Multi-Leg Amplitudes at One Loop

Methods, Algorithms, Automatization and Numerics

CAPP 2005, Zeuthen

Thomas Binoth

binoth@physik.uni-wuerzburg.de

Lectures 3 and 4

Content:

- Monday 4/4/05
  - Introduction
  - Amplitude organization
  - Reduction of one-loop N-point scalar integrals
- Tuesday 5/4/05
  - Reduction of one-loop N-point tensor integrals
  - Numerical evaluation of Feynman diagrams
  - Summary and Outlook

Reduction of N-point tensor integrals by subtraction

$$I_N^{\mu_1 \ldots \mu_r} = \int d^n\kappa \frac{k^{\mu_1} \ldots k^{\mu_r}}{\prod_{j=1}^{N} D_j}$$

Reduction of N-point tensor integrals by subtraction

$$I_N^{\mu_1 \dots \mu_r} = \int d^n\kappa \frac{k^{\mu_1} \dots k^{\mu_r}}{\prod_{j=1}^N D_j}$$

Make ansatz for reduction formula:

$$k^{\mu_1} = \left(k^{\mu_1} - \sum_{j=1}^N C_j^{\mu_1} D_j \right) + \sum_{j=1}^N C_j^{\mu_1} D_j$$

Reduction of N-point tensor integrals by subtraction

$$I_N^{\mu_1\ldots\mu_r} = \int d^n\kappa \frac{k^{\mu_1}\ldots k^{\mu_r}}{\prod_{j=1}^N D_j}$$

Make ansatz for reduction formula:

$$k^{\mu_1} = \left(k^{\mu_1} - \sum_{j=1}^N C_j^{\mu_1} D_j\right) + \sum_{j=1}^N C_j^{\mu_1} D_j$$

$\Rightarrow$ "pinched" graphs of rank $r-1$ plus rest:

$$I_N^{\mu_1\ldots\mu_r} = \sum_{j=1}^N C_j^{\mu_1} I_{N-1,j}^{\mu_2\ldots\mu_r} + T_{Remainder}$$

$$T_{Rem.} = \int d^n\kappa \frac{Q^{\mu_1} k^{\mu_2}\ldots k^{\mu_r}}{\prod_{j=1}^N D_j}$$

- Feynman parametrize and make shift: $k \to k + R$
- Denominator: $\mathcal{D} = k^2 - x \cdot S \cdot x/2$.
- Numerator, using $\Delta_{lj} = r_l - r_j$, $S_{jj} = 2m_j^2$:

$$
\begin{aligned}
Q'^{\mu} &= k^{\mu} + R^{\mu} - \sum_j C_j^{\mu} \Big[[k + \sum_l x_l (r_l - r_j)]^2 - m_j^2 \Big] \\
&= k^{\mu} + R^{\mu} - \sum_j C_j^{\mu} \Big[k^2 + 2 \sum_l x_l k \cdot \Delta_{lj} \\
&\quad + \sum_{l,m} x_l x_m \, \Delta_{lj} \cdot \Delta_{mj} - m_j^2 \Big]
\end{aligned}
$$

- Feynman parametrize and make shift: $k \to k + R$
- Denominator: $\mathcal{D} = k^2 - x \cdot S \cdot x/2$.
- Numerator, using $\Delta_{lj} = r_l - r_j$, $S_{jj} = 2m_j^2$:

$$
\begin{aligned}
Q'^{\mu} &= k^{\mu} + R^{\mu} - \sum_j C_j^{\mu} \Big[[k + \sum_l x_l (r_l - r_j)]^2 - m_j^2 \Big] \\
&= k^{\mu} + R^{\mu} - \sum_j C_j^{\mu} \Big[k^2 + 2 \sum_l x_l k \cdot \Delta_{lj} \\
&\quad + \sum_{l,m} x_l x_m \, \Delta_{lj} \cdot \Delta_{mj} - m_j^2 \Big]
\end{aligned}
$$

$$
2\Delta_{lj} \cdot \Delta_{mj} = G_{lm} - G_{lj} - G_{mj} + G_{jj} = S_{lm} - S_{lj} - S_{mj} + S_{jj}
$$

$$
\begin{aligned}
Q'^{\mu} &= k^{\mu} + R^{\mu} - \sum_j C_j^{\mu} \Big[k^2 + x \cdot S \cdot x/2 \\
&\quad + 2 \sum_l x_l k \cdot \Delta_{lj} - \sum_l x_l \; S_{lj} \Big] \\
&= \sum_l x_l (r_l^{\mu} + \sum_j C_j^{\mu} \; S_{lj}) - (\sum_j C_j^{\mu})(k^2 + M^2) \\
&\quad + k_{\nu} [g^{\mu\nu} - 2(\sum_j C_j^{\mu}) \sum_l x_l r_l^{\nu} + 2 \sum_j C_j^{\mu} r_j^{\nu}]
\end{aligned}
$$

$$
\begin{aligned}
Q'^{\mu} &= k^{\mu} + R^{\mu} - \sum_j C_j^{\mu} \Big[k^2 + x \cdot S \cdot x/2 \\
&\quad + 2 \sum_l x_l k \cdot \Delta_{lj} - \sum_l x_l \; S_{lj} \Big] \\
&= \sum_l x_l (r_l^{\mu} + \sum_j C_j^{\mu} \; S_{lj}) - (\sum_j C_j^{\mu})(k^2 + M^2) \\
&\quad + k_{\nu} [g^{\mu\nu} - 2(\sum_j C_j^{\mu}) \sum_l x_l r_l^{\nu} + 2 \sum_j C_j^{\mu} r_j^{\nu}]
\end{aligned}
$$

- First term $= 0 \Leftrightarrow$ $\boxed{\sum_l S_{jl} C_l^{\mu} + r_j^{\mu} = 0}$
- Second term leads to higher dim. integral $\Rightarrow$ IR finite!

