On Beam Energy Measurement Using Resonance Absorption

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In this abstract I follow the method proposed by R.A.Melikian to use resonance absorption in a static magnetic field for measuring the energy of a electron beam in an accelerator setup. I want to understand and numerically evaluate the formulae presented in order to determine the relative strength of absorbed intensity and the relative error obtainable in later measurements.

1 Introduction

We want to measure the energy of an electron/positron beam as precisely as possible, at least up to a relative error of 10^{-4}. Mainly, two types of measurements demand this precise knowledge of the center of mass energy $\sqrt{s}$ (see Ref. [2]):

- *1. Lineshape scan of threshold energy* e.g. measuring the top quark mass by scanning $t\bar{t}$ production threshold at $\sqrt{s} = 350GeV$
- *2. Direct reconstruction of particle resonance* e.g. continuum measurement in order to find the Higgs boson mass by reconstructing the Higgstrahlung process $e^+e^- \to HZ$.

Traditional methods mostly use magnetic spectrometers in the following way:

The beam is deflected via three magnets. Two auxilliary magnets - bending the beam out of the track and back into it - and one spectrometer magnet - deflecting the beam by an angle Θ (see fig. 1). This angle depends on the electron's momentum. Beam Position Monitors measure the position of the beam at nanometer-precision. If the angle Θ of deflection and the magnetic field strength $\int_{magnet} Bdl$ are known to high precision, use:

$$E_b = \frac{c \cdot e}{\Theta} \int_{magnet} Bdl$$

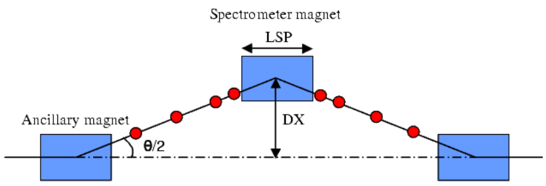

Fig. 1: *Schematics of the spectrometer setup, the blocks being the magnets, the circles the Beam Position Monitors - taken from [2]*

Ref. [1] proposes a different approach which we want to follow and discuss here.

2 Relativistic Landau Levels

Energy levels in a magnetic field:

Electrons/positrons, before the interaction point a static magnetic field $\vec{B} = (0, 0, B_z)$ arbitrarily set in z-direction, at the speed of light.

Basically, the whole idea is about resonance absorption, for that we need level splitting. So we first have to calculate the energy eigenvalues of electrons within a B-field.

Let's begin with [4] §32, which is helpful.

First of all, we need the DIRAC equation, as the particles are relativistic, and minimal substitution for the electromagnetic field:

$$i\hbar\frac{\delta}{\delta t}\overrightarrow{\psi}(\overrightarrow{r}) = \Big(c\overrightarrow{\alpha}\cdot(\frac{\hbar}{i}\overrightarrow{\nabla}-e\overrightarrow{A})+m_0c^2\beta\Big)\psi \quad (1)$$

Our vector potential in the Landau Gauge (from $\overrightarrow{B} = rot\overrightarrow{A}$) we set as $\overrightarrow{A} = B_0x\overrightarrow{e_y}$.

And we make the following stationary ansatz with two 2-component spinors: $\psi(r) = \exp\frac{iEt}{t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$, which upon insertion into (1) gives the following for the left hand side.

$$i\hbar\frac{\delta}{\delta t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = E\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

For the right hand side however we need the dirac matrices in the standard form:

$$\alpha^i = \begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}, \beta = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

So that.

$$\Big(c\overrightarrow{\alpha}\cdot(\frac{\hbar}{i}\overrightarrow{\nabla}-e\overrightarrow{A})+m_0c^2\beta\Big)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$
$$= \Big(\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})$$
$$+m_0c^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Big)\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$
$$= \begin{pmatrix}m_0c^2 & \sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\\ \sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A}) & -m_0c^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

This results then in a 2-component mixed differential equation.

