Symbolic Summation

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Introduction

Task

- Given expression $g(n)$ (depending on n) find expression $f(n)$, such that

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  - Hypergeometric summation
  - Harmonic summation
  - Beyond

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  - Harmonic summation
  - Beyond
- Examples in particle physics
- Summary

Polynomial summation

Examples

- Polynomials

$$\sum_{i=0}^{n-1} i = \frac{1}{2}n(n-1)$$

$$\sum_{i=0}^{n-1} i^2 = \frac{1}{6}n(n-1)(2n-1)$$

$$\sum_{i=0}^{n-1} i^3 = \frac{1}{4}n^2(n-1)^2$$

$$\sum_{i=0}^{n-1} i^4 = \frac{1}{30}n(n-1)(2n-1)(3n^2-3n-1)$$

Difference operator

- Introduce operator Δ with $(\Delta f)(n) = f(n+1) - f(n)$
- If $g = (\Delta f)$, then (for $a, b \in \mathbf{N}, a \leq b$)

$$\sum_{i=a}^{b-1} g(i) \qquad = \sum_{i=a}^{b-1} \big(f(i+1) - f(i)\big)$$

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- Consecutive cancellation of summands: telescoping
- Symbolic summation problem
  $g = (\Delta f)$ with $f = (\sum g)$, operator Δ is left inverse $\Delta(\sum f) = f$
- Cf. symbolic integration (differential operator D)

$$g = Df = \frac{d}{dx} f \quad \longrightarrow \quad \int_a^b dx g(x) = f(b) - f(a)$$

Difference operator (cont'd)

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Rising and falling factorials

- Define rising factorials as $f^{\overline{m}} = f(x)f(x+1)\dots f(x+m-1)$ (also known as Pochhammer symbols $(x)_m$)

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Rising and falling factorials

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Rising and falling factorials

- Define falling factorials as $f^{\underline{m}} = f(x)f(x-1)\dots f(x-m+1)$
- Then, with falling factorials

$$\Delta(x^{\underline{m}}) \;=\; mx^{\underline{m-1}} \qquad\qquad \sum_{i=0}^{n-1} i^{\underline{m}} \;=\; \frac{1}{m+1} n^{\underline{m+1}}$$

- Conversion of polynomial powers x^m
(decomposition with Stirling numbers of second kind $\left\{ \begin{matrix} m \\ i \end{matrix} \right\}$)

$$x^m = \sum_{i=0}^{m} \left\{ \begin{matrix} m \\ i \end{matrix} \right\} x^{\underline{i}}$$

  - Stirling numbers of second kind denote # of ways to partition n things in k non-empty sets

Examples

- Polynomials

$$\sum_{i=0}^{n-1} i = \sum_{i=0}^{n-1} i^{\underline{1}} = \frac{1}{2}n^{\underline{2}} = \frac{1}{2}n(n-1)$$

$$\sum_{i=0}^{n-1} i^2 = \sum_{i=0}^{n-1} (i^{\underline{2}} + i^{\underline{1}}) = \frac{1}{3}n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{i=0}^{n-1} i^3 = \sum_{i=0}^{n-1} (i^{\underline{3}} + 3i^{\underline{2}} + i^{\underline{1}}) = \frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} = \frac{1}{4}n^2(n+1)^2$$

Hypergeometric summation

Definition

- Hypergeometric function ${}_mF_n$

$$ {}_mF_n\left(\begin{array}{c} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{array}\middle|\, z\right) = \sum_{i\geq 0} \frac{a_1^{\overline{i}}\ldots a_m^{\overline{i}}}{b_1^{\overline{i}}\ldots b_n^{\overline{i}}}\,\frac{z^i}{i!} $$

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Examples

$$ {}_0F_0\left(\middle| z \right) = \sum_{i \geq 0} \frac{z^i}{i!} = \mathrm{e}^z $$

$$ {}_2F_1\left(\begin{matrix} a, 1 \\ 1 \end{matrix} \middle| z \right) = \sum_{i \geq 0} a^{\overline{i}} \frac{z^i}{i!} = \frac{1}{(1-z)^a} $$

$$ {}_2F_1\left(\begin{matrix} 1, 1 \\ 2 \end{matrix} \middle| z \right) = z \sum_{i \geq 0} \frac{1^{\overline{i}} 1^{\overline{i}}}{2^{\overline{i}}} \frac{z^i}{i!} = -\ln(1-z) $$

Ratios

- A term g_n is hypergeometric, if the ratio $r(n)$ of two consecutive terms is a rational function of n.

