The FORM version of MINCER

Abstract

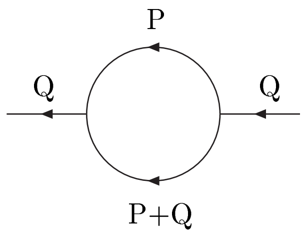
The program MINCER [1] for massless three-loop Feynman diagrams of the propagator type has been reprogrammed in the language of FORM [2]. The new version is thoroughly optimized for extremely large computations. It has been used for some calculations that were previously not practical.

1 The basic algorithms

Let us first sketch the basic algorithms that are to be used for the computation of massless diagrams of the propagator type. Although most of these formulae have been presented before in the literature we like to repeat them here for several reasons: it will allow us to establish a uniform notation and the literature contains many misprints. It will also make it easier to understand the later sections.

1.1 One loop integrals

The basic one loop diagram is given by:



The basic one loop diagram

The integral is given by [10]

$$\int \frac{d^D P}{(2\pi)^D} \frac{\mathcal{P}_n(P)}{P^{2\alpha}(P-Q)^{2\beta}} =$$
$$\frac{1}{(4\pi)^2}(Q^2)^{D/2-\alpha-\beta} \sum_{\sigma\geq 0}^{[n/2]} G(\alpha,\beta,n,\sigma) Q^{2\sigma} \left\{ \frac{1}{\sigma!}\left(\frac{\Box}{4}\right)^{\sigma} \mathcal{P}_n(P) \right\}_{P=Q}, \tag{1}$$

in which

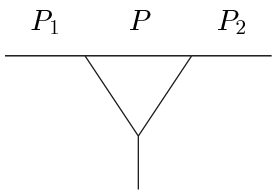
$$\mathcal{P}_n(P) = P_{\mu_1} P_{\mu_2} \cdots P_{\mu_n}. \tag{2}$$

D is the dimension of space-time and is also given by $D = 4 - 2\epsilon$, $\Box = \partial^2/\partial P_\mu \partial P_\mu$ and G can be expressed in terms of Γ-functions:

$$G(\alpha,\beta,n,\sigma) = (4\pi)^{\epsilon} \frac{\Gamma(\alpha+\beta-\sigma-D/2)\Gamma(D/2-\alpha+n-\sigma)\Gamma(D/2-\beta+\sigma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta+n)}. \tag{3}$$

1.2 Two loop integrals

The two loop integrals come in three topologies. Each of these topologies needs its own treatment. The algorithm to deal with the more complicated topologies is based on integration by parts. In its simplest form it is called the rule of the triangle [10] [11].



The triangle subgraph

We define the integral

$$I(n,\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2) = \int d^DP\, \frac{P_{\mu_1}\cdots P_{\mu_n}}{(P^2)^{\alpha_0}((P+p_1)^2)^{\beta_1}(p_1^2)^{\alpha_1}((P+p_2)^2)^{\beta_2}(p_2^2)^{\alpha_2}}, \tag{4}$$

This integrals can be studied by first considering the identity:

$$\int d^DP \left\{ \frac{\partial}{\partial P_\mu} \frac{P_\mu \quad \mathcal{P}_n(P)}{(P^2)^{\alpha_0}((P+p_1)^2)^{\beta_1}(p_1^2)^{\alpha_1}((P+p_2)^2)^{\beta_2}(p_2^2)^{\alpha_2}} \right\} = 0. \tag{5}$$

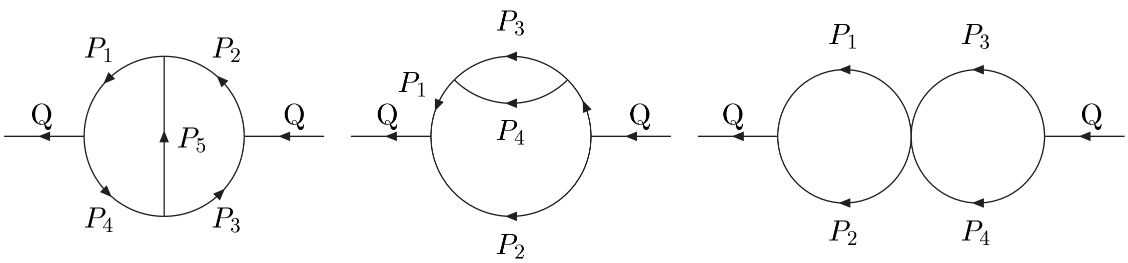
Working out the derivative one obtains the recursion relation:

$$\begin{aligned} I(n,\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2) = (& \\ &+\beta_1(I(n,\alpha_0-1,\beta_1+1,\beta_2,\alpha_1,\alpha_2) - I(n,\alpha_0,\beta_1+1,\beta_2,\alpha_1-1,\alpha_2)) \\ &+\beta_2(I(n,\alpha_0-1,\beta_1,\beta_2+1,\alpha_1,\alpha_2) - I(n,\alpha_0,\beta_1,\beta_2+1,\alpha_1,\alpha_2-1)) \\ &)/(D+n-2\alpha_0-\beta_1-\beta_2). \end{aligned} \tag{6}$$

By using this relation as a recursion relation one can eventually remove at least one of the denominators in most of the non-trivial topologies. It is however possible to improve upon this by 'solving' the recursion. This gives the formula [12]

$$\begin{aligned} & I(n,\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2) = \\ &+ \sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{\alpha_0-1}\sum_{k=0}^{\alpha_0-j-1} (-1)^{\alpha_1+i} I(n,\alpha_0-j-k,\beta_1+\alpha_1+j,\beta_2+i+k,0,\alpha_2-i) \\ &\quad \frac{\Gamma(\alpha_1+i+j+k)\Gamma(\beta_1+\alpha_1+j)\Gamma(D+1+n-2\alpha_0-\beta_1-\beta_2-\alpha_1-i)}{\Gamma(D+1+n-2\alpha_0-\beta_1-\beta_2+j+k)\Gamma(\alpha_1)\; i!\; j!\; k!} \\ &\quad \times\Gamma(\beta_2+k+i) \\ &+ \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{\alpha_2-1}\sum_{k=0}^{\alpha_0} (-1)^{i+j} I(n,0,\beta_1+i+k,\beta_2+\alpha_0+j-k,\alpha_1-i,\alpha_2-j) \\ &\quad \frac{\alpha_0\Gamma(\alpha_0+i+j)\Gamma(D+n-2\alpha_0-\beta_1-\beta_2-i-j)\Gamma(\beta_1+k+i)}{(\alpha_0-k)!\; i!\; j!\; k!\; \Gamma(D+n-\alpha_0-\beta_1-\beta_2)} \\ &\quad \times\Gamma(\alpha_0+\beta_2-k+j) \\ &+ \sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{\alpha_0-1}\sum_{k=0}^{\alpha_0-j-1} (-1)^{i+\alpha_2} I(n,\alpha_0-j-k,\beta_1+i+k,\beta_2+\alpha_2+j,\alpha_1-i,0) \\ &\quad \frac{\Gamma(D+1+n-2\alpha_0-\beta_1-\beta_2-\alpha_2-i)\Gamma(i+j+k+\alpha_2)\Gamma(\beta_1+i+k)}{\Gamma(D+1+n-2\alpha_0-\beta_1-\beta_2+k+j)\Gamma(\alpha_2) i!\; j!\; k!} \\ &\quad \times\Gamma(\beta_2+\alpha_2+j). \end{aligned} \tag{7}$$

The two loop topologies are given by



The two loop topologies T1, T2 and T3

The topologies T2 and T3 can be solved by applying equation 1 twice. The topology T1 is treated by applying the rule of the triangle. This will eliminate the denominator belonging to one of the lines and the resulting integrals are either of the type T2 or of the type T3. There is one complication that arises when the power of the center line in T1 is not an integer. In that case the application of the rule of the triangle is useless as it cannot reduce the power of the center line (α_0) to zero. In that case we use different recursion relations:

$$\begin{aligned}(\beta_1-1)&I(n,\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2)Q^2 = \\ &+(\alpha_0+\beta_1+2\beta_2-D-1)I(n,\alpha_0,\beta_1-1,\beta_2,\alpha_1,\alpha_2) \\ &+(\beta_1-1)I(n,\alpha_0,\beta_1,\beta_2-1,\alpha_1,\alpha_2) \\ &+\alpha_0(I(n,\alpha_0+1,\beta_1-1,\beta_2-1,\alpha_1,\alpha_2) \\ &\quad -I(n,\alpha_0+1,\beta_1-1,\beta_2,\alpha_1,\alpha_2-1)).\end{aligned} \tag{8}$$

and

$$\begin{aligned}(\alpha_0-1+\epsilon)&I(0,\alpha_0,1,1,1,1) = \\ &-(\alpha_0-2+2\epsilon)I(0,\alpha_0-1,1,1,1,1)Q^{-2} \\ &-(3-3\epsilon-2\alpha_0)G(1,\alpha_0,0,0)G(1,\alpha_0+\epsilon,0,0)Q^{-2(2+\alpha_0)}.\end{aligned} \tag{9}$$

