(k^2)\chi^0(\gamma) - \bar{\alpha}_s^2(k^2)\chi^1(\gamma)$, which for $\gamma = 1/2$ leads to an intercept of $2.8\bar{\alpha}_s(k^2)(1 - 6.5\bar{\alpha}_s(k^2))$. This is clearly a disaster, implying a very unconvergent expansion for the intercept.

However, this simple approach requires modification. The *eigenvalue* is not a real eigenvalue since it depends on k^2 – the running coupling has broken scale invariance, changing the whole form of the BFKL equation. Even for the LO BFKL equation, the $\ln(k^2/\Lambda^2)$ associated with a running coupling turns the γ-space BFKL equation into a differential equation. This can formally be solved, and then inverted back to k^2-space. At leading twist, the solution factorizes into a k^2-dependent part $g(k^2, N)$ and a Q_0^2-dependent part [6]. The

latter is ambiguous since the γ-dependent integrand has a cut along the contour of integration of the inverse transformation. This is because the growth of the coupling in the infrared coupled with the strong infrared diffusion leads to an infrared divergence in the BFKL equation, and hence a renormalon contribution. Therefore the running coupling equation has no prediction for the input for the gluon, this being sensitive to nonperturbative physics. Conversely, $g(k^2, N)$ is completely well-defined, being insensitive to infrared diffusion. However, it is influenced the ultraviolet diffusion, and hence by scales greater than k^2, where the coupling is weak.

Calculating $dg(k^2, N)/d\ln(k^2)$ one extracts an unambiguous anomalous dimension $\Gamma(N, \alpha_s(k^2))$ governing the evolution of the gluon. Considering for the moment the use of just the LO kernel, $\Gamma(N, \alpha_s(k^2))$ turns out to be the usual LO BFKL anomalous dimension $\gamma^0_{gg}(\bar{\alpha}_s/N)$, with α_s evaluated at k^2, plus an infinite series of corrections which are a power series in $\beta_0\alpha_s(k^2)$ compared to the LO. Transforming to x space one finds that each term in the series behaves like $x^{-\lambda(k^2)}$ as $x \to 0$, but with accompanying powers of $\alpha_s(k^2)\ln(1/x)$ which grow as the power of $\beta_0\alpha_s(k^2)$ grows. Hence a resummation of running coupling dependent terms is necessary. The series is too complicated to sum exactly, so some prescription must be used. I choose the BLM prescription [7], which adjusts the scale of the coupling in the LO expression to $\tilde{k}^2$ so that using $\alpha_s^{eff} \equiv \alpha_s(\tilde{k}^2) = \alpha_s(k^2) - \beta_0 \ln(\tilde{k}^2/k^2)\alpha_s^2(k^2) + \cdots$ in the LO splitting function one generates precisely the NLO β_0-dependent correction to the splitting function. Choosing $\tilde{k}^2$ in this manner one can find an exact expression for the scale choice in the effective coupling, and in the small x limit it is $\ln(\tilde{k}^2/\Lambda^2) = \ln(k^2/\Lambda^2) + 3.6(\alpha_s(k^2)\ln(1/x))^{1/2}$, (see [8] for details), which is entirely consistent with influence from the diffusion into the ultraviolet. The effective coupling is shown in fig. 1.

It is possible to check whether this prescription is sensible. If the summation of the complete series is generated by this effective scale then the whole series is given by using $\alpha_s(\tilde{k}^2) = \alpha_s(k^2) - \beta_0 \ln(\tilde{k}^2/k^2)\alpha_s^2(k^2) + \cdots$ in the LO splitting function. This can be checked explicitly for the $\mathcal{O}((\beta_0\alpha_s(k^2))^2)$ term. The agreement between the term generated in this manner and that calculated explicitly is excellent until very small x indeed, i.e., the $\mathcal{O}((\beta_0\alpha_s(k^2))^2)$ term leads to a modification to the scale of $\ln(\tilde{k}^2/\Lambda^2) = \ln(k^2/\Lambda^2) + 3.6(\alpha_s(k^2)\ln(1/x))^{1/2} - 1.2\beta_0\alpha_s(k^2)(\alpha_s(k^2)\ln(1/x))^2$. Checks at higher order in $\beta_0\alpha_s(k^2)$ give similar results.

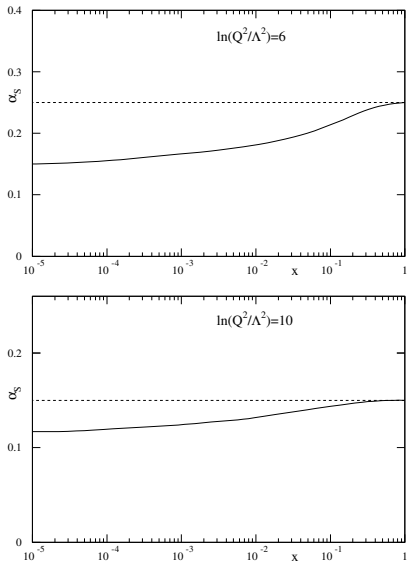

Figure 1. α_s^{eff} as a function of x compared to the constant values for two choices of $\ln(Q^2/\Lambda^2)$.