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$$\frac{\begin{pmatrix} m \\ n+1 \end{pmatrix}}{\begin{pmatrix} m \\ n \end{pmatrix}} = \frac{\Gamma(m+1)\Gamma(n+1)\Gamma(m-n+1)}{\Gamma(n+2)\Gamma(m-n)\Gamma(m)} = \frac{-n+m}{n+1}$$

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- Given a hypergeometric term g, is there hypergeometric term f such that $\Delta f = g$?

$$f_{n+1} - f_n = g_n$$

Gospers algorithm

- Gospers algorithm for indefinite hypergeometric summation determines f_n from a given recursion

$$f_n = f_{n-1} + g_{n-1} = f_{n-2} + g_{n-1} + g_{n-2} = \cdots = f_0 + \sum_{k=0}^{n-1} g_k$$

  - Idea: recursive algorithm; telescoping

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- Solve recursion with ansatz $f_n = y(n) g_n$ and (unknown) rational function $y(n)$

$$f_{n+1} - f_n = g_n \qquad \longrightarrow \qquad r(n) y(n+1) - y(n) = 1$$

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- **Upshot**
  - Solve **first-order linear recursion** for $y(n)$

Gospers algorithm (cont'd)

- Given $f_{n+1} - f_n = g_n$ and ansatz $f_n = y(n) g_n$ with rational function $y(n)$, then $r(n)y(n+1) - y(n) = 1$

Gospers algorithm (cont'd)

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- Let $r(n) = \dfrac{a(n)}{b(n)} \dfrac{c(n+1)}{c(n)}$ with polynomials $a(n), b(n), c(n)$ and $\gcd(a(n), b(n+k)) = 1$

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- Ansatz for $y(n)$ becomes $y(n) = \dfrac{b(n-1)}{c(n)} x(n)$ with (unknown) polynomial $x(n)$
- Solve for non-zero $x(n)$

$$a(n) x(n+1) - b(n-1) x(n) = c(n)$$

If non-zero $x(n)$ exists, hypergeometric recursion is summable.

Wilf-Zeilberger algorithm

- WZ algorithm
  - definite hypergeometric summation
  - telescoping

Wilf-Zeilberger algorithm

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Examples

- Definite vs. indefinite summation

$$\sum_k \binom{n}{k} = \sum_k \binom{n}{k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Harmonic summation

- Harmonic sums $S_{m_1,\ldots,m_k}(n)$ [see lecture by Blümlein]
  - recursive definition $S_{\pm m_1,\ldots,m_k}(n) = \sum_{i=1}^{n} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$

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- Particle physics
  - dimensional regularization $D = 4 - 2\epsilon$ requires expansion of the Gamma-function around positive integers values ($n \geq 0$)

$$\frac{\Gamma(n+1+\epsilon)}{\Gamma(1+\epsilon)} \quad = \quad \Gamma(n+1)\, \exp\left(- \sum_{k=1}^{\infty} \epsilon^k \, \frac{(-1)^k)}{k} \, S_k(n) \right)$$

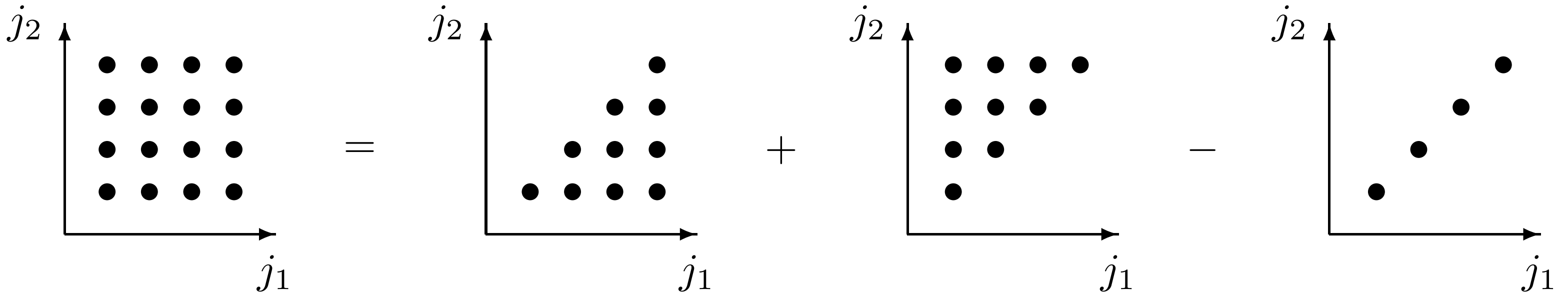
Algorithms for harmonic sums

- Multiplication (Hopf algebra)
  - basic formula (recursion)