Last term of the form:

$$\sim \quad \Gamma(N) \int d^D\kappa \, d^N x \, \delta(1 - \sum_l x_l) \tau_\nu^{\mu_1} \frac{k^\nu (k+R)^{\mu_2} \ldots (k+R)^{\mu_r}}{(k^2 - M^2)^N}$$

Last term of the form:

$$\sim \quad \Gamma(N) \int d^D\kappa \, d^N x \, \delta(1 - \sum_l x_l) \tau_\nu^{\mu_1} \frac{k^\nu (k+R)^{\mu_2} \dots (k+R)^{\mu_r}}{(k^2 - M^2)^N}$$

Case $r = 1$**:** integral$= 0$ symmetric integration

Case $r = 2$**:** integral $\sim \int d^n\kappa \; k^\nu k^{\mu_2}/(k^2 - M^2)^N = A g^{\nu\mu_2}$
$\rightarrow nA = \int d^n\kappa k^2/(k^2 - M^2)^N \sim$ higher dim. integral

Case $r \geq 3$**:** similar

Last term of the form:

$$\sim \quad \Gamma(N) \int d^D\kappa \, d^N x \, \delta(1 - \sum_l x_l) \tau_\nu^{\mu_1} \frac{k^\nu (k+R)^{\mu_2} \ldots (k+R)^{\mu_r}}{(k^2 - M^2)^N}$$

Case $r = 1$**:** integral$= 0$ symmetric integration

Case $r = 2$**:** integral $\sim \int d^n\kappa \; k^\nu k^{\mu_2}/(k^2 - M^2)^N = A g^{\nu\mu_2}$

$\rightarrow nA = \int d^n\kappa k^2/(k^2 - M^2)^N \sim$ higher dim. integral

Case $r \geq 3$**:** similar

- Last term is also IR finite !
- Reduction induces IR subtractions !

Solution for $\sum_l S_{jl} C_l^\mu = -r_j^\mu$ for general N:

Case $N \leq 5$**:** $\det(G) \neq 0$, $\det(S) \neq 0$

$$\boxed{C_j^\mu = -\sum_l (S^{-1})_{jl} r_l^\mu}$$

Case $N \geq 6$**:** $\det(G) = 0$

Solution for $\sum_l S_{jl} C_l^\mu = -r_j^\mu$ for general N:

Case $N \leq 5$**:** $\det(G) \neq 0$, $\det(S) \neq 0$

$$C_j^\mu = -\sum_l (S^{-1})_{jl} r_l^\mu$$

Case $N \geq 6$**:** $\det(G) = 0$

Ansatz: $$\sum_{j=1}^{N-1} G_{lj} C_j^\mu = -r_l^\mu \, , \sum_{j=1}^{N} v_j C_j^\mu = 0 \, , \sum_{j=1}^{N} C_j^\mu = 0$$

Solution for $\sum_l S_{jl} C_l^\mu = -r_j^\mu$ for general N:

Case $N \le 5$: $\det(G) \neq 0$, $\det(S) \neq 0$

$$C_j^\mu = -\sum_l (S^{-1})_{jl} r_l^\mu$$

Case $N \ge 6$: $\det(G) = 0$

Ansatz: $$\sum_{j=1}^{N-1} G_{lj} C_j^\mu = -r_l^\mu \; , \; \sum_{j=1}^{N} v_j C_j^\mu = 0 \; , \; \sum_{j=1}^{N} C_j^\mu = 0$$

Solution: $(w_j = v_j - v_N)$

$$C_j^\mu = -\sum_{l=1}^{N-1} \left[H_{jl} r_l^\mu - K_{jl} w_l \frac{w \cdot H \cdot r^\mu}{w \cdot K \cdot w} \right] , \; j \in \{1, \ldots, N-1\}$$

$$C_N^\mu = -\sum_{l=1}^{N-1} C_j^\mu$$

The fate of higher dimensional integrals for $N \geq 6$:

Because:

$$\sum_j K_{ij} r_j^\mu = \sum_j (\delta_{ij} - (H \cdot G)_{ij}) r_j^\mu$$
$$= r_i^\mu - \sum_j (R^T \cdot (R \cdot R^T)^{-1} \cdot R)_{ij} (R^T \cdot E^\mu)_j = 0$$
$$\Rightarrow Q'^\mu = k_\nu \left[g^{\nu\mu} - 2\, r^\nu \cdot H \cdot r^\mu \right]$$

Contracted with ext. vector $A^\mu = \sum_{j=1}^4 \alpha_j r_j^\mu$: $Q' \cdot A = 0$!!!

The fate of higher dimensional integrals for $N \geq 6$:

Because:

$$\sum_j K_{ij} r_j^\mu = \sum_j (\delta_{ij} - (H \cdot G)_{ij}) r_j^\mu$$

$$= r_i^\mu - \sum_j (R^T \cdot (R \cdot R^T)^{-1} \cdot R)_{ij} (R^T \cdot E^\mu)_j = 0$$

$$\Rightarrow Q'^\mu = k_\nu \left[g^{\nu\mu} - 2\, r^\nu \cdot H \cdot r^\mu \right]$$

Contracted with ext. vector $A^\mu = \sum_{j=1}^4 \alpha_j r_j^\mu$: $Q' \cdot A = 0$!!!

Reduction of $N \geq 6$ point tensor integrals trivial:

$$I_N^{\mu_1 \dots \mu_r} = \sum_{j=1}^N C_j^{\mu_1} I_{N-1,j}^{\mu_2 \dots \mu_r}$$

Formfactor representation for the case $N=5$ ($j_l \in \{1,\ldots,4\}$):

$$
\begin{aligned}
I_5^{\mu_1} &= \sum T^{5,1}_{j_1}\, r^{\mu_1}_{j_1} \\
I_5^{\mu_1\mu_2} &= T^{5,2}_{00}\, g^{\mu_1\mu_2} + \sum T^{5,2}_{j_1 j_2}\, r^{\mu_1}_{j_1} r^{\mu_2}_{j_2} \\
I_5^{\mu_1\mu_2\mu_3} &= \sum T^{5,3}_{00 j_3} \left(g^{\mu_1\mu_2} r^{\mu_3}_{j_3} + 2 \text{ p.} \right) \\
&+ \sum T^{5,3}_{j_1 j_2 j_3}\, r^{\mu_1}_{j_1} r^{\mu_2}_{j_2} r^{\mu_3}_{j_3} \\
I_5^{\mu_1\mu_2\mu_3\mu_4} &= T^{5,4}_{0000} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} + 2 \text{ p.} \right) \\
&+ \sum T^{5,4}_{00 j_3 j_4} \left(g^{\mu_1\mu_2} r^{\mu_3}_{j_3} r^{\mu_4}_{j_4} + 5 \text{ p.} \right) \\
&+ \sum T^{5,4}_{j_1 j_2 j_3 j_4}\, r^{\mu_1}_{j_1} r^{\mu_2}_{j_2} r^{\mu_3}_{j_3} r^{\mu_4}_{j_4} \\
I_5^{\mu_1\mu_2\mu_3\mu_4\mu_5} &= \sum T^{5,5}_{0000 j_5} \left(g^{\mu_1\mu_2} g^{\mu_3\mu_4} r^{\mu_5}_{j_5} + 14 \text{ p.} \right) \\
&+ \sum T^{5,5}_{00 j_3 j_4 j_5} \left(g^{\mu_1\mu_2} r^{\mu_3}_{j_3} r^{\mu_4}_{j_4} r^{\mu_5}_{j_5} + 9 \text{ p.} \right) \\
&+ \sum T^{5,5}_{j_1 j_2 j_3 j_4 j_5}\, r^{\mu_1}_{j_1} r^{\mu_2}_{j_2} r^{\mu_3}_{j_3} r^{\mu_4}_{j_4} r^{\mu_5}_{j_5}
\end{aligned}
$$