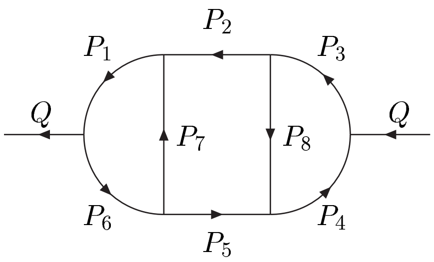
In both cases we have one integral left that cannot be solved by these means. It can be computed to order ϵ^6 with the use of symmetry considerations [13]:

$$I(0,1+\epsilon,1,1,1,1) = (6\zeta_3+9\zeta_4\epsilon+102\zeta_5\epsilon^2+240\zeta_6\epsilon^3-186\zeta_3^2\epsilon^3+\cdots)(1-2\epsilon) \tag{10}$$

1.3 Three loop integrals

For three loop integrals there are 14 topologies that have to be considered. Some of these are rather easy but there are 5 topologies that need special attention. One of them is outright complicated.

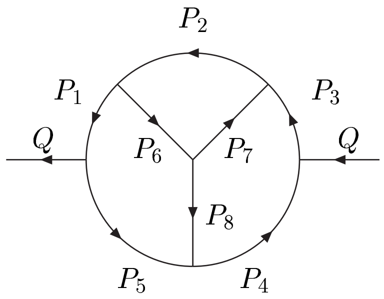
The first topology to consider is the ladder topology, also called LA:



The ladder or LA topology

Diagrams of this type can be simplified by applying the rule of the triangle to either the loop with P_7, P_1 and P_6, or to the loop with P_8, P_4 and P_3. Assuming that we use the first triangle we will get then a diagram that misses either P_7 in the denominator (topology O3) or it misses P_2 or P_5 in the denominator (topology FA). The resulting topologies are treated later in this section.

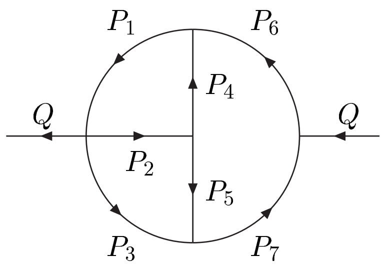
The second basic topology is the Benz topology, also called BE:



The Benz or BE topology

Again we can apply the rule of the triangle to simplify such a diagram. The triangle contains the momenta P_2, P_6 and P_7. After this we have diagrams that miss either P_2 (topology O1) or they miss P_1 or P_3 (topology BU). These topologies are treated later in this section.

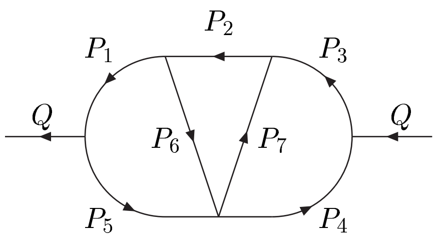
The topology BU is a derivative topology. This means that it is a topology that can be obtained by removing one or more denominators in a more complicated topology with the same number of loops. It is given by:



The topology BU

Also this topology can be simplified with an application of the rule of the triangle. In this case the triangle is either the triangle with the momenta P_4, P_1 and P_2 or the triangle with the momenta P_5, P_3 and P_2. The choice of the triangle depends on which triangle will give the fewest number of terms. Assuming that we used the rule for the first triangle we have diagrams that miss either P_4 in the denominator (topology O2), or they miss P_6 in the denominator (topology O4) or they miss P_5 in the denominator (topology O2). These topologies are treated later in this section.

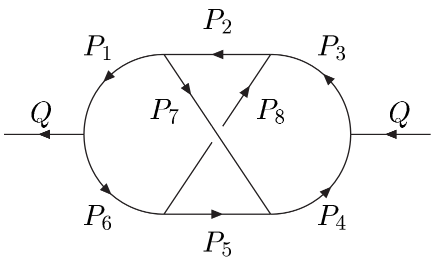
Another derivative topology is the FA topology. It is given by



The topology FA

and it is basically the Benz topology with momentum P_8 missing. We apply again the rule of the triangle and have then either topology O1 left or topology O2.