$$
\begin{aligned}
S_{m_1,\dots,m_k}(n) \times S_{m'_1,\dots,m'_l}(n) \quad = \quad & \sum_{j_1=1}^{n} \frac{1}{j_1^{m_1}} S_{m_2,\dots,m_k}(j_1) S_{m'_1,\dots,m'_l}(j_1) \\
& + \sum_{j_2=1}^{n} \frac{1}{j_2^{m'_1}} S_{m_1,\dots,m_k}(j_2) S_{m'_2,\dots,m'_l}(j_2) \\
& - \sum_{j=1}^{n} \frac{1}{j^{m_1+m'_1}} S_{m_2,\dots,m_k}(j) S_{m'_2,\dots,m'_l}(j)
\end{aligned}
$$

- Proof uses decomposition

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{n} \sum_{i=1}^{j} a_{ij} - \sum_{i=1}^{n} a_{ii}
$$



Algorithms for harmonic sums (cont'd)

- Convolution (sum over $n-j$ and j)

$$\sum_{j=1}^{n-1} \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

- Conjugation

$$-\sum_{j=1}^{n} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j)$$

- Binomial convolution (sum over binomial, $n-j$ and j)

$$-\sum_{j=1}^{n-1} \binom{n}{j} (-1)^j \frac{1}{j^{m_1}} S_{m_2,\ldots,m_k}(j) \frac{1}{(n-j)^{n_1}} S_{n_2,\ldots,n_l}(n-j)$$

Beyond

- Generalized sums $S(n; m_1, ..., m_k; x_1, ..., x_k)$
  - recursive definition

$$S(n; m_1, ..., m_k; x_1, ..., x_k) = \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} S(i; m_2, ..., m_k; x_2, ..., x_k)$$

  - multiple scales $x_1, ..., x_k$
  - depth k, weight $w = m_1 + ... + m_k$

Example

- Powers of logarithm $\ln(1-x)$

$$\sum_{j=1}^{\infty} \frac{x^j}{j!} \Gamma(j-\epsilon) \quad = \quad \sum_{j=1}^{\infty} \frac{x^j}{j} - \epsilon \sum_{j=1}^{\infty} \frac{x^j}{j} S_1(j-1) + \epsilon^2 \ldots$$

$$= \quad -\ln(1-x) - \epsilon \frac{1}{2} \ln(1-x)^2 + \epsilon^2 \ldots$$

Algorithms for nested sums

- Same structures as for harmonic sums, in particular
  - multiplication

$$S(n; m_1, ...; x_1, ...) \times S(n; m_1', ...; x_1', ...)$$

  - convolution
  - conjugation
  - binomial convolution
- Recursive algorithms analogous to harmonic sums solve multiple nested sums

Higher transcendental functions

- Expansion of higher transcendental functions in small parameter
  - expansion parameter ϵ occurs in the rising factorials (Pochhammer symbols)
- Hypergeometric function

$$ {}_2F_1(a,b;c,x_0) = \sum_{i=0}^{\infty} \frac{a^{\overline{i}} b^{\overline{i}}}{c^{\overline{i}}} \frac{x_0^i}{i!} $$

- First Appell function

$$ F_1(a,b_1,b_2;c;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c^{\overline{m_1+m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

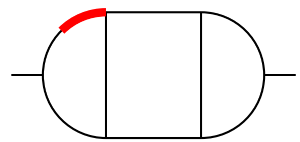
- Second Appell function

$$ F_2(a,b_1,b_2;c_1,c_2;x_1,x_2) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{a^{\overline{m_1+m_2}} b_1^{\overline{m_1}} b_2^{\overline{m_2}}}{c_1^{\overline{m_1}} c_2^{\overline{m_2}}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} $$

Examples in particle physics

Feynman integrals

- Scalar diagram with external momenta P and Q Four-point function with underlying ladder topology [see lecture by Vermaseren]



$$= \int \prod_{n}^{3} d^D l_n \, \frac{1}{(P - l_1)^2} \, \frac{1}{l_1^2 \ldots l_8^2}$$

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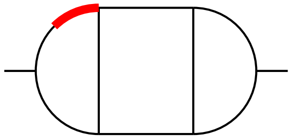

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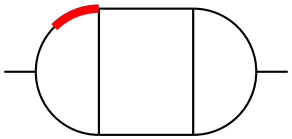
- N-th moment: coefficient of $(2P \cdot Q)^N$



$$= \frac{(2\,P \cdot Q)^N}{(Q^2)^{N+\alpha}} \, C_N$$

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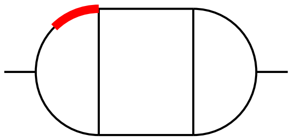

