QCD Parton Evolution

Fabian Spill

Theory group, DESY Zeuthen
Email: spill@physik.hu-berlin.de

In hard hadronic scattering reactions one uses Parton Distribution Functions to parametrise the parton content of the hadron. Even though one cannot derive the Parton Distribution functions in perturbative QCD, one can calculate their evolution with the hard momentum scale. Practically this is done by computer software like the programme "partonevolution" discussed in this paper.

1 Introduction

1.1 Deep Inelastic Scattering and the Parton Model

Scattering of hadrons at high energies probes the parton content, i.e. the quarks and gluons. The lowest order diagram for electron-proton-scattering, as it takes place at HERA, is shown in Fig.(1). Here the electron has momentum k before and k' after the interaction, the protons initial

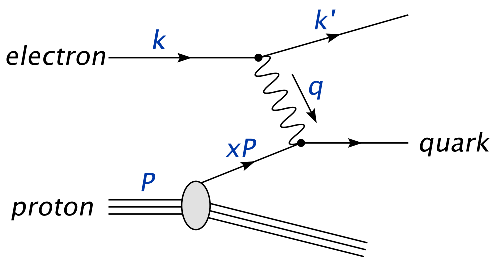

Fig. 1: *e-p-scattering*

momentum is P, hence we have a momentum transfer of $q = k' - k$. Due to confinement the quarks in the final state hadronize, so one measures hadronic jets, which have a total momentum $P' = P + q$. We denote the momentum fraction carried by the quark by $x = \frac{p}{P}$, ignoring the momentum $p_\perp$ transverse to the direction of motion of the hadron, take $Q^2 = -q^2 > 0$ as the variable for the momentum transfer and introduce the Parton distribution functions(PDF) $q(x, Q^2)$ which are probability density functions, i.e. $q(x, Q^2)dx$ is the probability of finding a quark q with momentum fraction in the interval $[x, x+dx]$. At higher energies we allow for quark-antiquark pair creation, so we do not just have u or d quarks in our nucleon but also s, c, b, t and the antiquarks $\overline{q}$. Furthermore it was measured that $\int x \sum_{flavours}(q(x, Q^2) + \overline{q}(x, Q^2)) \approx 0.5$, so by momentum conservation about half of the momentum is carried by the gluons, whose distibution is $g(x, Q^2)$. In the naive parton model the structure functions are given by the simple sum of the quark and antiquark densities, i.e. $F_2 = \sum_{flavours}(q + \overline{q})$ and the PDFs do not depend on the energy scale Q^2. This effect is called scaling. If we allow for QCD corrections the situation changes and the PDFs become scale dependent. This is called scaling violation and will be discussed in the following chapter.

1.2 QCD effects and the Altarelli-Parisi equation

To take QCD effects into account we consider processes like gluon radiation or pair production from gluons. The lowest order diagrams for those processes are shown in Fig.(2). The probability

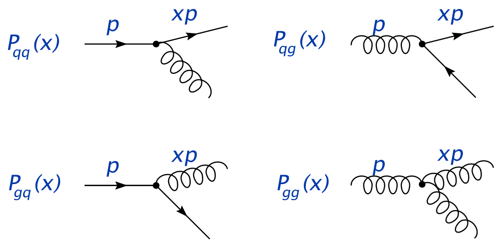

Fig. 2: *the splitting functions*

densities for those processes are given by the splitting functions $P_{gg}, P_{gq}, P_{qg}, P_{qq}$ which can be calculated in perturbative QCD by expanding them in powers of the strong coupling constant. At lowest order this leads to diagrams like Fig.(3). We also get a scale dependence of our PDFs