By far the most complicated topology is the non-planar topology, also called the NO topology:



The non-planar or NO topology

In this diagram there are no triangles, so we need additional relations. The first step in a reduction is to reduce the dotproducts in the numerator to squares of the 8 momenta P_1 to P_8, and in addition $Q.Q$ and either $Q.P_2$ or $P_2.P_5$. The last dotproduct causes problems, but it is inevitable, as there are 10 independent variables before integration and only 9 squares. The second step in the reduction removes this mixed dotproduct (or denominators which effectively simplifies the topology). There are two ways to do this. The first method was introduced in [1]. We define the integral

$$N'(\alpha_1,\cdots,\alpha_8,\alpha_9) = \int \frac{d^Dp_1\,d^Dp_2\,d^Dp_3\ (Q.p_2)^{\alpha_9}}{(p_1.p_1)^{\alpha_1}\cdots(p_8.p_8)^{\alpha_8}}. \tag{11}$$

and the recursion is given by

$$\begin{aligned} N'(\alpha_1,\cdots,\alpha_9) \quad = \quad & \Big(+\frac{1}{2}(D-1-\alpha_1-2\alpha_2-\alpha_3+\alpha_9)\mathbf{9}^- \\ & +\frac{1}{2}\alpha_1\mathbf{1}^+\mathbf{7}^-\mathbf{9}^- - \frac{1}{2}\alpha_1\mathbf{1}^+\mathbf{2}^-\mathbf{9}^- \\ & +\frac{1}{2}\alpha_3\mathbf{3}^+\mathbf{8}^-\mathbf{9}^- - \frac{1}{2}\alpha_3\mathbf{3}^+\mathbf{2}^-\mathbf{9}^- \\ & +\frac{\alpha_9-1}{2}\mathbf{2}^-\mathbf{9}^-\mathbf{9}^-\Big) \Big/ \left(2D-1+\alpha_9-\sum_{i=1}^{8}\alpha_i\right). \end{aligned} \tag{12}$$

The notation here is somewhat schematic: $\mathbf{1}^+\mathbf{7}^-\mathbf{9}^-$ means that in N the α_1 should be incremented by one, the α_7 should be decremented by one and the α_9 should also be lowered by one. This equation can be applied as many times as needed to eliminate all powers of $Q.p_2$. Note that there will never occur negative powers of $Q.p_2$ because the term that could generate such a negative power will have a zero coefficient.

The second scheme comes from [11]. For this we define the integral

$$N(\alpha_1,\cdots,\alpha_8,\alpha_9) = \int \frac{d^Dp_1\,d^Dp_2\,d^Dp_3\ (p_2.p_5)^{\alpha_9}}{(p_1.p_1)^{\alpha_1}\cdots(p_8.p_8)^{\alpha_8}}. \tag{13}$$

First we apply the recursion

$$
\begin{aligned}
N(\alpha_1,\cdots,\alpha_9) \quad = \quad & +\frac{1}{2}(\mathbf{3^-9^-} - \mathbf{2^-9^-} - \mathbf{8^-9^-}) \\
& + \frac{1}{2(\alpha_6-1)}(D-1-\alpha_1-2\alpha_2-\alpha_8+\alpha_9)\,\mathbf{6^-9^-} \\
& + \frac{\alpha_1}{2(\alpha_6-1)}(\mathbf{1^+6^-7^-9^-} - \mathbf{1^+2^-6^-9^-}) \\
& + \frac{\alpha_8}{2(\alpha_6-1)}(\mathbf{3^-6^-8^+9^-} - \mathbf{2^-6^-8^+9^-}).
\end{aligned} \tag{14}
$$

till either $\alpha_9 = 0$, or $\alpha_6 = 1$, or one of the other denominators is completely removed. In the first case we have reached our objective of removing the mixed dotproduct and in the last case the topology has been simplified. Hence we can concentrate on the second. case. By applying a symmetry transformation to equation 14 we can use it to make α_1, α_3 and α_4 equal to 1 (or remove α_9 or remove one of the other denominators). Hence when we are done with this recursion four of the denominators have only a single power. The next relation is

$$
\begin{aligned}
& N(1,\alpha_2,1,1,\alpha_5,1,\alpha_7,\alpha_8,\alpha_9) = \\
& \qquad -\tfrac{D-1+\alpha_9-2\alpha_2-\alpha_7-\alpha_8}{2(\alpha_5-1)}\mathbf{5^-9^-} + \tfrac{\alpha_9-1}{2(\alpha_5-1)}\mathbf{2^-5^-9^-9^-} \\
& + \tfrac{\alpha_7}{2(\alpha_5-1)}(\mathbf{2^-5^-7^+9^-} - \mathbf{1^-5^-7^+9^-}) + \tfrac{\alpha_8}{2(\alpha_5-1)}(\mathbf{2^-5^-8^+9^-} - \mathbf{3^-5^-8^+9^-}).
\end{aligned} \tag{15}
$$