$T^{5,r}_{j_1\cdots j_r}$ only needed for $j_1 \le j_2 \le \cdots \le j_r$.

Example $I_5^{\mu_1\mu_2}$:

Feynman parametrization and momentum integration:

$$T_{00}^{5,2} = \frac{1}{2} I_5^{n+2} \quad , \quad T_{j_1 j_2}^{5,2} = I_5^n(j_1 j_2)$$

$$I_N^D(j_1 j_2) = (-1)^N \, \Gamma(N - D/2) \int d^N x \delta(1 - \sum_l x_l) \frac{x_{j_1} x_{j_2}}{(x \cdot S \cdot x/2)^{N-D/2}}$$

what happens to the $(n+2m)$-dimensional integrals?

Example $I_5^{\mu_1\mu_2}$:

Feynman parametrization and momentum integration:

$$T_{00}^{5,2} = \frac{1}{2} I_5^{n+2} \quad , \quad T_{j_1 j_2}^{5,2} = I_5^n(j_1 j_2)$$

$$I_N^D(j_1 j_2) = (-1)^N \, \Gamma(N - D/2) \int d^N x \delta(1 - \sum_l x_l) \frac{x_{j_1} x_{j_2}}{(x \cdot S \cdot x/2)^{N-D/2}}$$

what happens to the $(n+2m)$-dimensional integrals?

Apply formalism and use $g^{\mu\nu}|_{d=4} = \sum_{jl} (b_l b_j - 2(S^{-1})_{lj}) \, r_l^\mu r_j^\nu$

$$T_{00}^{5,2} = -\frac{1}{2} \sum_j b_j I_{4,j}^{n+2} + \mathcal{O}(\epsilon)$$

$$T_{j_1 j_2}^{5,2} = \sum_l ((S^{-1})_{l j_1} b_{j_2} + (S^{-1})_{l j_2} b_{j_1} - 2(S^{-1})_{j_1 j_2} b_l + b_j (S_j^{-1})_{j_1 j_2}) I_{4,j}^{n+2}$$
$$+ \frac{1}{2} \sum_l \sum_{m \in \{1,\ldots,5\}/\{l\}} ((S^{-1})_{l j_2} (S_j^{-1})_{m j_1} + (S^{-1})_{l j_1} (S_j^{-1})_{m j_2}) I_{3,m,k}^n$$

[S_j^{-1} is 5 by 5 matrix with jth colum and row $= 0$

and remaining entries filled with $minor(j,S)^{-1}$.]

Representation of form factors for $N \leq 5$:

Form factors can be expressed by I_3^n, I_4^{n+2}, I_4^{n+4} and

$$
\begin{aligned}
I_3^n(j_1,\ldots,j_r) &= -\Gamma\left(3-\frac{n}{2}\right)\int_0^1 d^3x\,\delta(1-\sum_{l=1}^{3} x_l)\,\frac{x_{j_1}\ldots x_{j_r}}{(x\cdot S\cdot x/2)^{3-n/2}}\;, \\
I_4^{n+2}(j_1,\ldots,j_r) &= \Gamma\left(3-\frac{n}{2}\right)\int_0^1 d^4x\,\delta(1-\sum_{l=1}^{4} x_l)\,\frac{x_{j_1}\ldots x_{j_r}}{(x\cdot S\cdot x/2)^{3-n/2}}\;, \\
I_4^{n+4}(j_1) &= \Gamma\left(2-\frac{n}{2}\right)\int_0^1 d^4x\,\delta(1-\sum_{l=1}^{4} x_l)\,\frac{x_{j_1}}{(x\cdot S\cdot x/2)^{2-n/2}}\;,
\end{aligned}
$$

Representation of form factors for $N \leq 5$:

Form factors can be expressed by I_3^n, I_4^{n+2}, I_4^{n+4} and

$$
\begin{aligned}
I_3^n(j_1,\dots,j_r) &= -\Gamma\left(3-\frac{n}{2}\right)\int_0^1 d^3x\,\delta(1-\sum_{l=1}^{3}x_l)\,\frac{x_{j_1}\dots x_{j_r}}{(x\cdot S\cdot x/2)^{3-n/2}}\,,\\
I_4^{n+2}(j_1,\dots,j_r) &= \Gamma\left(3-\frac{n}{2}\right)\int_0^1 d^4x\,\delta(1-\sum_{l=1}^{4}x_l)\,\frac{x_{j_1}\dots x_{j_r}}{(x\cdot S\cdot x/2)^{3-n/2}}\,,\\
I_4^{n+4}(j_1) &= \Gamma\left(2-\frac{n}{2}\right)\int_0^1 d^4x\,\delta(1-\sum_{l=1}^{4}x_l)\,\frac{x_{j_1}}{(x\cdot S\cdot x/2)^{2-n/2}}\,,
\end{aligned}
$$

- "Basis" integrals needed for $r \leq 3$ only
- Formalism has no inverse Gram determinants
- IR divergences isolated in 3-point functions
- Ideal starting point for numerical evaluation

Projection operators for form factors:

In applications with $N \leq 4$ form factors are preferably further reduced to scalar integrals I_1^n, I_2^n, I_3^n, I_4^{n+2}. The prize to pay are inverse Gram determinants leading to numerical instabilities at the phase space boundaries. These problems are typically not prohibitive for $N \leq 4$.