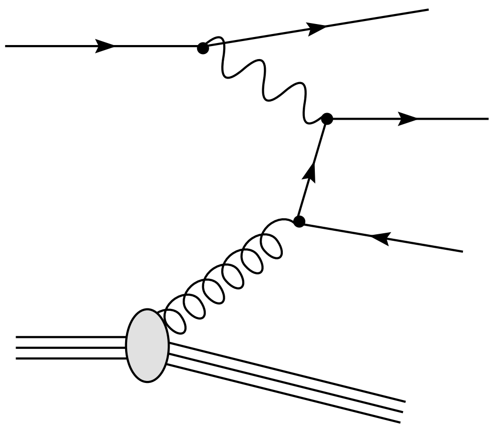

Fig. 3: *photon gluon fusion*

which is described by the Altarelli-Parisi equation[1][1]

$$\frac{\partial}{\partial \ln Q^2} f_a(x, Q^2) = \sum_j P_{ab}(x, Q^2) \otimes f_b(x, Q^2) \ . \tag{1}$$

Here a and b denote the different partons,i.e. quarks and gluons and $\otimes$ stands for the Mellin convolution defined by

$$A(x) \otimes B(x) := \int_0^1 dy \int_0^1 dz \delta(x - yz) A(y) B(z)$$

We will deal with the solution of the evolution equation within Perturbation Theory in the next chapter.

2 Perturbative QCD

2.1 The running coupling constant

To solve the evolution equation we want to expand it in powers of the strong coupling constant α_s. Before doing that we have to note that α_s itself depends on the scale Q^2, it actually decreases with

[1] this equation is also known as evolution equation or DGLAP(Dokshitzer-Gribov-Lipatov-Altarelli-Parisi) equation

increasing Q^2. This leads to asymptotic freedom and is crucial for the application of perturbation theory at high energy scales. We take $a_s = \frac{\alpha_s}{4\pi}$ as the variable to get rid of powers of 4π which would appear later and introduce the beta function

$$\frac{d\,a_{\mathrm{s}}}{d\ln Q^2} = \beta(a_{\mathrm{s}}) \tag{2}$$

as the function which describes the scale dependence of a_s. This function itself depends on a_s, so to solve this equation we expand beta in powers of a_s. This reads

$$\beta_{\mathrm{N^mLO}}(a_{\mathrm{s}}) = -\sum_{k=0}^{m} a_{\mathrm{s}}^{k+2}\,\beta_k \ , \tag{3}$$

where N^mLO stands for m times Next-to-Leading order. Today one knows the coefficients of the beta function up to N^3LO, and letting n_f be the number of quark flavours at scale Q^2 one gets for the first three

$$\begin{aligned}\beta_0 &= 11 - 2/3\,n_f \\ \beta_1 &= 102 - 38/3\,n_f \\ \beta_2 &= 2857/2 - 5033/18\,n_f + 325/54\,n_f^2 \ .\end{aligned} \tag{4}$$

We note that β_0 is positive for $n_f \leq 16$, which has the important physical consequence of a decreasing coupling constant with rising scale, which was mentioned above. This leads to the phenomenon of asymptotic freedom. Furthermore we note that n_f increases by one at each flavour threshold, hence the β_i's are also not constant and in general imply discontinuities of the coupling constant (details depend on the renormalisation scheme). Instead of numerically solving equation (2), one can, as it is done by the Particle Data Group[2], expand the solution in powers of $\frac{1}{L} = \frac{1}{\ln \frac{Q^2}{\Lambda^2_{QCD}}}$, where Λ^2_{QCD} is the QCD scale parameter. This reads to NLO

$$a_s = \frac{1}{L} - \frac{\beta_1 \ln L}{L^2 \beta_0} \ . \tag{5}$$

2.2 Mellin N-space and expansion of the moments

In this section we present the solution of equation (1) in the so called N-space. Taking the Mellin transform

$$A^N := \int_0^1 x^{N-1} A(x) dx \ , \tag{6}$$

of equation (1) has the advantage of turning the righthand side into a simple product, we get:

$$\frac{\partial}{\partial \ln Q^2} f_a^N(Q^2) = \sum_a P_{ab}^N(Q^2) f_b^N(Q^2) \tag{7}$$

