Multi-Leg Amplitudes at One Loop

Methods, Algorithms, Automatization and Numerics.

CAPP 2005, Zeuthen

Thomas Binoth

binoth@physik.uni-wuerzburg.de

University of Würzburg

Content:

- Introduction
- Amplitude organization
- Reduction of one-loop N-point scalar integrals
- Reduction of one-loop N-point tensor integrals
- Numerical evaluation of Feynman diagrams
- Summary and Outlook

Introduction

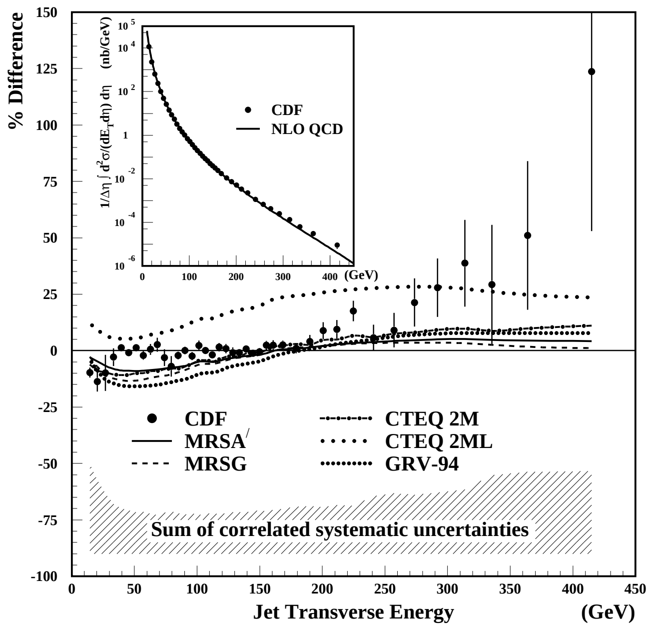
The decade of hadron colliders at the TeV scale

Tevatron: $P\bar{P}$ collider at Fermilab:

Run I: $\sqrt{s} = 1.8$ TeV (1992-1996)

Run II: $\sqrt{s} = 1.96$ TeV (2001-2007 $\pm$ x)

- discovery of the top quark (1994/95)
- electroweak, b and jet physics
- new physics searches: leptoquarks, SUSY, extra dimensions...



CDF, Abe et al., Phys. Rev. Lett. 77, 438 (1996).

The decade of hadron colliders at the TeV scale

LHC: PP collider at CERN, $\sqrt{s} = 14$ TeV, start 2007 $+$ x

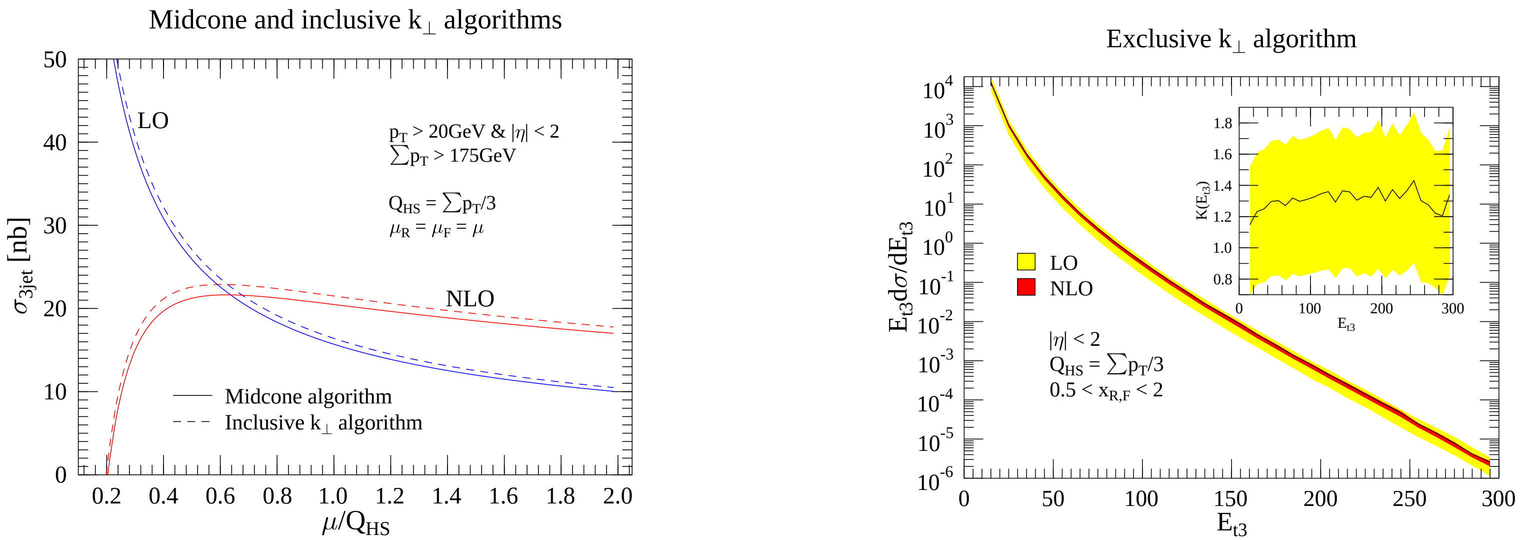
- Higgs mechanism, electroweak symmetry breaking
- low energy SUSY models
- strong interaction scenarios like Technicolour theories
- models with space time dimensions > 4
- ...