Projection operators for form factors:

In applications with $N \leq 4$ form factors are preferably further reduced to scalar integrals I_1^n, I_2^n, I_3^n, I_4^{n+2}. The prize to pay are inverse Gram determinants leading to numerical instabilities at the phase space boundaries. These problems are typically not prohibitive for $N \leq 4$.

Notations now as later used in FORM code:

$$TI(N, D, R, \mu_1, \ldots, \mu_R, r_1, \ldots, r_N, m_1^2, \ldots, m_N^2) = \int d^D\kappa \frac{k^{\mu_1} \ldots k^{\mu_R}}{\prod_{j=1}^N (q_j^2 - m_j^2)}$$

Case N=2: $s = r_1 \cdot r_1 \neq 0$:

Case N=2: $s = r_1 \cdot r_1 \neq 0$:

$$
\begin{aligned}
TI(D,2,1,\mu_1,r_1,0,m_1,m_2) &= TC21(1,D,s,m_1,m_2)\, r_1^{\mu_1} \\
TI(D,2,2,\mu_1,\mu_2,r_1,0,m_1,m_2) &= TC22(0,0,D,s,m_1,m_2)\, g^{\mu_1\mu_2} \\
&+ TC22(1,1,D,s,m_1,m_2)\, r_1^{\mu_1} r_1^{\mu_2}
\end{aligned}
$$

Case N=2: $s = r_1 \cdot r_1 \neq 0$:

$$
\begin{aligned}
TI(D,2,1,\mu_1,r_1,0,m_1,m_2) &= TC21(1,D,s,m_1,m_2)\, r_1^{\mu_1} \\
TI(D,2,2,\mu_1,\mu_2,r_1,0,m_1,m_2) &= TC22(0,0,D,s,m_1,m_2)\, g^{\mu_1\mu_2} \\
&+ TC22(1,1,D,s,m_1,m_2)\, r_1^{\mu_1} r_1^{\mu_2}
\end{aligned}
$$

The two linear operators

$$
\mathcal{R}^\mu = \frac{r^\mu}{s} \quad , \quad \mathcal{P}^{\mu\nu} = \left(g^{\mu\nu} - \frac{r^\mu r^\nu}{s} \right)
$$

$$
\mathcal{P} \cdot \mathcal{R} = 0 \ , \ \mathcal{P}\mathcal{P} = \mathcal{P} \quad , \quad \mathcal{R} \cdot r = 1 \ , \ tr(\mathcal{P}) = (n-3)
$$

can be used to project tensor integrals onto form factors:

$$
\begin{aligned}
TC21(1,n,s,m_1^2,m_2^2) &= \mathcal{R}_\mu I_2^\mu \\
TC22(0,0,n,s,m_1^2,m_2^2) &= \frac{1}{n-3} \mathcal{P}_{\mu\nu} I_2^{\mu\nu} \\
TC22(1,1,n,s,m_1^2,m_2^2) &= \left(\mathcal{R}_\mu \mathcal{R}_\nu - \frac{1}{s} \frac{1}{n-3} \mathcal{P}^{\mu\nu} \right) I_2^{\mu\nu}
\end{aligned}
$$

Case N=3:

Invariants: $s_1 = r_1 \cdot r_1$, $s_2 = r_2 \cdot r_2 - 2r_1 \cdot r_2 + r_2 \cdot r_2$, $s_3 = r_2 \cdot r_2$, m_j^2

Case N=3:

Invariants: $s_1 = r_1 \cdot r_1$, $s_2 = r_2 \cdot r_2 - 2r_1 \cdot r_2 + r_2 \cdot r_2$, $s_3 = r_2 \cdot r_2$, m_j^2

$$TI(D,3,1,\mu_1,r_1,r_2,0,m_1,m_2,m_3) = TC31(1)\, r_1^{\mu_1} + TC31(2)\, r_2^{\mu_1}$$

$$TI(D,3,2,\mu_1,\mu_2,r_1,r_2,0,m_1,m_2,m_3) = TC32(0,0)\, g^{\mu_1\mu_2}$$
$$+TC32(1,1)\, r_1^{\mu_1} r_1^{\mu_2} + TC32(1,2)\, (r_1^{\mu_1} r_2^{\mu_2} + r_1^{\mu_2} r_2^{\mu_1}) + TC32(2,2)\, r_2^{\mu_1} r_2^{\mu_2}$$

$$TI(D,3,3,\mu_1,\mu_2,\mu_3,r_1,r_2,0,m_1,m_2,m_3) =$$
$$TC33(0,0,1)\, (g^{\mu_1\mu_2} r_1^{\mu_3} + 2 \text{ perms.}) + \mathrm{TC33}(0,0,2)\, (\mathrm{g}^{\mu_1\mu_2} \mathrm{r}_2^{\mu_3} + 2 \text{ perms.})$$
$$+TC33(1,1,1)\, r_1^{\mu_1} r_1^{\mu_2} r_1^{\mu_3} + TC33(1,1,2)\, (r_1^{\mu_1} r_1^{\mu_2} r_2^{\mu_3} + 2 \text{ perms.})$$
$$+TC33(1,2,2)\, (r_1^{\mu_1} r_2^{\mu_2} r_2^{\mu_3} + 2 \text{ perms.}) + \mathrm{TC33}(2,2,2)\, \mathrm{r}_2^{\mu_1} \mathrm{r}_2^{\mu_2} \mathrm{r}_2^{\mu_3}$$

Case N=3:

Invariants: $s_1 = r_1 \cdot r_1$, $s_2 = r_2 \cdot r_2 - 2r_1 \cdot r_2 + r_2 \cdot r_2$, $s_3 = r_2 \cdot r_2$, m_j^2

$$TI(D,3,1,\mu_1,r_1,r_2,0,m_1,m_2,m_3) = TC31(1)\; r_1^{\mu_1} + TC31(2)\; r_2^{\mu_1}$$

$$TI(D,3,2,\mu_1,\mu_2,r_1,r_2,0,m_1,m_2,m_3) = TC32(0,0)\; g^{\mu_1\mu_2}$$
$$+TC32(1,1)\; r_1^{\mu_1} r_1^{\mu_2} + TC32(1,2)\; (r_1^{\mu_1} r_2^{\mu_2} + r_1^{\mu_2} r_2^{\mu_1}) + TC32(2,2)\; r_2^{\mu_1} r_2^{\mu_2}$$

$$TI(D,3,3,\mu_1,\mu_2,\mu_3,r_1,r_2,0,m_1,m_2,m_3) =$$
$$TC33(0,0,1)\; (g^{\mu_1\mu_2} r_1^{\mu_3} + 2 \text{ perms.}) + \mathrm{TC33}(0,0,2)\; (\mathrm{g}^{\mu_1\mu_2} \mathrm{r}_2^{\mu_3} + 2 \text{ perms.})$$
$$+TC33(1,1,1)\; r_1^{\mu_1} r_1^{\mu_2} r_1^{\mu_3} + TC33(1,1,2)\; (r_1^{\mu_1} r_1^{\mu_2} r_2^{\mu_3} + 2 \text{ perms.})$$
$$+TC33(1,2,2)\; (r_1^{\mu_1} r_2^{\mu_2} r_2^{\mu_3} + 2 \text{ perms.}) + \mathrm{TC33}(2,2,2)\; \mathrm{r}_2^{\mu_1} \mathrm{r}_2^{\mu_2} \mathrm{r}_2^{\mu_3}$$