Dropping the indices a and b we start expanding the moments of the splitting functions, with i denoting the expansion index:

$$P^N(Q^2) = a_{\mathrm{s}} \sum_{i=0}^{\infty} a_{\mathrm{s}}^i P_i^N \tag{8}$$

Now we rewrite equation (7) in terms of the coupling constant and get

$$\frac{\partial}{\partial \ln a_{\mathrm{s}}} f_i^N(Q^2) = \frac{P^N}{\beta(a_{\mathrm{s}})} f^N(Q^2). \tag{9}$$

Into this equation we insert the expansions for P and for a_{s} and expand again in a_{s}, so to NLO we get:

$$\frac{\partial f^N(Q^2)}{\partial \ln a_{\mathrm{s}}} = -\frac{P_0^N}{\beta_0 a_{\mathrm{s}}} - \frac{1}{\beta_0}(P_1^N - \frac{\beta_1}{\beta_0} P_0^N) f^N(Q^2). \tag{10}$$

To solve this equation we introduce the evolution operator E^N as the operator evolving f^N from one energy scale Q_0 to another scale Q, i.e.

$$f^N(Q^2) = E^N(a_\mathrm{s}(Q^2), a_\mathrm{s}(Q_0^2))f^N(Q_0^2) \tag{11}$$

Thus the evolution operator follows the same differential equation as the PDFs.

One problem remaining in N-space is, after the easy solution of the problem, the inverse transformation. The N appearing in the last equations is complex, and the inverse transformation involves a contour through the complex plane. In its simplest form this is just a straight line from $c - i\infty$ to $c + i\infty$, where c is any number right of the rightmost pole of the Mellin transform. The inverse transformation is then given by

$$A(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z} A^N dN \ . \tag{12}$$

Now it turns out that the functions we are dealing with just have poles on the real axis, hence by the residue theorem we can deform this contour as much as we want as long as we don't intersect the real axis. This actually is used for a better numerical convergence of the integral in section (3.1).

2.3 Flavour decomposition

Equation (7) in fact represents a system of $2n_f + 1$ (n_f for all quarks and antiquarks and one for the gluon) differential equations. But we have a number of additional constraints. Since the strong interaction is independent of the flavour the quark-gluon and the gluon-quark splitting functions are flavour independent and we get

$$P_\mathrm{gq} \ \equiv \ P_{\mathrm{gq}_i} \ = \ P_{\mathrm{g}\bar{\mathrm{q}}_i} \ , \quad P_\mathrm{qg} \ \equiv \ n_f\, P_{\mathrm{q}_i\mathrm{g}} \ = \ n_f\, P_{\bar{\mathrm{q}}_i\mathrm{g}} \ . \tag{13}$$

From that follows that the so-called non-singlet distributions $q_i - q_j$, $q_i - \bar{q}_j$, $\bar{q}_i - \bar{q}_j$ and combinations of them do not couple to the gluon density. Further symmetries are given for the quark-quark, quark-antiquark and antiquark-antiquark splitting functions, however, here one gets special terms for the valence quark cases:

$$\begin{aligned} P_{\mathrm{q}_i\mathrm{q}_k} \ &= \ P_{\bar{\mathrm{q}}_i\bar{\mathrm{q}}_k} \ &= \ \delta_{ik}P_\mathrm{qq}^\mathrm{v} + P_\mathrm{qq}^\mathrm{s} \\ P_{\mathrm{q}_i\bar{\mathrm{q}}_k} \ &= \ P_{\bar{\mathrm{q}}_i\mathrm{q}_k} \ &= \ \delta_{ik}P_{\mathrm{q}\bar{\mathrm{q}}}^\mathrm{v} + P_{\mathrm{q}\bar{\mathrm{q}}}^\mathrm{s} \ . \end{aligned} \tag{14}$$

Hence we get three types of non-singlet distributions evolving with different splitting functions, that are the flavour asymmetries $q_{\mathrm{ns},ik}^{\pm} = q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$ evolving with $P_\mathrm{ns}^{\pm} = P_\mathrm{qq}^\mathrm{v} \pm P_{\mathrm{q}\bar{\mathrm{q}}}^\mathrm{v}$ and the total valence distribution $q_\mathrm{ns}^\mathrm{v} = \sum_{r=1}^{n_f}(q_r - \bar{q}_r)$ evolving with $P_\mathrm{ns}^\mathrm{v} = P_\mathrm{qq}^\mathrm{v} - P_{\mathrm{q}\bar{\mathrm{q}}}^\mathrm{v} + n_f(P_\mathrm{qq}^\mathrm{s} - P_{\mathrm{q}\bar{\mathrm{q}}}^\mathrm{s}) \equiv P_\mathrm{ns}^- + P_\mathrm{ns}^\mathrm{s}$.

