Harmonic Sums and Mellin Transforms

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The finite and infinite harmonic sums form the general basis for the Mellin transforms of all individual functions $f_i(x)$ describing inclusive quantities such as coefficient and splitting functions which emerge in massless field theories. We discuss the mathematical structure of these quantities.

1. INTRODUCTION

The splitting and coefficient functions in massless QED and QCD can be evaluated in terms of Nielsen-integrals [1]

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dz}{z} \log^{n-1}(z) \log^p(1-zx) \quad (1)$$

and their Mellin convolutions

$$f_1(x) \otimes f_2(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) f_1(x_1) f_2(x_2) \quad (2)$$

up to the level of the second order in the coupling constant. Alternatively to the x–space representation one may consider the Mellin transform [2] of these expressions

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x) \ . \quad (3)$$

Here N refers either to the even or odd positive integers depending on the quantity being studied. The Mellin transforms can be expressed in terms of polynomials of finite multiple (alternating or non–alternating) harmonic sums

$$S_{k_1,\ldots,k_m}(N) = \prod_{i=1}^{m} \sum_{n_i=1}^{n_{i-1}} \frac{(\mathrm{sign}(k_i))^{n_i}}{n_i^{|k_i|}}, \quad (4)$$

with $n_0 \equiv N$. The finite harmonic sums form classes which are given by their transcendentality $t = \sum_{l=1}^{m} |k_l|$. This term was chosen to characterize the behavior of these sums for $N \to \infty$. In this case the values of the sums, if existing, are described by a transcendental number of rank t, i.e. $\log 2, \zeta(2), \zeta(3)$ or $\log 2 \cdot \zeta(2)$, etc. for $t = 1, 2, 3, \ldots$ All harmonic sums have a linear representation in terms of polynomials of harmonic sums of lower transcendentality and a Mellin transform of a higher function, which is up to two–loop order always representable as a product of Nielsen integrals with a polynomial argument structure. Moreover algebraic relations between harmonic sums of a given transcendentality allow for a considerable reduction of the basic set of functions which has to be calculated and leads to further structural simplifications as well.

2. LINEAR REPRESENTATIONS

The number of alternating and non-alternating harmonic sums of transcendentality k is $n_S(k) = 2 \cdot 3^{k-1}$, yielding a total number of sums of $3^k - 1$. The harmonic sums obey representations in terms of multiple integrals which are obtained from

$$S_k(N) = \int_0^1 dx \frac{[-\log(x)]^{k-1}}{(k-1)!} \frac{x^N - 1}{x - 1} \quad (5)$$

$$S_{-k}(N) = \int_0^1 dx \frac{[-\log(x)]^{k-1}}{(k-1)!} \frac{(-x)^N - 1}{x + 1} \quad (6)$$

consecutively. These integrals are expressed by harmonic sums of lower degree and a Mellin transform which is not further reducible on the linear level, cf. [3] for details. In this representation also infinite harmonic sums occur, which can be represented as linear combinations of basic transcendentals as $\log(2), \zeta(2), \zeta(3), \mathrm{Li}_4(1/2)$ up to the level of transcendentality 4. These sums were given in explicit form in Ref. [3]. Already up to transcendentality 4 Mellin transforms of rather

complicated functions, as e.g.

$$F_1(x) = \mathrm{S}_{1,2}\left(\frac{1-x}{2}\right) + \mathrm{S}_{1,2}(1-x) \tag{7}$$
$$-\mathrm{S}_{1,2}\left(\frac{1-x}{1+x}\right) + \mathrm{S}_{1,2}\left(\frac{1}{1+x}\right)$$
$$-\log(2)\mathrm{Li}_2\left(\frac{1-x}{2}\right)$$
$$+\frac{1}{2}\log^2(2)\log\left(\frac{1+x}{2}\right) - \log(2)\mathrm{Li}_2\left(\frac{1-x}{1+x}\right)$$

emerge. These Mellin transforms, however, turn out to be reducible using the algebraic relations given below, i.e. $F_1(x)$ can be obtained as a linear combination of Mellin convolutions of more elementary functions and simpler argument structure. This highlights the importance of algebraic relations between the harmonic sums, to which we are turning now.

3. ALGEBRAIC RELATIONS

The finite harmonic sums of order k are related by algebraic equations. They are obtained studying sums of harmonic sums with permutations in the set of their indices. The simplest relation is due to EULER [4] for two indices

$$S_{m,n} + S_{n,m} = S_m S_n + S_{m\wedge n}, \tag{8}$$

where
$m_1 \wedge m_2 \wedge ... m_k = \prod_{l=1}^{k} \mathrm{sign}(m_k) \sum_{l=1}^{k} |m_k|$.
The corresponding relations for three- and four-fold sums are [3]

$$\sum_{\mathrm{perm}} S_{l,m,n} = S_l S_m S_n + \sum_{\mathrm{perm}} S_l S_{m\wedge n} + S_{l\wedge m\wedge n} \tag{9}$$

and

$$\sum_{\mathrm{perm}} S_{k,l,m,n} = S_k S_l S_m S_n + \sum_{\mathrm{perm}} S_k S_l S_{m\wedge n} + \sum_{\mathrm{perm}} S_{k\wedge l} S_{m\wedge n} + 2\sum_{\mathrm{perm}} S_k S_{l\wedge m\wedge n} + 6 S_{k\wedge l\wedge m\wedge n}. \tag{10}$$

For non-alternating sums and $N \to \infty$ relation (9) was given in [5] before.