The projection operators and their properties are ($j \in 1,2,\ H = G^{-1}$)

$$\mathcal{R}_j^\mu = 2 \sum_{i=1}^{2} r_i^\mu H_{ij} \quad , \quad \mathcal{P}^{\mu\nu} = \left[g^{\mu\nu} - 2 \sum_{i,j=1}^{2} r_i^\mu H_{ij} r_j^\nu \right]$$
$$\mathcal{P}^{\mu\nu} \mathcal{R}_\nu = 0 \quad , \quad \mathcal{P}_\rho^\mu \mathcal{P}^{\rho\nu} = \mathcal{P}^{\mu\nu}\ ,\ tr(\mathcal{P}) = (n-2)\ ,\ \mathcal{R}_j^\mu r_{l\mu} = \delta_{jl}$$

Projection operators for $N = 3$:

Projection operators for $N = 3$:

$$
\begin{aligned}
\mathcal{R}_{j\,\mu} I_3^{\mu} &= TC31(j) \\
\frac{1}{n-2}\mathcal{P}_{\mu\nu} I_3^{\mu\nu} &= TC32(0,0) \\
\left(\mathcal{R}_{j_1\,\mu_1}\mathcal{R}_{j_2\,\mu_2} - \frac{2}{n-2}H_{j_1 j_2}\mathcal{P}_{\mu_1\mu_2}\right) I_3^{\mu_1\mu_2} &= TC32(j_1,j_2) \\
\frac{1}{n-2}\mathcal{P}_{\mu_1\mu_2}\mathcal{R}_{j_3\,\mu_3} I_3^{\mu_1\mu_2\mu_3} &= TC33(0,0,j_3) \\
\left(\mathcal{R}_{j_1\,\mu_1}\mathcal{R}_{j_2\,\mu_2}\mathcal{R}_{j_3\,\mu_3} - \frac{2}{n-2}\mathcal{P}_{\mu_1\mu_2}[H_{j_1 j_2}\mathcal{R}_{j_3\,\mu_3} + 2\text{ c.p.}]\right)\mathrm{I}_3^{\mu_1\mu_2\mu_3} &= TC33(j_1,j_2,j_3)
\end{aligned}
$$

Case $N = 4$:

10 invariants: $s = (p_1 + p_2)^2, t = (p_2 + p_3)^2, s_j = p_j^2, m_j^2$,
3,7,13,22 ffs for $R = 1, 2, 3, 4$, respectivly.

Case $N = 4$:

10 invariants: $s = (p_1 + p_2)^2, t = (p_2 + p_3)^2, s_j = p_j^2, m_j^2,$
3,7,13,22 ffs for $R = 1, 2, 3, 4$, respectivly.

$$
\begin{aligned}
\mathcal{R}_{j\,\mu} I_4^{\mu} &= TC41(j) \\
\frac{1}{n-3}\mathcal{P}_{\mu\nu} I_4^{\mu\nu} &= TC42(0,0) \\
\left[\mathcal{R}_{j_1\,\mu_1}\mathcal{R}_{j_2\,\mu_2} - \frac{2}{n-3} H_{j_1 j_2}\mathcal{P}_{\mu_1\mu_2}\right] I_4^{\mu_1\mu_2} &= TC42(j_1, j_2) \\
\frac{1}{n-3}\mathcal{P}_{\mu_1\mu_2}\mathcal{R}_{j_3\,\mu_3} I_3^{\mu_1\mu_2\mu_4} &= TC43(0,0,j_3) \\
\left[\mathcal{R}_{j_1\,\mu_1}\mathcal{R}_{j_2\,\mu_2}\mathcal{R}_{j_3\,\mu_3} - \frac{2}{n-3}\mathcal{P}_{\mu_1\mu_2}[H_{j_1 j_2}\mathcal{R}_{j_3\,\mu_3} + 2\text{ p.}]\right] \mathrm{I}_4^{\mu_1\mu_2\mu_3} &= TC43(j_1, j_2, j_3) \\
\frac{1}{(n-1)(n-3)}\mathcal{P}_{\mu_1\mu_2}\mathcal{P}_{\mu_3\mu_4} I_4^{\mu_1\mu_2\mu_3\mu_4} &= TC44(0,0,0,0) \\
\frac{1}{n-3}\left[\mathcal{P}_{\mu_1\mu_2}\mathcal{R}_{j_3\,\mu_3}\mathcal{R}_{j_4\,\mu_4} - \frac{2}{n-1} H_{j_3 j_4}\mathcal{P}_{\mu_1\mu_2}\mathcal{P}_{\mu_3\mu_4}\right] I_4^{\mu_1\mu_2\mu_3\mu_4} &= TC44(0,0,j_3,j_4 \\
\Big[\mathcal{R}_{j_1\,\mu_1}\mathcal{R}_{j_2\,\mu_2}\mathcal{R}_{j_3\,\mu_3}\mathcal{R}_{j_4\,\mu_4} - 2\frac{\mathcal{P}_{\mu_1\mu_2}}{(n-3)}[H_{j_1 j_2}\mathcal{R}_{j_3\,\mu_3}\mathcal{R}_{j_4\,\mu_4} + 5\text{ p.}] & \\
-\frac{4}{(n-1)(n-3)}\mathcal{P}_{\mu_1\mu_2}\mathcal{P}_{\mu_3\mu_4}\ [H_{j_1 j_2} H_{j_3 j_4} + 2\text{ p.}]\Big] \mathrm{I}_4^{\mu_1\mu_2\mu_3\mu_4} &= TC44(j_1, j_2, j_3, j
\end{aligned}
$$

Numerical evaluation of Feynman parameter integrals

A numerically stable evaluation of the integrals

$$I_N^D(j_1, \ldots, j_r) \quad = \quad (-1)^N \Gamma(N - \frac{D}{2}) \int_0^\infty d^N x \, \delta(1 - \sum_{l=1}^{N} x_l) \frac{x_{j_1} \ldots x_{j_r}}{(x \cdot \mathcal{S} \cdot x/2 + i\varepsilon)^{N-D/2}}$$

would completly solve the 1-loop problem!!!