This recursion will make $\alpha_5 = 1$ (or again remove α_9 or another denominator) and a symmetric version of it will do the same for α_2. After that we need two more recursions:

$$
\begin{aligned}
& N(1,1,1,1,1,1,\alpha_7,\alpha_8,\alpha_9) = \\
& \quad +\frac{1}{2}(2\,\mathbf{1^-9^-} + 2\,\mathbf{3^-9^-} - 2\,\mathbf{2^-9^-} - \mathbf{7^-9^-} - Q^2\,\mathbf{9^-}) \\
& \quad +\frac{1}{2}\frac{\alpha_9-1}{\alpha_8-1}(\mathbf{1^-8^-9^-9^-} - \mathbf{2^-8^-9^-9^-} - \mathbf{7^-8^-9^-9^-}) \\
& \quad +\frac{D-1-2\alpha_7-\alpha_8}{2(\alpha_8-1)}\mathbf{8^-9^-} + \frac{1}{\alpha_8-1}(\mathbf{1^-2^+8^-9^-} - \mathbf{2^+7^-8^-9^-}),
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
& N(1,1,1,1,1,1,\alpha_7,1,\alpha_9) = \\
& \quad +\frac{1}{2}(2\,\mathbf{4^-9^-} + 2\,\mathbf{6^-9^-} - 2\,\mathbf{5^-9^-} - \mathbf{8^-9^-} - Q^2\,\mathbf{9^-}) \\
& \quad +\frac{1}{2}\frac{\alpha_9-1}{\alpha_7-1}(\mathbf{6^-7^-9^-9^-} - \mathbf{5^-7^-9^-9^-} - \mathbf{7^-8^-9^-9^-}) \\
& \quad +\frac{D-3-\alpha_7}{2(\alpha_7-1)}\mathbf{7^-9^-} + \frac{1}{\alpha_7-1}(\mathbf{5^+6^-7^-9^-} - \mathbf{5^+7^-8^-9^-}).
\end{aligned} \tag{17}
$$

They will make α_8 and α_7 equal to one. This brings us to the last recursion

$$
\begin{aligned}
N(1,1,1,1,1,1,1,1,\alpha_9) \quad = \quad & \mathbf{3^-9^-} - \mathbf{8^-9^-} + Q^2\frac{1+2\epsilon-\alpha_9}{2(\alpha_9-1-4\epsilon)}\mathbf{9^-} \\
& + \frac{Q^2}{\alpha_9-1-4\epsilon}(\mathbf{2^-3^+9^-} - \mathbf{3^+8^-9^-}).
\end{aligned} \tag{18}
$$

After this all integrals that are left have either a simpler topology or have $\alpha_9 = 0$.

The advantage of the first reduction scheme is its simplicity, while the second scheme is faster because at each step (except for the last) it does not only reduce the value of α_9, but the sum of the other α's is also reduced and hence it leaves simpler integrals behind. The simplicity of the programming made that the first method was used in the original Mincer program [1]. In the FORM version of Mincer we have selected the second method. This will be discussed in a later section.

After eliminating α_9 we still have to try to remove one of the other α's. First we apply:

$$\begin{aligned} Q^2 N(\alpha_1,\cdots,\alpha_8) &= \mathbf{1}^- + \frac{-2D-1+2\alpha_1+2\alpha_2+\alpha_5+\alpha_6+2\alpha_7+\alpha_8}{\alpha_6-1}\mathbf{6}^- \\ &+ \frac{\alpha_8}{\alpha_6-1}\mathbf{6}^-\mathbf{8}^+(\mathbf{2}^- - \mathbf{3}^-) + \frac{\alpha_5}{\alpha_6-1}\mathbf{6}^-\mathbf{5}^+(\mathbf{7}^- - \mathbf{4}^-). \end{aligned} \tag{19}$$

or a symmetric version of it to make α_6, α_1, α_3 and α_4 equal to one (or remove one of the denominators). Note that when α_9 is not present the N' and the N integrals are identical. Then we use