Harmonic sums with the same index, but arbitrary length of the index set are related to determinant–structures, cf. [3], and can be calculated in terms of products of single harmonic sums only, as e.g.

$$\underbrace{S_{-1,\ldots,-1}}_{k} = \frac{1}{k}\sum_{l=1}^{k} S_{(-1)^l|l|}\underbrace{S_{-1,\ldots,-1}}_{k-l} \tag{11}$$

$$\underbrace{S_{1,\ldots,1}}_{k} = \frac{1}{k}\sum_{l=1}^{k} S_l \underbrace{S_{1,\ldots,1}}_{k-l}. \tag{12}$$

Starting with 3-fold harmonic sums more algebraic relations can be obtained by partial permutations of the index set. For 3-fold alternating or non-alternating harmonic sums 3 relations are obtained [6,3], which cover the case (9) and result from the combinations of

$$T = S_{a,b,c} + S_{a,c,b} - S_{a\wedge b,c} - S_{a\wedge c,b} - S_{a,b\wedge c} + S_{a\wedge b\wedge c} \tag{13}$$
$$T = S_c S_{a,b} - S_{c,a,b} + S_{c,a\wedge b} - S_c S_{a\wedge b} \tag{14}$$
$$T = S_b S_{a,c} - S_{b,a,c} + S_{b,a\wedge c} - S_b S_{a\wedge c} \tag{15}$$
$$T = S_{b,c,a} + S_{c,b,a} - S_{b\wedge c,a} - S_c S_{b,a} + S_b S_{a,c} - S_b S_{a\wedge c}. \tag{16}$$

Using these relations the number of Mellin transforms occurring in the linear representations can be reduced substantially. Moreover Mellin transforms of more complicated functional structure are recognized as transforms of convolutions of much more elementary functions. Up to 2–loop order only simple, reducible variants of harmonic sums of the type $S_{\pm1,\pm1,\pm1,\pm1}(N)$ occur. The remaining linear representations can be represented by the Mellin-transforms of the basic functions given below.

For the analytic continuation of the Mellin moments $\mathbf{M}[f_i(x)](N)$ in addition to the well–known relations for single harmonic sums only the Mellin transforms of 24 basic functions have to be analytically continued, see [7].

In Ref. [3] a systematic evaluation of the Mellin transforms of the individual functions representing the polarized and unpolarized coefficient functions and anomalous dimensions up to 2–loop order (cf. e.g. [8]) are given. They can be expressed through harmonic sums recursively, which are linear functions of the Mellin transforms of the func-

tions listed below. About 80 functions $f_i(x)$ occur. One example is

$$\mathbf{M}\left[\frac{1}{1+z}\left[\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)\right]\right](N) =$$
$$(-1)^{N-1}\Biggl\{S_{1,1,-2}(N-1)-S_{1,-1,2}(N-1)$$
$$+S_{-1,1,2}(N-1)-S_{-1,-1,-2}(N-1)$$
$$+2\zeta(2)S_{1,-1}(N-1)+\frac{1}{4}\zeta(2)S_1^2(N-1)$$
$$-\frac{1}{4}\zeta(2)S_{-1}^2(N-1)-\zeta(2)S_1(N-1)S_{-1}(N-1)$$
$$-\zeta(2)S_{-2}(N-1)-\left[\frac{7}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]$$
$$\times S_1(N-1)+\left[\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_{-1}(N-1)$$
$$-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{19}{40}\zeta^2(2)+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2\Biggr\}$$

4. CONCLUSIONS

A systematic study of the finite harmonic alternating and non–alternating sums has been performed up to the four–fold sums, which have been evaluated in explicit form in the linear representation. Algebraic relations were used to reduce this set to a representation over a much smaller set of functions. Whereas in the N–space representations these relations are truly algebraic, the corresponding relations in the x–space representation are given by sums of multiple Mellin convolutions. In this representation the algebraic relations lead to essential structural simplifications both concerning the contributing functions as well as their argument structure. The corresponding Mellin transforms were evaluated in explicit form.

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$\frac{\log(1+x)}{x+1}$	$\frac{\log^2(1+x)-\log^2(2)}{x-1}$
$\frac{\log^2(1+x)}{x+1}$	$\frac{\mathrm{Li}_2(x)}{x+1}$
$\frac{\mathrm{Li}_2(x)-\zeta(2)}{x-1}$	$\frac{\mathrm{Li}_2(-x)}{x+1}$
$\frac{\mathrm{Li}_2(-x)+\zeta(2)/2}{x-1}$	$\frac{\log(x)\mathrm{Li}_2(x)}{x+1}$
$\frac{\log(x)\mathrm{Li}_2(x)}{x-1}$	$\frac{\mathrm{Li}_3(x)}{x+1}$
$\frac{\mathrm{Li}_3(x)-\zeta(3)}{x-1}$	$\frac{\mathrm{Li}_3(-x)}{x+1}$
$\frac{\mathrm{Li}_3(-x)-3\zeta(3)/4}{x-1}$	$\frac{\mathrm{S}_{1,2}(x)}{x+1}$
$\frac{\mathrm{S}_{1,2}(x)-\zeta(3)}{x-1}$	$\frac{\mathrm{S}_{1,2}(-x)-\zeta(3)/8}{x-1}$
$\frac{\mathrm{S}_{1,2}(-x)}{x+1}$	$\frac{\mathrm{S}_{1,2}(x^2)}{x+1}$
$\frac{\mathrm{S}_{1,2}(x^2)-\zeta(3)}{x-1}$	$\log(1-x)\frac{\mathrm{Li}_2(-x)}{x+1}$
$\frac{\log(1+x)-\log(2)}{x-1}\mathrm{Li}_2(-x)$	$\frac{\log(1-x)\mathrm{Li}_2(x)}{1+x}$
$\frac{\log(1+x)-\log(2)}{x-1}\mathrm{Li}_2(x)$	$\frac{\log(1+x)}{1+x}\mathrm{Li}_2(x)$

List of the basic functions