LHC may give hints on physics we did not even think about !!!

Keep tuned for the unexpected !!!

Scale uncertainties:

Example: 3 jet cross section at NLO



Z. Nagy, Phys.Rev. D68 (2003).

> Higher order QCD calculations are mandatory to soften scale dependence of phenomenological predictions !!!

Observables have to be defined infrared safe !!!

i.e. insensitive to emission of an extra soft/collinear quark or gluon.

Jet rates at the LHC

Number of jets:	3	4	5	6	7	8
σ/nb	91.4	6.54	0.46	0.032	0.002	0.0002

$p_T(jet) > 60$ GeV, $\theta_{ij} > 30^0$, $|\eta_j| < 3$

[Draggiotis,Kleiss,Papadopoulos, EPJ C24 (2002)]

Multi–particles/jet production plays a very important role !!!

Problems with leading order predictions:

- Scale dependence: N–jet cross sections behave $\sim \alpha_s(\mu)^N$
  $\Rightarrow$ To have predictions for jet rates NLO corrections have to be included
- Peripheral phase space regions: degenerate partonic configurations at LO are sensitive to extra parton emission $\Rightarrow$ estimates for backgrounds — especially after severe cuts — may be considerably underestimated
- Jet structure: the more partons are in the amplitudes the more precise the jet structure is described

Amplitude organization

Amplitude should be partitioned in structures or SMEs = "standard matrix elements" respecting

- colour structure
- gauge symmetries
- Bose symmetries

Compensations have to happen inside coefficients to a given structure

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Compensations have to happen inside coefficients to a given structure

- Algebraic approach $\Rightarrow$ explicite cancellations of denominators
- Numeric approach $\Rightarrow$ splitting of problem into pieces leads to better numerical performance

Example: $gg \to hhh$

Kinematics: $g(p_1, \lambda_1) + g(p_2, \lambda_2) + h(p_3) + h(p_4) + h(p_5) \to 0$

Amplitude: $\Gamma(gghhh \to 0) = \varepsilon_{1\,\mu_1}^{\lambda_1} \varepsilon_{2\,\mu_2}^{\lambda_2} \mathcal{M}^{\mu_1 \mu_2}$

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Scattering tensor $\mathcal{M}^{\mu_1\mu_2}$ has decomposition:
$(p_5 = -p_1 - p_2 - p_3 - p_4,\ j_1, j_2 \in \{1,2,3,4\})$

$$\mathcal{M}^{\mu_1\mu_2} \;=\; A\, g^{\mu_1\mu_2} + \sum B_{j_1 j_2}\, p^{\mu_1}_{j_1}\, p^{\mu_2}_{j_2}$$