The sea consisting of the sum of all quark and antiquarks distributions is given by $q_\mathrm{s} = \sum_{r=1}^{n_f}(q_r + \bar{q}_r)$, it couples maximally to the gluons and evolves with the equation

$$\frac{d}{d\ln Q^2}\begin{pmatrix} q_\mathrm{s} \\ g \end{pmatrix} = \begin{pmatrix} P_\mathrm{qq} & P_\mathrm{qg} \\ P_\mathrm{gq} & P_\mathrm{gg} \end{pmatrix} \otimes \begin{pmatrix} q_\mathrm{s} \\ g \end{pmatrix} \ . \tag{15}$$

P_{qq} is given by

$$P_\mathrm{qq} \ = \ P_\mathrm{ns}^+ + n_f(P_\mathrm{qq}^\mathrm{s} + P_{\bar{\mathrm{q}}\mathrm{q}}^\mathrm{s}) \ \equiv \ P_\mathrm{ns}^+ + P_\mathrm{ps} \ , \tag{16}$$

that means it is composed of a non-singlet and a pure-singlet term.

3 Partonevolution

3.1 Design of the programme

The programme "partonevolution", which was first written by Weinzierl[3] in C++, numerically solves the evolution equation in N-space. As one can see from equation (11) we need a functional

form of $f(x, Q_0^2)$ for some fixed scale Q_0^2 as an input. Due to the physical nature of the PDFs we assume a form like $xf(x, Q_0^2) = \sum_i A_i x^{\alpha_i}(1-x)^{\beta_i}$, so the user input consists of the coefficients A_i, α_i, β_i. Unfortunately, those input distributions cannot be calculated in perturbative QCD, instead one gets them from a fit to experimental data. One hopes that eventually one will also be able to calculate them with the help of Lattice QCD.
Another user input is the value of the strong coupling constant $\alpha_s(Q_0^2)$ at the initial scale Q_0^2. After choosing the final scale Q^2, x and the required order to which the calculation should be performed (for the moment, just LO and NLO are implemented) the programme calculates the value of $q(x, Q^2)$ for the non-singlet or both values for $q_s(x, Q^2), g(x, Q^2)$ in the singlet case. The contour used for the inverse Mellin transform is the parabola described by Kosower[4], it is optimized in such a way that it gives a good numerical performance also for just a few support points chosen for the numerical evaluation of the integral with the Gauss-Laguerre quadrature formula. Additional user inputs are the number of points for the evaluation of the Gauss-Laguerre quadrature formula, the masses of the heavy quarks (which also enables one to go to fixed flavour evolution by just setting the masses in such a way that none lies between Q_0 and Q) and the method of how to calculate α_s, that is either the numerical solution of equation (2) or taking the truncated solution to the required order as in equation (5). Since the programme itself consists just of a library we have summarized the user input functions in Appendix A.