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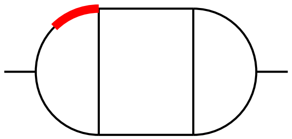


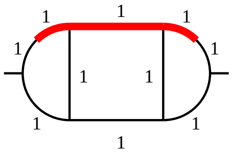
$$= \frac{(2\, P \cdot Q)^N}{(Q^2)^{N+\alpha}} \, C_N$$

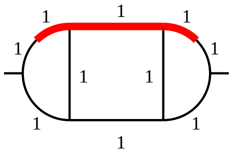
- Taylor expansion

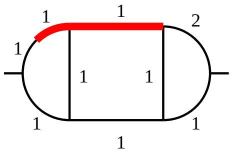
$$\frac{1}{(P-l_1)^2} = \sum_i \frac{(2P \cdot l_1)^i}{(l_1^2)^{i+1}} \quad \longrightarrow \quad \frac{(2P \cdot l_1)^N}{(l_1^2)^N}$$

Difference equations

- Single-step difference equation in N
  - extremely simple example



$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$


$+\frac{2}{N+2}$


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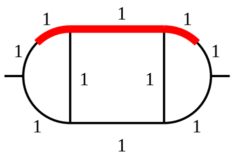
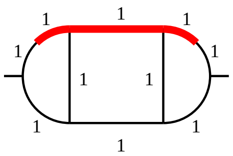
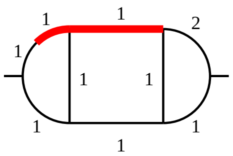


$$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$$



$$+\frac{2}{N+2}$$

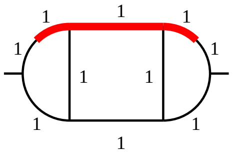
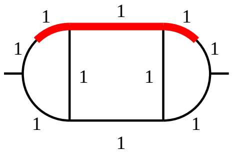
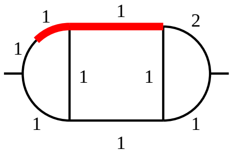


- Formal equation

$$\mathbf{I}(\mathbf{N}) \quad = \quad -\frac{N+3+3\epsilon}{N+2}\,\mathbf{I}(\mathbf{N}-\mathbf{1}) + \frac{2}{N+2}\,\mathbf{G}(\mathbf{N})$$

Difference equations

- Single-step difference equation in N
  - extremely simple example

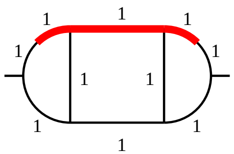
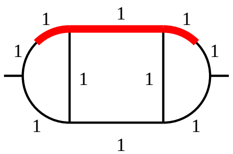
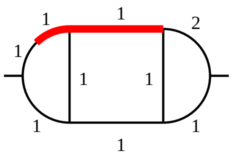
 $= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$  $+\,\frac{2}{N+2}$ 

- Formal equation, formal solution

$$\mathbf{I(N)} \quad = \quad (-1)^N \,\frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \,\frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

Difference equations

- Single-step difference equation in N
  - extremely simple example

 $= -\frac{N+3+3\varepsilon}{N+2}\frac{2\,p\cdot q}{q^2}$  $+\frac{2}{N+2}$ 

- Formal equation, formal solution, input to solution

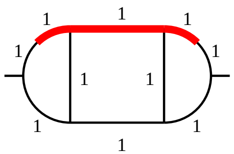
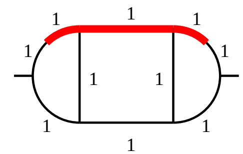
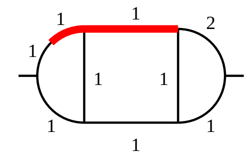
$$\mathbf{I}(\mathbf{N}) = (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I}(\mathbf{0}) + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G}(\mathbf{i})$$

$$\mathbf{I}(\mathbf{0}) = -\frac{2}{3}\frac{1}{\epsilon^2} + \frac{23}{3}\frac{1}{\epsilon} - 42$$

$$\mathbf{G}(\mathbf{i}) = \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \ldots$$

Difference equations

- Single-step difference equation in N
  - extremely simple example

 $= -\dfrac{N+3+3\varepsilon}{N+2}\,\dfrac{2\,p\cdot q}{q^2}$  $+\,\dfrac{2}{N+2}$ 

- Formal equation, formal solution, input to solution

$$\mathbf{I(N)} = (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

$$\mathbf{I(0)} = -\frac{2}{3}\frac{1}{\epsilon^2} + \frac{23}{3}\frac{1}{\epsilon} - 42$$

$$\mathbf{G(i)} = \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \ldots$$

- **Upshot**
  - automatic build-up of **nested sums**
  - efficient implementation in FORM

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
  3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
  410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
  2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
  den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
   - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
  2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
  N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
  *den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
  R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
   - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
  )*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
  18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
  den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
   + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
  S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
   + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
  ) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
  den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
  *N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
  3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
  den(2 + N)^4 - 50*den(2 + N)^5 );
```

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
 - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
*den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
 - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
)*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
 + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
 + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
*N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
den(2 + N)^4 - 50*den(2 + N)^5 );
```

- Result for $\mathbf{I}(\mathbf{N})$ in the G-scheme

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Symbolic summation

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Examples in particle physics

- Feynman diagram calculations and more ...