$\to$ 17 coefficients

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$\to$ 17 coefficients

(4-dim.) Transversality: $p_1 \cdot \varepsilon_1 = p_2 \cdot \varepsilon_2 = 0$

$$\Rightarrow\ j_1 \in \{2, 3, 4\}\ ,\ j_2 \in \{1, 3, 4\}$$

$\to$ 10 coefficients

Solving Ward identities

WI (1): $\mathcal{M}^{\varepsilon_1 p_2} = 0$

WI (2): $\mathcal{M}^{p_1 \varepsilon_2} = 0$

Solving Ward identities

$$\text{WI (1):} \qquad \mathcal{M}^{\varepsilon_1 p_2} = 0$$
$$\text{WI (2):} \qquad \mathcal{M}^{p_1 \varepsilon_2} = 0$$

Solve WI (1):

$$\begin{aligned} 0 &= \varepsilon_2 \cdot p_1 \Big[B_{21}\, p_1 \cdot p_2 + B_{31}\, p_1 \cdot p_3 + B_{41}\, p_1 \cdot p_4 + A \Big] \\ &+ \varepsilon_2 \cdot p_3 \Big[B_{23}\, p_1 \cdot p_2 + B_{33}\, p_1 \cdot p_3 + B_{43}\, p_1 \cdot p_4 \Big] \\ &+ \varepsilon_2 \cdot p_4 \Big[B_{24}\, p_1 \cdot p_2 + B_{34}\, p_1 \cdot p_3 + B_{44}\, p_1 \cdot p_4 \Big] \end{aligned}$$

$\rightarrow$ 3 independent equations! E.g. solve for B_{21}, B_{23}, B_{24}.

Solving Ward identities

$$\mathcal{M}^{\varepsilon_1\varepsilon_2} = -\frac{A}{s_{12}}\big(2\,\varepsilon_1\cdot p_2\;\varepsilon_2\cdot p_1 - s_{12}\;\varepsilon_1\cdot\varepsilon_2\big)$$

$$-2\frac{B_{31}}{s_{12}}\varepsilon_2\cdot p_1\big(p_3\cdot p_1\;\varepsilon_1\cdot p_2 - p_3\cdot\varepsilon_1\;p_1\cdot p_2\big) - 2\frac{B_{33}}{s_{12}}\varepsilon_2\cdot p_3\big(p_3\cdot p_1\;\varepsilon_1\cdot p_2 - p_3\cdot\varepsilon_1\;p_1\cdot p_2\big)$$

$$-2\frac{B_{34}}{s_{12}}\varepsilon_2\cdot p_4\big(p_3\cdot p_1\;\varepsilon_1\cdot p_2 - p_3\cdot\varepsilon_1\;p_1\cdot p_2\big) - 2\frac{B_{41}}{s_{12}}\varepsilon_2\cdot p_1\big(p_4\cdot p_1\;\varepsilon_1\cdot p_2 - p_4\cdot\varepsilon_1\;p_1\cdot p_2\big)$$

$$-2\frac{B_{43}}{s_{12}}\varepsilon_2\cdot p_3\big(p_4\cdot p_1\;\varepsilon_1\cdot p_2 - p_4\cdot\varepsilon_1\;p_1\cdot p_2\big) - 2\frac{B_{44}}{s_{12}}\varepsilon_2\cdot p_4\big(p_4\cdot p_1\;\varepsilon_1\cdot p_2 - p_4\cdot\varepsilon_1\;p_1\cdot p_2\big)$$

Solving Ward identities

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use field strength tensor $\mathcal{F}_j^{\mu\nu} = p_j^\mu\varepsilon_j^\nu - \varepsilon_j^\mu p_j^\nu$:

$$\begin{aligned} p_1\cdot p_2\,\mathcal{M}^{\varepsilon_1\varepsilon_2} &= -A\frac{1}{2}\mathrm{tr}(\mathcal{F}_1\mathcal{F}_2) \\ &\quad + p_2\cdot\mathcal{F}_1\cdot p_3\Big(B_{31}\varepsilon_2\cdot p_1 + B_{33}\varepsilon_2\cdot p_3 + B_{34}\varepsilon_2\cdot p_4\Big) \\ &\quad + p_2\cdot\mathcal{F}_1\cdot p_4\Big(B_{41}\varepsilon_2\cdot p_1 + B_{43}\varepsilon_2\cdot p_3 + B_{44}\varepsilon_2\cdot p_4\Big) \end{aligned}$$

gauge symmetry for gluon 1 manifest!