$$\begin{pmatrix}E\varphi\\ E\chi\end{pmatrix} = \begin{pmatrix}m_0c^2\varphi+\sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\chi\\ m_0c^2\chi+\sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\varphi\end{pmatrix}$$

$$\begin{pmatrix}(E-m_0c^2)\varphi\\ (E+m_0c^2)\chi\end{pmatrix} = \begin{pmatrix}\sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\chi\\ \sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\varphi\end{pmatrix} \quad (2)$$

By multiplying (2), top, with $(E+m_0c^2)$, we then can replace $(E+m_0c^2)\chi$ with the bottom part and receive an equation of φ alone.

$$(E^2-m_0^2c^4)\varphi = \sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\sigma^i\cdot(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\varphi \quad (3)$$

In order to solve this equation, we utilise:

$$\overrightarrow{\sigma}\cdot\overrightarrow{\alpha}\overrightarrow{\sigma}\cdot\overrightarrow{\beta} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\overrightarrow{\alpha}\cdot\overrightarrow{\beta}+i\overrightarrow{\sigma}(\overrightarrow{\alpha}\times\overrightarrow{\beta}) \quad (4)$$

Which is simply proven by calculation with $\overrightarrow{\alpha}\cdot\overrightarrow{\sigma} = \Sigma_{i=1}^3 a_i\sigma^i$, where σ^i are the Pauli matrices.

To simplify the eq. (3) we perform an auxiliary calculation using eq. (4)

$$(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})^2$$
$$+i\overrightarrow{\sigma}\underbrace{(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})\times(\frac{\hbar c}{i}\overrightarrow{\nabla}-ec\overrightarrow{A})}_{\text{mind: }\overrightarrow{\nabla}\times\overrightarrow{\nabla}=\overrightarrow{A}\times\overrightarrow{A}=0}$$
$$= c^2(\frac{\hbar}{i}\overrightarrow{\nabla}-e\overrightarrow{A})^2+i\overrightarrow{\sigma}\frac{e\hbar c^2}{i}\underbrace{[\overrightarrow{\nabla}\times\overrightarrow{A}]_+}_{=2\cdot rot\overrightarrow{A}=2\overrightarrow{B}}$$
$$= c^2(\frac{\hbar}{i}\overrightarrow{\nabla}-e\overrightarrow{A})^2-2e\hbar c^2\overrightarrow{\sigma}\overrightarrow{B}$$

and obtain.

$$(E^2-m_0^2c^4)\varphi = \Big[c^2(\frac{\hbar}{i}\overrightarrow{\nabla}-e\overrightarrow{A})^2-2e\hbar c^2\overrightarrow{\sigma}\overrightarrow{B}\Big]\varphi$$
$$= \Big[(\frac{\hbar c}{i})^2\Delta-\frac{e\hbar c^2}{i}(\underbrace{\underbrace{\overrightarrow{\nabla}\overrightarrow{A}+\overrightarrow{A}\overrightarrow{\nabla}}_{=0+2\cdot\overrightarrow{A}\overrightarrow{\nabla}}}_{=2\cdot B_0x\frac{\delta}{\delta y}})+\underbrace{e^2c^2\overrightarrow{A}^2}_{(ecB_0x\overrightarrow{e_y})^2}-\dots$$
$$= \Big[(\frac{\hbar c}{i})^2\Delta$$
$$-2\frac{e\hbar c^2}{i}B_0x\frac{\delta}{\delta y}+(ecB_0x)^2-2e\hbar c^2\overrightarrow{\sigma}\overrightarrow{B}\Big]\varphi$$