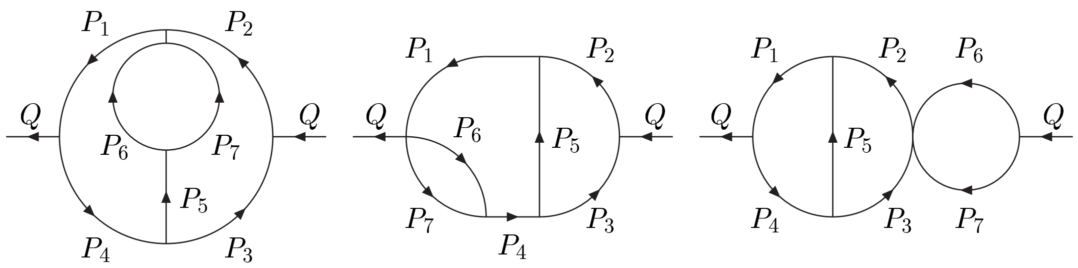
$$\begin{aligned} &(\alpha_5-1)(2-D+2\alpha_5)N(1,\alpha_2,1,1,\alpha_5,1,\alpha_7,\alpha_8) = \\ &\quad(-2D+4+\beta)Q^2\mathbf{5}^-(+4-3D+2\beta+\mathbf{3}^-\mathbf{4}^+ + \mathbf{1}^-\mathbf{6}^+) \\ &\quad+(\alpha_5-1)(\mathbf{7}^-\mathbf{4}^+ + \mathbf{8}^-\mathbf{6}^+) \\ &\quad-(3-2D+\beta+\alpha_5)(\mathbf{5}^-\mathbf{4}^+ + \mathbf{5}^-\mathbf{6}^+), \end{aligned} \tag{20}$$

with $\beta = \alpha_2+\alpha_5+\alpha_7+\alpha_8$ to make α_5, α_2, α_8 and α_7 equal to one. At this point we have either killed off at least one denominator in which case the most complicated topologies that are left are of the type FA or BU, or we are faced with the integral

$$N(1,1,1,1,1,1,1,1) = 20\zeta_5 + \mathcal{O}(\epsilon). \tag{21}$$

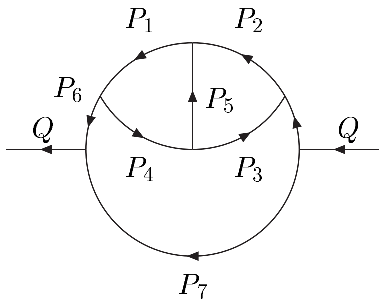
This integral was evaluated in [11]. The term in $O(\epsilon)$ has been computed in [22] but here we do not need it [19].

The leftover topologies have all a one loop two-point function subgraph that can be integrated directly. We still distinguish two groups. Diagrams that still need work at the two loop level (diagrams of type O) and diagrams that do not even need that work anymore (diagrams of type Y). The topologies of type O are