Solving Ward identity (2):

$\varepsilon_2 \to p_2 \Rightarrow \mathcal{F}_2 \to 0.$

$$\begin{aligned} 0 &= p_2 \cdot \mathcal{F}_1 \cdot p_3 \Big(B_{31}\, p_2 \cdot p_1 + B_{33}\, p_2 \cdot p_3 + B_{34}\, p_2 \cdot p_4 \Big) \\ &+ p_2 \cdot \mathcal{F}_1 \cdot p_4 \Big(B_{41}\, p_2 \cdot p_1 + B_{43}\, p_2 \cdot p_3 + B_{44}\, p_2 \cdot p_4 \Big) \end{aligned}$$

leads to 2 new linear equations.

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&+ p_2 \cdot \mathcal{F}_1 \cdot p_4 \Big(B_{41}\, p_2 \cdot p_1 + B_{43}\, p_2 \cdot p_3 + B_{44}\, p_2 \cdot p_4 \Big)
\end{aligned}
$$

leads to 2 new linear equations.
solve for B_{31}, B_{41} and use field strength tensor:

$$
\begin{aligned}
(p_1 \cdot p_2)^2 \, \mathcal{M}^{\varepsilon_1 \varepsilon_2} &= -A \frac{p_1 \cdot p_2}{4} \mathrm{tr}(\mathcal{F}_1 \mathcal{F}_2) \\
&+ B_{33}\;\; p_2 \cdot \mathcal{F}_1 \cdot p_3\;\; p_1 \cdot \mathcal{F}_2 \cdot p_3 + \;B_{34}\;\; p_2 \cdot \mathcal{F}_1 \cdot p_3\;\; p_1 \cdot \mathcal{F}_2 \cdot p_4 \\
&+ B_{43}\;\; p_2 \cdot \mathcal{F}_1 \cdot p_4\;\; p_1 \cdot \mathcal{F}_2 \cdot p_3 + \;B_{44}\;\; p_2 \cdot \mathcal{F}_1 \cdot p_4\;\; p_1 \cdot \mathcal{F}_2 \cdot p_4
\end{aligned}
$$

gauge invariance + momentum conservation

$\to$ 5 coefficients left (n-dim.)

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"Bianchi identity" $\Rightarrow$ $p_3 \cdot \mathcal{F}_1 \cdot p_4$ not independent:

$$p_1 \cdot p_2 \;\; p_3 \cdot \mathcal{F}_1 \cdot p_4 \;\; = \;\; p_1 \cdot p_3 \;\; p_2 \cdot \mathcal{F}_1 \cdot p_4 - p_1 \cdot p_4 \;\; p_2 \cdot \mathcal{F}_1 \cdot p_3$$

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- Fast road: Choose gauge $\varepsilon_1 \cdot p_2 = \varepsilon_2 \cdot p_1 = 0$!

$$\mathcal{M}^{\varepsilon_1 \varepsilon_2} \;\; = \;\; A\, \varepsilon_1 \cdot \varepsilon_2 + \sum_{j_1, j_2 \in \{3,4\}} B_{j_1 j_2}\, p_{j_1} \cdot \varepsilon_1 \; p_{j_2} \cdot \varepsilon_2$$

Gauge choice "solves" WIs directly!

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- In $D = 4$ and $N \geq 5$: $g_{\mu\nu}$ not independent object

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$p_1, \ldots, p_4$ lin. independent ("non-exceptional kinematics")

$$G_{j_1 j_2} = 2\, p_{j_1} \cdot p_{j_2}\ ,\ H = G^{-1}$$
$$\varepsilon_1 \cdot \varepsilon_2 = 2 \sum_{j_1 j_2 \in \{1,2,3,4\}} H_{j_1 j_2} \varepsilon_1 \cdot p_{j_1} \varepsilon_2 \cdot p_{j_2}$$

$\Rightarrow$ "A" is not independent coeff. in $D = 4$
source of $1/\det(G)$ in N-point amplitudes.