Numerical evaluation of Feynman parameter integrals

A numerically stable evaluation of the integrals

$$I_N^D(j_1,\ldots,j_r) \quad = \quad (-1)^N \Gamma(N-\frac{D}{2}) \int_0^\infty d^N x\, \delta(1-\sum_{l=1}^N x_l) \frac{x_{j_1}\ldots x_{j_r}}{(x\cdot\mathcal{S}\cdot x/2+i\varepsilon)^{N-D/2}}$$

would complety solve the 1-loop problem!!!

First deal with singularities:

- UV divergencies:
  - Absorbed in $\Gamma(N-\frac{D}{2})$ factor.
  - At one-loop only overall UV divergencies.
  - IR/UV divergencies "algebraically" separated.
  - At most one single pole in $1/\epsilon$.

- IR divergencies:
  - End point zeros of $x \cdot S \cdot x$ lead to non-integrable singularities,
  - at most double pole in $1/(n-4)$.
  - poles extractable by reduction techniques (see above) (or ”iterated sector decomposition”).
  - In the method ”reduction by subtraction” only the 3-point functions contain IR divergencies $\Rightarrow$ easy to evaluate algebraically!

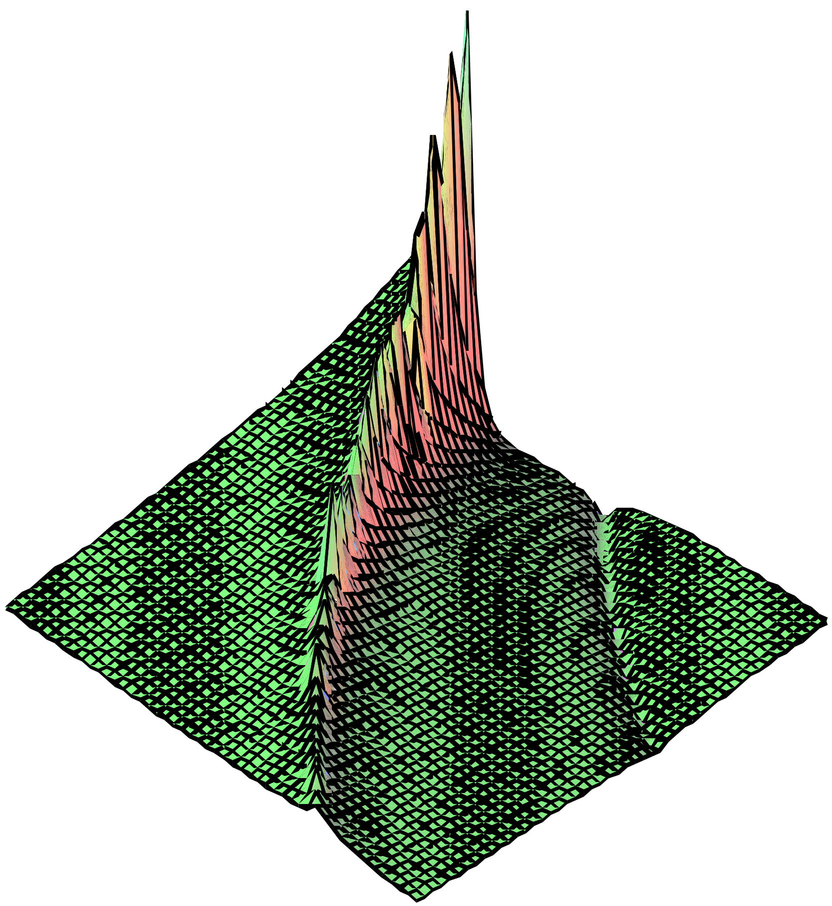
- IR divergencies:
  - End point zeros of $x \cdot S \cdot x$ lead to non-integrable singularities,
  - at most double pole in $1/(n-4)$.
  - poles extractable by reduction techniques (see above) (or "iterated sector decomposition").
  - In the method "reduction by subtraction" only the 3-point functions contain IR divergencies $\Rightarrow$ easy to evaluate algebraically!
- Kinematical singularities:
  - zeros of $x \cdot S \cdot x$ lead to integrable singularities $\Leftrightarrow$ poles in N-dim. complex x-space.
  - In general numerical methods already fail for N=3!

Illustration for an 2-dim. integrable singularity:

Typically:

square root singularity: $dz_1/\sqrt{z_1 - f(z_2, \ldots, z_N)}$

logarithmic singularity: $dz_1 \log(|z_1 - f(z_2, \ldots, z_N)|)$



Contour deformation for parameter integrals:

To get rid of $\delta(1 - \sum x_l)$ make sector decomposition:

$$1 = \sum_{l=1}^{N} \theta(x_l > x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_N)$$

Integral decays into N terms:

$$I_N^D(j_1, \ldots, j_r) = (-1)^N \Gamma(N - D/2) \sum_{l=1}^{N} J_l(N, D, j_1, \ldots, j_r)$$

Contour deformation for parameter integrals:

To get rid of $\delta(1 - \sum x_l)$ make sector decomposition:

$$1 = \sum_{l=1}^{N} \theta(x_l > x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_N)$$

Integral decays into N terms:

$$I_N^D(j_1, \ldots, j_r) = (-1)^N \Gamma(N - D/2) \sum_{l=1}^{N} J_l(N, D, j_1, \ldots, j_r)$$

In sector l make variable transform: $x_j = t_j x_l$ for $(j < l)$, $x_j = t_{j-1} x_l$ for $(j > l)$, integrate out x_l with δ distribution. Defining $\vec{T} = (t_1, \ldots, t_{l-1}, 1, t_l, \ldots, t_{N-1})$ gives:

$$J_l(N, D, j_1, \ldots, j_r) = \int_0^1 d^{N-1}t \left(\sum_{j=1}^{N} T_j\right)^{N-D-r} \frac{T_{j_1} \ldots T_{j_r}}{\left(T \cdot \mathcal{S} \cdot T/2 - i\delta\right)^{N-D/2}}$$

Denominator $Q(t) = T \cdot S \cdot T$ singular if

$$Q(t) = \frac{1}{2} \sum_{j,k=1}^{N-1} A_{jk} t_j t_k + \sum_{j=1}^{N-1} B_j t_j + C = 0$$

A, B, C defined by S.