Literature

- Text books
  - *Modern Computer Algebra*, J. von zur Gathen, J. Gerhard
  - *Concrete Mathematics*, R. L Graham, D. E. Knuth, O. Pataschnik
  - *A=B*, M. Petkovsek, H. S. Wilf, D. Zeilberger
- Research articles
  - *Harmonic sums, Mellin transforms and integrals*, J. Vermaseren; hep-ph/9806280
  - *Nested sums, expansion of transcendental functions and multi-scale multi-loop integrals*, S. Moch, P. Uwer, S. Weinzierl; hep-ph/0110083

Software

- Commercial programs
  - *Mathematica*
  - *Maple*
- Freeware/Add-on packages
  - *Mathematica*, *Maple*
    - Several packages for hypergeometric summation
      [see for instance `www.cis.upenn.edu/~wilf/AeqB.html`]
  - GiNaC
    - *nestedsums*, S. Weinzierl
  - Form
    - *Summer6*, J. Vermaseren
    - *XSummer*, S. Moch, P. Uwer to be published

Exercises 1

- Use *Mathematica* or *Maple* for polynomial summmation.
- Check some of the examples for hypergeometric summation with *Mathematica* or *Maple* like

$$\sum_{i \geq 0} a^{\overline{i}} \frac{z^i}{i!} = \frac{1}{(1-z)^a}$$

$$-z \sum_{i \geq 0} \frac{1^{\overline{i}} 1^{\overline{i}}}{2^{\overline{i}}} \frac{z^i}{i!} = \ln(1-z)$$

- Try to evaluate the sum $\sum_{j_1=1}^{N} \frac{1}{j_1} S_1(j_1)$ in *Mathematica* or *Maple*.

  What happens?

Exercises 2

- Use the FORM package summer6.h for harmonic summmation.
  - Evaluate the product of harmonic sums $S_2(N)S_1(N)^2$. Use the procedure `basis.prc`.
    ```
    #-
    #include summer6.h
    .global
    L exampleproduct = S(R(2),N)*S(R(1),N)^2;
    #call basis(S)
    Print;
    .end
    ```
  - Check your result with the following sequence of calls.
    ```
    Multiply, replace_(N,<some_number>);
    #call subesses(S)
    ```

Exercises 3

- Use the FORM package summer6.h for harmonic summmation.
  - Evaluate the sum $\sum_{j_1=1}^{N} \frac{1}{j_1} S_1(j_1)$ Use the procedure `summer.prc`.

```
#-
#include summer6.h
.global
L examplesum = sum1(j1,1,N)*den(j1)*S(R(1),j1);
#call summer(1)
Print;
.end
```

  - Compute examples for the convolution and conjugation of harmonic sums. Use the notation `fac(N)`, `invfac(N)` and `sign(N)` with the (obvious) meaning $N!$, $\frac{1}{N!}$ and $(-1)^N$

Exercises 4

- Program the expansion of the Gamma-function around any integer value in FORM. To do so, use the result for expansions around positive integers ($n \geq 0$)

$$\frac{\Gamma(n+1+\epsilon)}{\Gamma(1+\epsilon)} = \Gamma(n+1)\exp\left(-\sum_{k=1}^{\infty} \epsilon^k \frac{(-1)^k)}{k} S_k(n)\right)$$

  For expansions around negative integers ($n \leq 0$) use the well-known relation

$$\frac{\Gamma(-n+1+\epsilon)}{\Gamma(1+\epsilon)} = (-1)^n \frac{\Gamma(-\epsilon)}{\Gamma(n-\epsilon)}$$

- Use the FORM package summer6.h and your Gamma-function expansion to solve the difference equation

$$\mathbf{I(N)} = -\frac{N+3+3\epsilon}{N+2}\mathbf{I(N-1)} + \frac{2}{N+2}\mathbf{G(N)}$$

  with the boundary conditions for $\mathbf{I(0)}$ and $\mathbf{G(N)}$ given in the lecture.