Only 4 coefficients left: $B_{33}, B_{34}, B_{43}, B_{44}$

Solving Bose symmetry for gluons and Higgs bosons: $\mathcal{S}_2 \otimes \mathcal{S}_2$

(Eliminating p_5 reduces $\mathcal{S}_2 \otimes \mathcal{S}_3$ to $\mathcal{S}_2 \otimes \mathcal{S}_2$)

$$\mathcal{M}^{\varepsilon_1 \varepsilon_2}(p_1, p_2) = \mathcal{M}^{\varepsilon_2 \varepsilon_1}(p_2, p_1) \quad , \quad \mathcal{M}^{\varepsilon_1 \varepsilon_2}(p_3, p_4) = \mathcal{M}^{\varepsilon_1 \varepsilon_2}(p_4, p_3)$$

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$$\begin{aligned} B_{33}(p_1, p_2) &= B_{33}(p_2, p_1) \quad , \quad B_{44}(p_1, p_2) = B_{44}(p_2, p_1) \\ A(p_1, p_2) &= A(p_2, p_1) \quad , \quad B_{43}(p_1, p_2) = B_{34}(p_2, p_1) \\ B_{43}(p_3, p_4) &= B_{34}(p_4, p_3) \quad , \quad B_{44}(p_3, p_4) = B_{33}(p_4, p_3) \\ A(p_3, p_4) &= A(p_4, p_3) \end{aligned}$$

$\rightarrow$ only 2 independent coefficients left!

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$\rightarrow$ only 2 independent coefficients left!

Tip for the practitioner:

Compute all coefficients and check Bose symmetry and WIs!

An example with fermions:

Kinematics: $b(p_1, \lambda_1) + \bar{b}(p_2, \lambda_2) + \gamma(p_3, \lambda_3) + h(p_4) \to 0$

Amplitude: $\Gamma(b + \bar{b} + \gamma + h \to 0) = \varepsilon^{\lambda_3}_{3\,\mu_3} \mathcal{M}^{\mu_3}$

use Dirac equation, transversality, momentum conservation:

$$\not{p}_1 u_1 = m_b u_1, \; \bar{v}_2 \not{p}_2 = -m_b \bar{v}_2,$$
$$p_3 \cdot \varepsilon_3 = 0, \; p_4 = -p_1 - p_2 - p_3$$

$$\Rightarrow \quad \mathcal{M}^{\varepsilon_3} = \sum_\alpha C^\alpha \bar{v}_2 \Gamma^\alpha u_1, \quad \Gamma^\alpha \in \{\not{p}_3, \not{\varepsilon}_3, \not{p}_3 \not{\varepsilon}_3\}$$

Solve Ward identity:

$$\mathcal{M}^{\varepsilon_3} = (C_1^1\, p_1 \cdot \varepsilon_3 + C_2^1\, p_2 \cdot \varepsilon_3)\, \bar{v}_2\, \not{p}_3\, u_1 + C^2\, \bar{v}_2\, \not{\varepsilon}_3\, u_1 + C^3 \bar{v}_2\, \not{p}_3 \not{\varepsilon}_3\, u_1$$

$$\mathcal{M}^{p_3} = 0 \iff C_1^1\, p_1 \cdot p_3 + C_2^1\, p_2 \cdot p_3 + C^2 = 0$$

Solve Ward identity:

$$\mathcal{M}^{\varepsilon_3} = (C_1^1\, p_1\cdot\varepsilon_3 + C_2^1\, p_2\cdot\varepsilon_3)\, \bar{v}_2\, \not{p}_3\, u_1 + C^2\, \bar{v}_2\, \not{\varepsilon}_3\, u_1 + C^3 \bar{v}_2\, \not{p}_3 \not{\varepsilon}_3\, u_1$$

$$\mathcal{M}^{p_3} = 0 \quad \Leftrightarrow \quad C_1^1\, p_1\cdot p_3 + C_2^1\, p_2\cdot p_3 + C^2 = 0$$

use $J^\mu = \bar{v}_2 \gamma^\mu u_1$ and field strength tensor:

$$\mathcal{M}^{\varepsilon_3} = C_1^1\, J\cdot\mathcal{F}_3\cdot p_1 + C_2^1\, J\cdot\mathcal{F}_3\cdot p_2 + \frac{1}{2} C^3 \bar{v}_2 \not{\mathcal{F}}_3 u_1$$

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$$\mathcal{M}^{p_3} = 0 \;\Leftrightarrow\; C_1^1\, p_1\cdot p_3 + C_2^1\, p_2\cdot p_3 + C^2 = 0$$

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- Manifest gauge invariant representations
  $\Rightarrow$ ideal starting point!
- for electroweak computations more involved.
- Complicated processes lead to big system of linear equations$\Rightarrow$ use Computer algebra!