The topologies O1, O2, O3

and



The topology O4

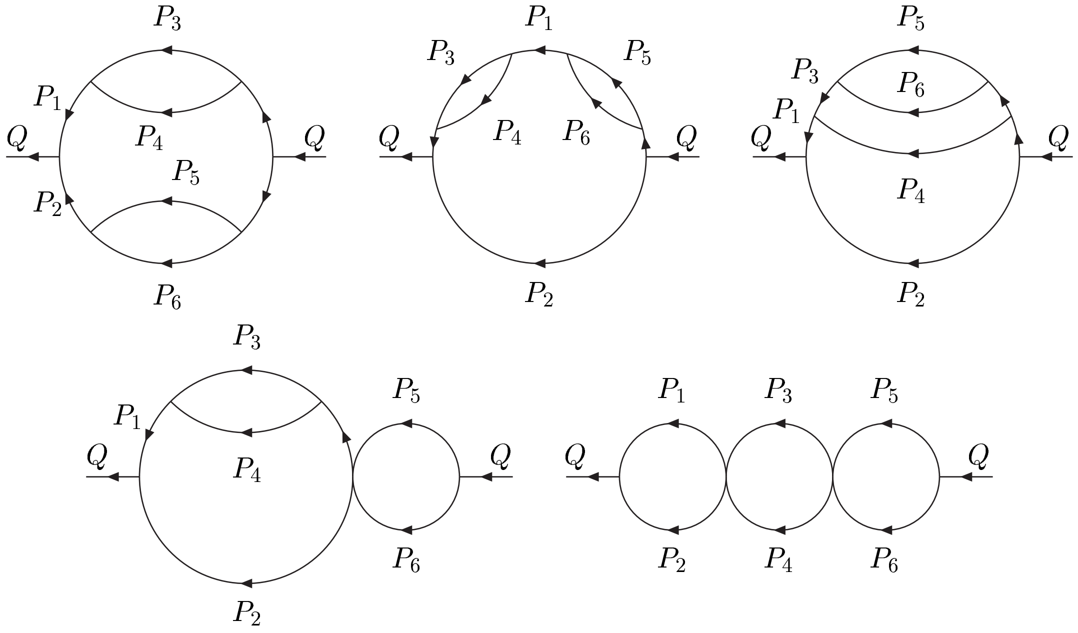
The topologies O1-O3 have a trivial two point subintegral (with momenta P_6 and P_7). The topology O4 requires more care. Based on scaling and Lorenz invariance arguments one can treat the integral over P_6 first. This is particularly simple when there are no dotproducts in the numerator. If there are such dotproducts, dotproducts between momenta in the two loop subgraph are also trivial, dotproducts involving P_7 can be rewritten with $P_7 = Q - P_6$, and $P_6 = P_1 - P_4$. This leaves as only complicated dotproducts the ones involving Q and P_{1-5}. Let us replace $Q.P_i$ by $xR(P_i)$. Then we can use[1]:

$$\begin{aligned}\frac{x^n d^D p_6}{(p_6^2)^{x_1}(p_7^2)^{x_2}} &= \frac{1}{(Q^2)^{x_1+x_2-D/2}}\sum_{s=0}^{n/2} G(x_1-s, x_2, n-2s, 0)\\ &\quad \sum_{j=0}^{n/2-s}\frac{\Gamma(n+2-\epsilon-2s)\Gamma(n+1-\epsilon-2s-j)(-1)^j}{\Gamma(n+2-\epsilon-s)\Gamma(n+1-\epsilon-2s)\ s!\ j!}\\ &\quad \times\left(\frac{Q^2\Box_R}{4}\right)^{s+j}\end{aligned} \tag{22}$$

The d'Alembertian $\Box_R$ acts only on the function R. If it is defined by $\Box_R = \partial^\mu\partial_\mu$ then $\partial^\mu R(p_i) = p_i^\mu$. The way to evaluate these derivatives is discussed in a later section.

The Y topologies are given by

[1]The relevant formula (A.16) from [11] contains a misprint.



The topologies Y1-Y5

In the case of these topologies one can directly apply equation 1 leaving a diagram of topology T2 or T3.

1.4 Harmonic projections

A harmonic tensor of order n in momentum P is defined as a totally symmetric tensor with n indices that is traceless and in which the indices are either coupled to Kronecker delta's or the momentum P. Such harmonic tensors are Lorenz invariants. Hence, if we have such a tensor of which the indices are contracted with momenta we are integrating over, after the integration we will still have such a tensor, even though the indices are now contracted with other momenta. We will use this property frequently. It has actually already been used in equations 1 and 22. We need it however in a very visible way when the moments of structure functions in lepton hadron scattering are studied. In that case we have an effective four-point function and two different external momenta Q and P. Such integrals cannot be dealt with directly. When we make an expansion in terms of P around zero we effectively eliminate one of the external momenta at the cost of much more complicated integrals. Because the moments of structure functions actually correspond to powers of $P.Q/Q.Q$ we can eliminate all terms that contain $P.P$ and the extraction of the n-th power of $P.Q/Q.Q$ corresponds to a harmonic projection.

$$P^{\mu_1} \cdots P^{\mu_n} T^{\mu_1 \cdots \mu_n} = (P.Q/Q.Q)^n H^{\mu_1 \cdots \mu_n} T^{\mu_1 \cdots \mu_n} \tag{23}$$

in which the left hand side represents our integral and H is the harmonic projection operator. This operator is again a harmonic tensor of degree n in Q. It has some rather nasty properties. The number of terms in it is a very rapidly increasing function of n:

n	terms
2	2
3	4
4	10
5	26
6	76
7	232
8	764
9	2620
10	9496
11	35696
12	140152

Number of terms in harmonic tensor of order n

When calculating longitudinal crosssections one has to make a longitudinal projection that involves two extra powers of P and hence the computation of a n=6 moment involves a harmonic projection with 8 indices. For a n=10 moment the harmonic projection operator has 9496 terms for the transverse projection and 140152 terms for the longitudinal projection! It should be notted that at the moment of projection we have already made a rather deep expansion in P, so there are already many terms. This presents us with quite a problem.