3.2 Numerical results

In the following we compare the results calculated by this programme with values from the similar programme "QCD Pegasus" by Vogt[5]. As initial conditions we took the values set up at the Les Houches meeting, see chapter 1.3 of reference[6]. For the parametrization of the PDFs we have

$$\begin{aligned}
xu_v(x, Q_0^2) &= 5.107200\, x^{0.8}\, (1-x)^3 \\
xd_v(x, Q_0^2) &= 3.064320\, x^{0.8}\, (1-x)^4 \\
xg\,(x, Q_0^2) &= 1.700000\, x^{-0.1}(1-x)^5 \\
x\bar{d}\,(x, Q_0^2) &= .1939875\, x^{-0.1}(1-x)^6 \\
x\bar{u}\,(x, Q_0^2) &= (1-x)\, x\bar{d}\,(x, Q_0^2) \\
xs\,(x, Q_0^2) &= x\bar{s}\,(x, Q_0^2) \\
&= 0.2\, x(\bar{u}+\bar{d}\,)(x, Q_0^2),
\end{aligned} \tag{17}$$

where the evolution is chosen to start at $Q_0^2 = 2GeV^2$ and the coupling constant is given by $\alpha_s(Q_0^2) = 0.35$. The heavy quark masses are taken as

$$m_{\rm c} = Q_0\,, \quad m_{\rm b} = 4.5 \text{ GeV}\,, \quad m_{\rm t} = 175 \text{ GeV}. \tag{18}$$

We calculated our values at a scale of $Q^2 = 10000GeV^2$ for $10^{-7} \leq x \leq 0.9$. For the Gauss-Laguerre quadrature formula we took the maximum of 30 points. At LO we got a very good agreement for the non-singlet distributions (agreement for the first 5 digits) and an error of less than 1% for the singlet gluon and sea distributions. At NLO we got a higher difference of less than 3% for both the non-singlet valence distributions and the quark and the sea distribution. For reasons not yet known the L_--distribution differs by a factor of three for $x = 10^{-7}$. In table (1) and table (2) in the Appendix we give the values of the programmes "partonevolution" and "QCD-Pegasus" at NLO.
The values calculated by "QCD-Pegasus" have already been compared to values from another independently developed programme (see again chapter 1.3 of reference[6]) and it was checked that the values from those programmes do not differ by more than one part in 10^5 at the "Les Houches" test. That is an indication that the difference of the values calculated by "partonevolution" and "QCD-Pegasus" are caused by numerical errors of "partonevolution". One source of these errors might be an insufficient number of points for the quadrature formula. Especially for the singlet distributions one still gets an improvement of about 5% for small x when changing from 20 to 30

points in the quadrature formula, so one might get even better values after implementing more support points. However, the values for the non-singlet distributions stabilize when changing from 20 to 30 points and there is still a bigger deviation to the values from "QCD-Pegasus", so there must be another, yet unidentified error source.

Acknowledgments

I would like to thank my supervisor, Sven-Olaf Moch, for his great support during my time at Desy Zeuthen. Furthermore, I would like to thank Karlheinz Hiller for organising the summer student programme, and finally thanks to all summer students for the good time we had together.

References

[1] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

[2] Particle Data Group, S. Eidelman *et al.*, Phys. Lett. **B592**, 1 (2004).

[3] S. Weinzierl, Comput. Phys. Commun. **148**, 314 (2002), hep-ph/0203112.

[4] D. A. Kosower, Nucl. Phys. **B506**, 439 (1997), hep-ph/9706213.

[5] A. Vogt, (2004), hep-ph/0408244.

[6] W. Giele *et al.*, (2002), hep-ph/0204316.

A Parameters for "partonevolution"

In the following we just list the beginning of an example c++-file where the initial parameters are set. Here we refer to the version of "partonevolution" modified by the author of this article. Not all of these options are included in version 1.0 by Weinzierl. A general manual how to use the library is found in reference [3].

```
//select how to evolve alpha
//possible values:truncate_in_one_over_L, exact
alpha_strong::method = alpha_strong::exact;


//choose to which order evolution should be performed
//possible values LO,NLO
evolutionkernel::order = evolutionkernel::LO;

//set the heavy quark masses in GeV
alpha_strong::charm_threshold = sqrt(2.0);
alpha_strong::bottom_threshold = 4.5;
alpha_strong::top_threshold = 175;


//set the initial scale
double Q0 = sqrt(2.0);

//additionally one has to set the following variable
//on the initial scale, this variable is used for
//the evolution of alpha0
alpha_strong::mu0 = Q0;

//set the value of alpha strong at Q0
alpha_strong::alpha0 = 0.35;

//the following few lines create an initial parton
//distribution for the valence u. The eta value has
//to be -1 for non-singlet valence distributions and
//1 otherwise.In the singlet case it is ignored anyway
  double Q = 100;
  int eta = -1;
  double A_u[1]     = { 5.107200};
  double alpha_u[1] = { 0.8 };
  double beta_u[1]  = { 3.0 };
  partondistribution u =
    partondistribution(1,Q0,A_u,alpha_u,beta_u,eta);
```