Reduction of N-point scalar integrals

$$
\begin{aligned}
I_N^n &= \int \frac{d^n k}{i\pi^{n/2}} \frac{1}{(q_1^2 - m_1^2 + i\delta)\ldots(q_N^2 - m_N^2 + i\delta)} \\
q_j &= k - r_j = k - p_1 - p_2 \cdots - p_j \\
D_j &= q_j^2 - m_j^2 + i\delta
\end{aligned}
$$

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Make ansatz for reduction formula:

$$
1 = \left(1 - \sum_{j=1}^{N} b_j D_j\right) + \sum_{j=1}^{N} b_j D_j
$$

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\end{aligned}
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$$

$\Rightarrow$ "pinched" graphs plus rest:

$$
I_N^n = \sum_{j=1}^{N} b_j I_{N-1,j}^n + I_{Remainder}
$$

Compute remainder term using Feynman parameters:

$$
\begin{aligned}
I_{Rem.} &= \int d^n\kappa \frac{1-\sum_{j=1}^{N} b_j D_j}{\prod_{j=1}^{N} D_j} \\
&= \Gamma(N) \int d^n\kappa \; dx_1 \ldots dx_n \; \delta(1-\sum_{l=1}^{N} x_l) \frac{1-\sum_{j=1}^{N} b_j D_j}{\mathcal{D}^N}
\end{aligned}
$$

Compute remainder term using Feynman parameters:

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$$= \Gamma(N) \int d^n\kappa \, dx_1 \ldots dx_n \, \delta(1-\sum_{l=1}^N x_l) \frac{1-\sum_{j=1}^N b_j D_j}{\mathcal{D}^N}$$

$$\mathcal{D} = \sum_{j=1}^N \left(x_j k^2 - 2x_j k\cdot r_j + x_j r_j \cdot r_j - x_j m_j^2\right) + i\delta$$

$$= (k-R)^2 - R^2 - \sum_{j=1}^N x_j (r_j \cdot r_j - m_j^2) + i\delta$$

using $\quad R = \sum_j x_j r_j \quad , \quad d^n\kappa = d^n k/(i\pi^{n/2})$

Make shift: $k \to k + R$, $R^2 = \sum_{j,l} x_l x_j r_l \cdot r_j =: x \cdot G \cdot x/2$

$$\begin{aligned}
\mathcal{D} &= k^2 - \sum_{j,l} x_l x_j r_l \cdot r_j + \sum_{j,l} x_l x_j r_j \cdot r_j - \sum_{j,l}^{N} x_l x_j m_j^2 + i\delta \\
&= k^2 - \frac{1}{2} \sum_{j,l} x_l x_j \left[-(r_j - r_l)^2 + m_l^2 + m_j^2 \right] + i\delta \\
&= k^2 - \frac{1}{2} x \cdot S \cdot x + i\delta
\end{aligned}$$

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$$\begin{aligned}\mathcal{D} &= k^2 - \sum_{j,l} x_l x_j r_l \cdot r_j + \sum_{j,l} x_l x_j r_j \cdot r_j - \sum_{j,l}^{N} x_l x_j m_j^2 + i\delta \\ &= k^2 - \frac{1}{2}\sum_{j,l} x_l x_j \left[-(r_j - r_l)^2 + m_l^2 + m_j^2\right] + i\delta \\ &= k^2 - \frac{1}{2} x \cdot S \cdot x + i\delta\end{aligned}$$

- S contains complete kinematical information of the graph: IR divergences, thresholds.
- for later: $S_{ij} = G_{ij} - v_i - v_j$
  $G_{ij} = 2r_i \cdot r_j$ "Gram" matrix, $v_i = r_i \cdot r_i - m_i^2$

Deal with numerator:

$$\mathcal{N} = 1 - \sum_j b_j (k^2 - 2k \cdot r_j + r_j \cdot r_j + 2k \cdot R + R^2 - 2R \cdot r_j - m_j^2)$$