One can also solve for $\Gamma(N, \alpha_s(k^2))$ numerically and compare with the transformed splitting function using the effective scale. Due to a zero in $g(k^2, N)$ (first noticed by Ciafaloni, Colferai and Salam), the anomalous dimension has a leading pole for $N > 0$, which for $\alpha_s(k^2) \approx 0.25$ is at $N \approx \alpha_s(k^2)$, leading to a splitting function $\sim x^{-\alpha_s(k^2)}$. However, the numerical $\Gamma(N, \alpha_s(k^2))$ is in very good agreement with the transformation of $p^0_{gg}(x, \alpha_s^{eff})$ until one gets extremely near the pole – the replacement $\ln(\tilde{k}^2/\Lambda^2) = \ln(k^2/\Lambda^2) + 3.6(\alpha_s(k^2)\ln(1/x))^{1/2}$ is extremely effective until very small x indeed. At $x \sim 10^{-8}$ the corrections from $\mathcal{O}((\beta_0\alpha_s(k^2))^2)$ and beyond act to stop the decrease of the coupling at even smaller x and freeze it at $\alpha_s^{eff,0}(k^2) \sim 0.4\alpha_s(k^2)$.

The validity of using the effective coupling can also be checked by solving the NLO BFKL equa-

tion with running coupling [9]. Now, as well as β_0-dependent corrections to the naive LO result there is also an $\mathcal{O}(\alpha_s)$ correction not involved with the running of the coupling, and ignoring β_0-dependent effects the NLO correction to the intercept of the splitting function is indeed $-6.5\bar{\alpha}_s(k^2)$ the LO. However, one may use the same type of prescription for fixing the scale in the NLO contribution to the splitting function as at LO, finding that, although it is not necessary, the α_s^{eff} appropriate for this NLO correction is the same as at LO [8]. This is both for the $\ln(\tilde{k}^2/\Lambda^2) = \ln(k^2/\Lambda^2) + 3.6(\alpha_s(k^2)\ln(1/x))^{1/2}$ behaviour for $x > 10^{-8}$ and the freezing below this – the $x \to 0$ NLO corrected splitting function does not behave like the naive $x^{-2.8\bar{\alpha}_s(k^2)(1-6.5\bar{\alpha}_s(k^2))}$, but like $x^{-\bar{\alpha}_s(k^2)(1-2.3\bar{\alpha}_s(k^2))}$ for $\alpha_s(k^2) \approx 0.25$. Hence, the perturbative expansion is more stable. Exact calculations at finite x show that the NLO correction to the splitting function leads to fairly small corrections to the evolution [8]. I note that other authors have considered partial resummations of the BFKL kernel which provide improved stability besides that associated with the running coupling [10,9]. These are necessary for single-scale processes, but I feel they are less important for structure functions than running coupling effects.

2. PHENOMENOLOGY.

In order to apply these results to a study of structure functions it is necessary to calculate the small x expansions for the physical splitting functions [11] which give the evolution of $F_2(x,Q^2)$ and $F_L(x,Q^2)$ in terms of each other, in order to avoid factorization scheme ambiguities. This results in α_s^{eff} of exactly the same form as above. In order to compare with data one can then combine these small x expansions with the normal expansions in powers of α_s in the manner in [12]. At present there is only sufficient information to work at LO in this combined expansion. However, it is possible to use effective scales at large x using similar considerations as above, finding a coupling which grows as $x \to 1$.

A global analysis of structure function data can be performed (with constraints applied for other data, e.g. prompt photon, high E_T jets), and the results are very good. The χ^2 for 1330 data points is 1339 compared to 1511 for the standard NLO in α_s MRST fit [13]. Not only is the fit improved but, whereas the input gluon (and therefore input $F_L(x,Q^2)$) in the conventional approach is valencelike at $Q^2 \sim 1\text{GeV}^2$, $F_L(x,Q^2)$ in this approach is the same general shape as $F_2(x,Q^2)$ at this low Q^2. The prediction for $F_L(x,Q^2)$ in the HERA range is smaller than that using the conventional approach, being similar to [12].

Hence, I believe that the correct way to take account of the $\ln(1/x)$ terms in the calculation of structure functions is to use the combined expansion for physical splitting functions, and the α^{eff} given by resumming β_0-dependent terms. This is preferred by current $F_2(x,Q^2)$ data, and predictions for other quantities are different from the conventional approach.

REFERENCES

1. H1 collaboration, Nucl. Phys. **B470** (1996) 3; Nucl. Phys. **B497** (1997) 3.
2. ZEUS collaboration, Zeit. Phys. **C69** (1996) 607; Zeit. Phys. **C72** (1996) 399.
3. L.N. Lipatov, Sov. Jour. Nucl. Phys. **23** (1976) 338; E.A. Kuraev, L.N. Lipatov, and V.S. Fadin, Sov. Jour. JETP **45** (1977) 199; Ya. Balitskii and L.N. Lipatov, Sov. Jour. Nucl. Phys. **28** (1978) 6.
4. V.S Fadin and L.N. Lipatov, Phys. Lett. **B429** (1998) 127.
5. G. Camici and M. Ciafaloni, Phys. Lett. **B430** (1998) 349.
6. J. Kwiecinski, Zeit. Phys. **C29** (1985) 561.
7. S.J. Brodsky, G.P. Lepage and P.B. MacKenzie, Phys. Rev. **D28**(1993) 228.
8. R.S. Thorne, Phys. Rev. **D** to appear, `hep-ph/9901331`.
9. M. Ciafaloni and D. Colferai, Phys. Lett. **B452** (1999) 372.
10. G. Salam, JHEP 9807 (1998) 19.
11. S. Catani, Proceedings of DIS 96, Rome, April 1996, p. 165 `hep-ph/9608310`; Zeit. Phys. **C75** (1997) 665.
12. R.S. Thorne, Phys. Lett. **B392** (1997) 463; Nucl. Phys. **B512** (1998) 323.
13. A.D. Martin *et. al.*, Eur. Phys. J. **C4** (1998) 463.