B Reference results

Tab. 1: *results from the programme "partonevolution"*

NLO, $Q^2 = 10^4$ GeV^2					
x	xu_v	xd_v	xL_-	$x\Sigma$	xg
10^{-7}	1.1173^{-4}	6.5597^{-5}	1.5073^{-6}	3.3696^{+2}	1.0752^{+3}
10^{-6}	5.6890^{-4}	3.3290^{-4}	8.5233^{-6}	1.6654^{+2}	4.9747^{+2}
10^{-5}	2.8036^{-3}	1.6343^{-3}	4.6840^{-5}	7.8333^{+1}	2.1946^{+2}
10^{-4}	1.3267^{-2}	7.6938^{-3}	2.4779^{-4}	3.4032^{+1}	8.8201^{+1}
10^{-3}	5.9126^{-2}	3.3982^{-2}	1.2284^{-3}	1.3308^{+1}	3.0057^{+1}
10^{-2}	2.8036^{-1}	1.3009^{-1}	5.1725^{-3}	4.5836^{+0}	7.7422^{+0}
0.1	5.5164^{-1}	2.7165^{-1}	9.9218^{-3}	1.4237^{+0}	8.5955^{-1}
0.3	3.4985^{-1}	1.2992^{-1}	2.9930^{-3}	5.2924^{-1}	8.0157^{-2}
0.5	1.2077^{-1}	3.1423^{-2}	3.7542^{-4}	1.5538^{-1}	7.7895^{-3}
0.7	2.0004^{-2}	3.0773^{-3}	1.3365^{-5}	2.3150^{-2}	3.7965^{-4}
0.9	3.4988^{-4}	1.7723^{-5}	8.6094^{-9}	3.6764^{-4}	1.2162^{-6}

Tab. 2: *results from the programme "QCD Pegasus"*

NLO, $Q^2 = 10^4$ GeV^2					
x	xu_v	xd_v	xL_-	$x\Sigma$	xg
10^{-7}	1.0927^{-4}	6.4125^{-5}	4.3925^{-6}	3.329^{+2}	1.1167^{+3}
10^{-6}	5.5533^{-4}	3.2498^{-4}	1.9829^{-5}	1.6489^{+2}	5.2289^{+2}_{*}
10^{-5}	2.7419^{-3}	1.5989^{-3}	8.5701^{-5}	7.7078^{+1}	2.2753^{+2}
10^{-4}	1.3039^{-2}	7.5664^{-3}	3.5582^{-4}	3.3429^{+1}	8.9513^{+1}
10^{-3}	5.8507^{-2}	3.3652^{-2}	1.4329^{-3}	1.3117^{+1}	3.0245^{+1}
10^{-2}	2.3128^{-1}	1.2978^{-1}	5.3472^{-3}	4.5440^{+0}	7.7491^{+0}
0.1	5.5324^{-1}	2.7252^{-1}	9.9709^{-3}	1.4265^{+0}	8.5586^{-1}
0.3	3.5129^{-1}	1.3046^{-1}	3.0061^{-3}	5.3159^{-1}	7.9625^{-2}
0.5	1.2130^{-1}	3.1564^{-2}	3.7719^{-4}	1.5609^{-1}	7.7265^{-3}
0.7	2.0101^{-2}	3.0932^{-3}	1.3440^{-5}	2.3265^{-2}	3.7574^{-4}
0.9	3.5232^{-4}_{*}	1.7855^{-5}	8.6800^{-9}	3.7020^{-4}	1.1954^{-6}