The first solution that we have used [**?**] was to avoid the harmonic projection completely. This was possible because it involved a 2 loop computation. When there are only integer powers of the denominators in the topology T1 we can keep a 'foreign' momentum like P. Even though this makes the integrals more complicated it seems to be the most economic method. Unfortunately, for three loop integrals this method cannot be used. The non-planar recursions cannot deal with an extra momentum, and neither can the two loop recursions when there is a denominator with a non-integer power. Hence we need a better solution.

Let us first look at what a harmonic tensor looks like:

$$H^{\mu_1\mu_2}(Q) = h_0^2 Q^{\mu_1} Q^{\mu_2} + h_2^2 \delta^{\mu_1\mu_2} Q.Q \tag{24}$$

The value of h_2^2 can be expressed in terms of h_0^2 with the condition that the tensor should be traceless. The normalization is given by the condition that

$$H^{\mu_1\mu_2}(Q) Q^{\mu_1} Q^{\mu_2} = (Q.Q)^2 \tag{25}$$

which is equivalent to the statement that $(P.Q)^n$ should be invariant under the whole procedure sketched above. If we define h_j^n as the coefficient of all terms in the n-th order tensor that have $j/2$ Kronecker delta's then the normalization condition is that

$$\sum_{j=0,jeven}^{n} h_j^n = 1 \tag{26}$$

The formula for the h_j^n is given by

$$h_j^n = (-1)^j 2^{n-j} \frac{\Gamma(D-2)\Gamma(n+D/2-1-j)}{\Gamma(n+D-2)\Gamma(D/2-1)} \tag{27}$$

which, as one can see, was already incorporated in eq 22. The FORM code for the tensor becomes now:

```
id  P.Q = Q.Q;
ToTensor,nosquare,T,P;
id  T(??) = sum_(j,0,nargs_(..),2,Q.Q^(j/2)*
            acc(hfac(nargs_(..),j))*distrib_(1,j,dd,T,..));
tovector,T,Q;
id  dd(??) = dd_(..);
```

Let us look at these statements in detail as most are quite powerful. The first statement reflects the normalization property that

$$H^{\mu_1\cdots\mu_n}Q^{\mu_n} = H^{\mu_1\cdots\mu_{n-1}}Q.Q \tag{28}$$

The second statement is rather new in FORM (the statements 2,4 and 5 can be used since version 2.2) and collects all occurrences of the vector P in a single tensor T. in older versions of FORM one would use

```
id  P = T(?);
repeat id T(?)*T(??) = T(.,..);
```

The fourth statement is exactly the opposite: the indices in the tensor T are connected to (multiple) occurrences of the vector Q. The third statement sums over the number of indices that will end up in Kronecker delta's and then the distrib_ function splits the list of indices in all different distributions of the indices for the delta's, and those that will end up in the vectors Q. The function dd_ distributes the indices in its arguments over the Kronecker delta's in all different ways. Its power is that when indices are contracted with identical vectors this function takes this into account with the appropriate factor. It does not generate the same term twice. The factor h^n_j is called hfac and it has been tabulated as powerseries in ϵ to save time. The use of the function acc has already been explained.

The above program by itself is no guarantee for speed. Although the functions distrib_ and dd_ can take care of many symmetries a straight forward application would still be very impratical for the higher moments. We need another observation: We a diagram of a topology like NO will be integrated there are 9 different momenta involved. This means that the tensor T can be contracted with 8 different momenta (we eliminated the contractions with Q). There are however only 3 independent momenta of those 8, so if we rewrite the momenta to three such momenta, before we try the harmonic projection there will be many identical momenta in T and the number of terms generated will be relatively modest. This is even more so for two loop integrals in which there are only two independent mementa (apart from Q). This way it turned out to be possible to redo the two-loop calculation in hardly more time than was needed for the calculation in which no harmonic projection was used, even for the n=10 moment in the longitudinal projection.

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An earlier version of FORM can be obtained for free by means of anonymous ftp from nikhefh.nikhef.nl. Version 1 is however insufficient to run the program described in this publication.

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