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Terms $\sim k^\mu$ give zero because of "symmetric integration":

$$\begin{aligned} \mathcal{N} &= 1 - (\sum_j b_j) \left(k^2 + x \cdot G \cdot x/2\right) - \sum_j b_j [r_j^2 - m_j^2 - 2R \cdot r_j] \\ &= -(\sum_j b_j) \left(k^2 + x \cdot S \cdot x/2\right) - (\sum_j b_j)(\sum_l x_l v_l) \\ &\quad -(\sum_j x_j)(\sum_l b_l v_l) + \sum_{l,j} b_l x_j G_{lj} + 1 \end{aligned}$$

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$$\Rightarrow \quad \mathcal{N} = -(\sum_j b_j)(k^2 + x \cdot S \cdot x/2) + \sum_j x_j (1 + \sum_l S_{jl} b_l)$$

Choose $b_{l\in\{1,\ldots,N\}}$ such that $\boxed{\sum_j S_{jl} b_l = -1}$

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Scalar integral in arbitrary dimensions:

$$\begin{aligned} I_N^D &= \int d^D\kappa \frac{1}{\prod_j D_j} \\ &= \Gamma(N) \int d^D\kappa\, dx_1 \dots dx_N\, \delta(1 - \sum_l x_l) \frac{1}{(k^2 - M^2)^N} \\ &= (-1)^N \Gamma(N - D/2) \int d^N x\, \delta(1 - \sum_l x_l)(M^2)^{D/2-N} \end{aligned}$$

$$\int d^n\kappa \frac{k^2 + M^2}{(k^2 - M^2)^N} = \int d^n\kappa \frac{(k^2 - M^2) + 2\ M^2}{(k^2 - M^2)^N}$$

$$= (-1)^N (M^2)^{n/2-N+1} / \Gamma(N)$$

$$\times[-(N-1)\Gamma(N-(n+2)/2) + 2\ \Gamma(N - n/2)]$$

$$= [N - n - 1](-1)^N \frac{\Gamma(N-(n+2)/2)}{\Gamma(N)}\ (M^2)^{(n+2)/2-N}$$

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$$\Rightarrow \boxed{I_{Rem.} \ = \ -(\sum_j b_j)(N-n-1)\ I_N^{n+2}}$$

Solution for $(S \cdot b)_l + 1 = 0$ for general N:

- S is regular for non-exceptional momenta and $N \leq 6$:
  $$b_j = -\sum_l (S^{-1})_{jl}$$
- G is regular for non-exceptional momenta and $N \leq 5$, rank$(G) = \min(N-1, 4)$
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Ansatz: $$\sum_{j=1}^{N-1} G_{lj} b_j = 0 \, , \; \sum_{j=1}^{N} b_j = 0 \, , \; \sum_{j=1}^{N} v_j b_j = 1$$

Solution using "Moore-Penrose" generalized inverse:

Each symmetric matrix G has a uniquely defined pseudo-inverse H defined by the properties:

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Construction of the Moore-Penrose inverse to the Gram matrix G for $N \geq 5$:

$$r_j^\mu = \sum_{m=1}^{4} R_{mi} E_m^\mu \ , \ \tilde{G}_{ij} = 2 \ E_i \cdot E_j$$

$$\Rightarrow \quad H = R^T \cdot (R \cdot R^T)^{-1} \cdot \tilde{G}^{-1} \cdot (R \cdot R^T)^{-1} \cdot R$$

Explicite solution:

$(b_1, \ldots, b_{N-1})$ is in kernel of G, dim(kernel)$=N-5$.
Construct basis of kernel: $\{K \cdot v/(v \cdot K \cdot v), U^{(1)}, \ldots, U^{(N-6)}\}$, with $v \cdot U^{(j)} = 0$. General solution:

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$$\begin{aligned} b_j &= \frac{(K \cdot v)_j + \sum_{k=1}^{N-6} \beta_k U_j^{(k)}}{(v \cdot K \cdot v)} \quad \text{for } j \in \{1, \ldots, N-1\} \\ b_N &= -\sum_{j=1}^{N-1} b_j \end{aligned}$$