View $Q(t)$ as value of complex 2-form $Q(x)$ along integration "contour", which is the whole $N-1$ dim. hyper cube.

Denominator $Q(t) = T \cdot S \cdot T$ singular if

$$Q(t) = \frac{1}{2} \sum_{j,k=1}^{N-1} A_{jk} t_j t_k + \sum_{j=1}^{N-1} B_j t_j + C = 0$$

A, B, C defined by S.

View $Q(t)$ as value of complex 2-form $Q(x)$ along integration "contour", which is the whole $N-1$ dim. hyper cube.

To avoid crossing a pole make Ansatz: $\vec{x} = \vec{t} - i\vec{\tau}(\vec{t})$:

$$Q(\vec{x}) = Q(\vec{t}) - \frac{1}{2} \sum_{j,k=1}^{N-1} A_{jk} \tau_j \tau_k - i \sum_{k=1}^{N-1} \tau_k \sum_{j=1}^{N-1} (A_{jk} t_j + B_k)$$

choose $\vec{\tau}$ such that always: $\mathrm{Im}(Q(x)) < 0$!

The following choice is doing the job (proof upon request):

$$\vec{x}(\vec{t}) = \vec{t} - i\,\vec{\tau}(\vec{t})$$

$$\tau_k = \lambda t_k^{\alpha}(1 - t_k)^{\beta} \sum_{j=1}^{N-1} (A_{jk} t_j + B_k)$$

contour deformation parametrized by: α, $\beta > 0$, $\lambda \geq 0$.

The following choice is doing the job (proof upon request):

$$\begin{aligned}\vec{x}(\vec{t}) &= \vec{t} - i\,\vec{\tau}(\vec{t}) \\ \tau_k &= \lambda t_k^{\alpha}(1-t_k)^{\beta}\sum_{j=1}^{N-1}(A_{jk}t_j + B_k)\end{aligned}$$

contour deformation parametrized by: α, β>0, $\lambda \geq 0$.
Invariance under homeomorphisms of the contour, $\mathcal{C}_\lambda$:

$$\int_{\mathcal{C}_0} d^{N-1}x f(x) = \int_{\mathcal{C}_\lambda} d^{N-1}x f(x)$$

In the given parametrisation:

$$\int_0^1 d^{N-1}t\ f(\vec{t}) = \int_0^1 d^{N-1}t\ \det\left(\frac{\partial x_i}{\partial t_j}\right)\ f(\vec{t} - i\vec{\tau}(\vec{t}))$$

$$\frac{\partial x_l}{\partial t_j} = \delta_{lj} - i\,\lambda\,t_l^{\alpha-1}(1-t_l)^{\beta-1}\Big[\delta_{lj}[\alpha(1-t_l)-\beta t_l]\Big(\sum_{k=1}^{N-1}A_{lk}t_k + B_l\Big) + t_l(1-t_l)A_{l}$$

Comments:

- $\lambda\nabla \cdot Q$ controls the size of the deformation, α, β control the smoothness of the deformation at the integration boundaries.

Comments:

- $\lambda\nabla \cdot Q$ controls the size of the deformation, α, β control the smoothness of the deformation at the integration boundaries.

- $Q = 0$ and $\nabla \cdot Q = 0$: exceptional kinematics $\Leftrightarrow$ anomalous threshold (related to $\det(S) = 0$).

  Split integration at: $t_j = -\sum_{l=1}^{N-1} A_{jl}^{-1} B_l$.

Comments:

- $\lambda \nabla \cdot Q$ controls the size of the deformation, α, β control the smoothness of the deformation at the integration boundaries.

- $Q = 0$ and $\nabla \cdot Q = 0$: exceptional kinematics $\Leftrightarrow$ anomalous threshold (related to $\det(S) = 0$).
  Split integration at: $t_j = -\sum_{l=1}^{N-1} A_{jl}^{-1} B_l$.

- UV divergence $\Leftrightarrow N \leq 2 + m$ $(D = 4 + 2m - 2\epsilon)$

$$J_l(N, D, j_1, \ldots, j_r) = \int_0^1 d^{N-1}t \left(\sum_{j=1}^{N} T_j\right)^{N-4-2m-r} \left(T_{j_1} \ldots T_{j_r}\right)\left(T \cdot \mathcal{S} \cdot T\right)^{2+m-N}$$
$$\times \left[1 - \epsilon \log\left(T \cdot \mathcal{S} \cdot T - i\delta\right) + 2\epsilon \log\left(\sum_{j=1}^{N} T_j\right) + \mathcal{O}(\epsilon^2)\right]$$

Comparison between numerical/algebraical method:

- All necessary code can be produced by "Code Generators" from algebraic programs
- Analytic representation of $I_4^{n+2}(j_1, \ldots, j_r) \sim 1/\det(G)^r$

Comparison between numerical/algebraical method:

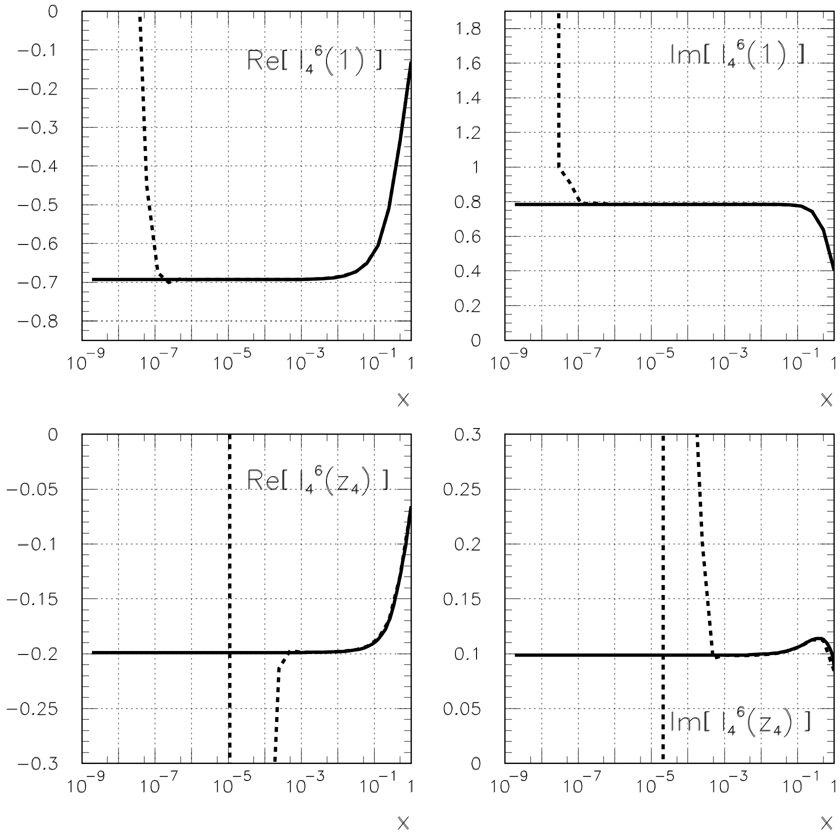
- All necessary code can be produced by "Code Generators" from algebraic programs
- Analytic representation of $I_4^{n+2}(j_1,\ldots,j_r) \sim 1/\det(G)^r$

Consider the following $2 \to 2$ kinematics:

$$
\begin{aligned}
p_1 &= (E(x), 0, 0, xM) \\
p_2 &= (E(x), 0, 0, -xM) \\
p_3 &= E(x)\,(1, 0, \sin(\theta), \cos(\theta)) \\
p_4 &= E(x)\,(1, 0, -\sin(\theta), -\cos(\theta)) \\
E(x) &= M\,\sqrt{1+x^2}
\end{aligned}
$$

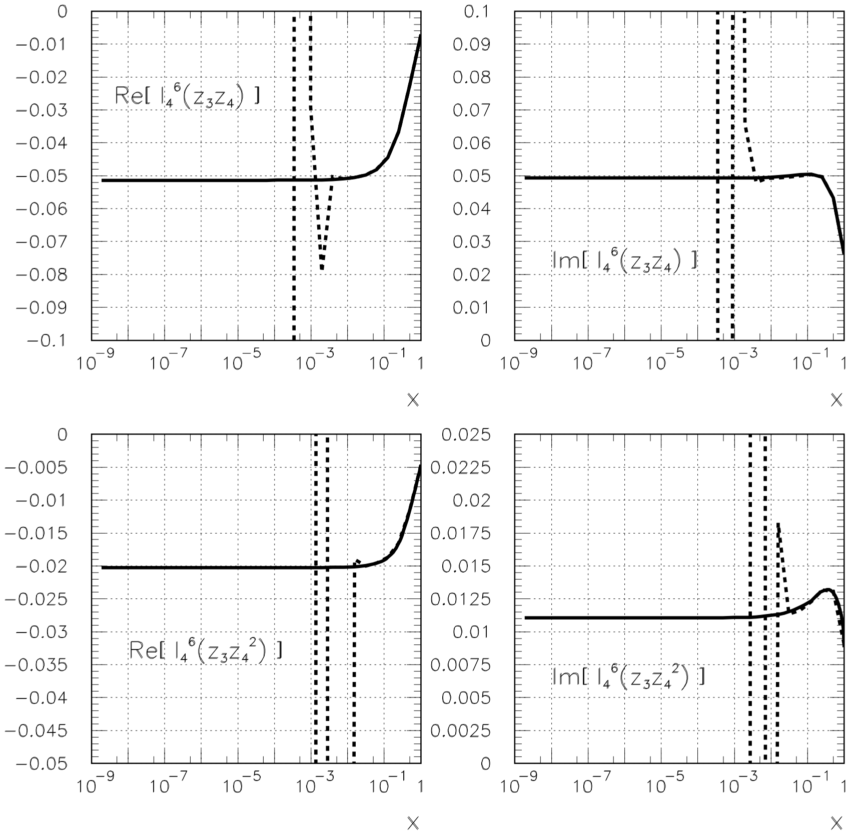
Phase space boundaries: $x \to 0$, $\theta \to 0 \Rightarrow \det(G) = 0$.

Real and imaginary parts of the basis integrals, $I_4^6(1)$, $I_4^6(z_4)$:



- Full line: numerical implementation
- Dashed line: algebraical implementation

Real and imaginary parts of the basis integrals, $I_4^6(z_3 z_4)$, $I_4^6(z_3 z_4^2)$:



Conclusion for the practitioner:

Use fast and accurate algebraic formulas for the "bulk" of the phase space, switch to slow but reliable numerical evaluation at the phase space boundary!

Summary:

One-loop $2 \to N$ matrix elements for BSM+SM processes

- Sort amplitudes by gauge, colour and analyticity structure
- Amplitude organization: decomposition of scattering tensor
  $\Rightarrow$ To be done once and forever for NV,$(N-2)V + f\bar{f}$,$(N-4)V + 2S + f\bar{f}$,...
- Generate graphs (and UV counterterms) for a given amplitude
  (e.g. QGRAPH, FeynArts, ...)
- Project onto amplitude coefficients
  deal with *reducible* scalar products first
  tensor reduction of *irreducible* $N \geq 5$ point tensor integrals
- n-dim.form factor representation of 1- to 4-point functions
  or use numeric approach from here on.

- Sort into basis set of two, three, four point scalar integrals
  ⇒ Basis set for each amplitude can be determined automatically
  ⇒ IR, UV divergent part can be isolated explicitly!
- Simplify coefficients analytically using algebraic programs
- Combine N-point loop amplitude with (N+1) tree level amplitude
  ⇒ IR divergencies cancel
- Evaluate cross section numerically
  ⇒ Phenomenological prediction!

Summary:

Loads of NLO calculations necessary for the LHC, e.g.

- $PP \to N$ jets
- $PP \to \gamma\gamma + 0, 1, 2$ jets
- $PP \to t\bar{t} + 0, 1, 2$ jets
- $PP \to WW, ZZ + 0, 1, 2$ jets
- $PP \to l^+l^- + 0, 1, 2$ jets
- $PP \to l^+l^- + 0, 1, 2$ jets
- $PP \to$???

Hopefully a future computer algebra package will do the work for us...

keep tuned and help!!!