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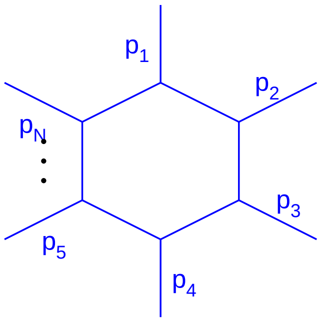
- $N = 6 \Rightarrow$ solution unique
- $N > 6 \Rightarrow \beta_k$'s can be chosen freely, e.g. $\beta_k = 0$.
- Exceptional kinematics: rank$(G) = d < \min(N-1, 4)$:
  $r_j^\mu = \sum_{m=1}^{d} R_{mi} E_m^\mu$ defines R, $\tilde{G}$ and H

General scalar integral reduction formula:

with $\sum_j b_j = -\det(G)/\det(S)$, $n = 4 - 2\epsilon$:

$I_N^n =$



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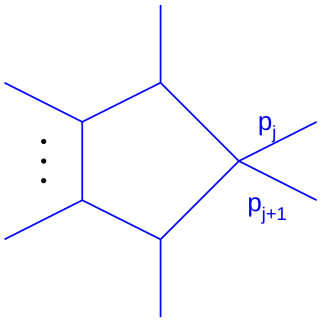

$$+ \begin{cases} -(1+2\epsilon)\frac{\det(G)}{\det(S)}\, I_N^{n+2} & , N = 4 \\ \mathcal{O}(\epsilon) & , N = 5 \\ 0 & , N \geq 6 \end{cases}$$

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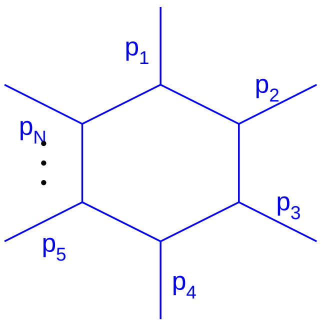

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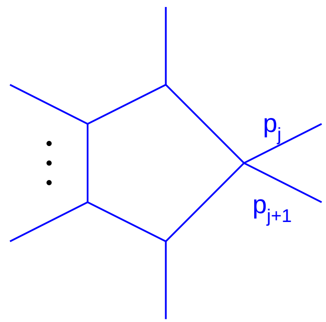

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By iteration:

Any N point integral can be represented by n–dimensional triangle functions and (n+2) dimensional box functions. The latter are infrared finite.
Reduction of N-point scalar integrals completly solved!

Infrared power counting:

Consider massless integrals with light-like external legs, $m_j^2 = p_j \cdot p_j = 0$:

$$I_N^n = \int \frac{d^n k}{i\pi^{n/2}} \frac{1}{q_1^2 \ldots q_N^2}$$

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Soft divergence: $q_j \to 0 \Leftrightarrow k \to r_j$. W.r.o.g. $k = \lambda k, \lambda \to 0$:

$$I_N^D \sim \int \frac{d^D k}{i\pi^{D/2}} \frac{\lambda^{D-4}}{k^2 k \cdot p_1 k \cdot p_N} \frac{1}{r_2^2 \ldots r_{N-2}^2}$$

$\Rightarrow$ integral divergent for $D \leq 4$, convergent for $D > 4$.

Collinear divergence:

$k^2 = 0$ ($k^0 \neq 0!$) necessary condition for collinear divergence.
Collinear limit $k \to p_j$, $p_j \cdot p_j = 0$, defined by
($n \cdot n = n \cdot k_T = p_j \cdot k_T = 0$):

$$k^\mu = z\, p_j + k_T^\mu - \frac{k_T \cdot k_T}{2zp_1 \cdot n} n^\mu$$

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Collinear divergence: $k \to z p_j \Leftrightarrow k_T = \lambda k_T, \lambda \to 0.$
W.r.o.g. $p_j = p_1 = E_1(1,0,0,1), n = p_6$:

$$I_N^D \sim \int \frac{dk^0 dk^3 d^{D-2} k_T}{i\pi^{D/2}} \frac{\lambda^{D-4}}{[k_0^2 - k_3^2 + i\delta] k_T^2} \times \text{(regular terms)}$$

$\Rightarrow k_T$ integral divergent for $D \leq 4$, convergent for $D > 4$.

(for $z \to 0$ one gets a soft divergence in addition.)