φ is a composite wave function of three space coordinates $\overrightarrow{r}$ and of spin ξ. Using the setup's symmetries, we apply a separation ansatz $\varphi(\overrightarrow{r}) = e^{i(k_yy+k_zz)}\phi(x)\xi$ and bring both $e^{i(k_yy+k_zz)}\cdot\xi$ onto the other side (in order to get rid of them[1]). However in doing so minding the operators Δ, $\frac{\delta}{\delta y}$ and $\overrightarrow{\sigma}$, using the eigenvalue equation of the spin eigenfunction $\overrightarrow{\sigma}\xi = m_s\xi$ ($m_s = \pm\frac{1}{2}$), $\frac{\delta}{\delta x_i}e^{i(k_{x_i}x_i)} = ik_{x_i}e^{i(k_{x_i}x_i)}$ and $\overrightarrow{\nabla}^2 = \Delta = \frac{\delta}{\delta x}^2+\frac{\delta}{\delta y}^2+\frac{\delta}{\delta z}^2$, we get:

$$(E^2-m_0^2c^4)\phi = \Big[\hbar^2c^2(k_y^2+k_z^2-\frac{\delta^2}{\delta x^2})$$
$$-2e\hbar c^2B_0xk_y+e^2c^2B_0^2x^2-2e\hbar c^2B_0m_s\Big]\phi$$

$$\frac{1}{c^2}(E^2-m_0^2c^4-\hbar^2c^2k_z^2+2e\hbar c^2B_0m_s)\phi =$$
$$\underbrace{[\hbar^2k_y^2-2\hbar k_yeB_0x+e^2B_0^2x^2-\hbar^2\frac{\delta^2}{\delta x^2}]}_{(\hbar k_y-eB_0x)^2-\hbar^2\frac{\delta^2}{\delta x^2}}\phi$$

Now using the substitution $x_0 = -\frac{\hbar}{eB_0}k_y$ and

[1] by multiplying with each complex conjugate and knowing $<\dots|\dots>=1$

$(x - x_0) = \chi$

$$
\begin{aligned}
&(\hbar k_y - eB_0 x)^2 - \hbar^2 \frac{\delta^2}{\delta x^2} \\
&= e^2 B_0^2 \underbrace{(x - x_0)}_{=\chi}{}^2 - \hbar^2 \frac{\delta^2}{\delta x^2} \\
&= e^2 B_0^2 \chi^2 + i^2 \hbar^2 \frac{\delta^2}{\delta \chi^2} \\
&= e^2 B_0^2 \chi^2 + p_\chi^2
\end{aligned}
$$

and dividing by $2m_0$ we obtain.

$$
\begin{aligned}
&\frac{1}{2m_0 c^2}(E^2 - m_0^2 c^4 - \hbar^2 c^2 k_z^2 + e\hbar c^2 B_0 m_s)\phi \\
&= \Big(\frac{p_\chi^2}{2m_0} + \frac{m_0}{2} \underbrace{\frac{e^2 B_0^2}{m_0^2}}_{=\omega_c} \chi^2 \Big)\phi(\chi) \quad (5)
\end{aligned}
$$

If we now compare on the right hand side the term in the brackets with the one-dimensional Hamiltonian of a harmonic oscillator (mass m_0, $\omega_c = \frac{eB_0}{m_0}$), the energy eigenvalues of the electron within a static magnetic field follow analogously:

$$
\hbar\omega(n + \frac{1}{2}) = \hbar \frac{eB_0}{m_0}(n + \frac{1}{2}) = \frac{1}{2m_0 c^2}(E^2 - m_0^2 c^4 - \hbar^2 c^2 k_z^2 + 2e\hbar c^2 B_0 m_s)
$$

$$
\boxed{E_{n,m_s} = \sqrt{\hbar \frac{eB_0}{m_0} m_0 c^2 (2n + 1 + 2m_s) + m_0^2 c^4 + c^2 p_z^2}} \quad (6)
$$

Eq. (6) is in perfect agreement with the formula given in Ref. [1]. To repeat: E_{n,m_s} is the energy level of the electron, m_0 its mass, e its charge, $p_z = \hbar k_z$ its momentum in z-direction, m_s its spin eigenvalue projected onto the z-axis, B_0 is the strength of the magnetic field and $\frac{eB}{m_0} = \omega_c$ is the cyclotron frequency.

To give you a *classical* picture, have a look at fig. 2. An electron in a magnetic field will perform a helical motion (if angle $\varphi \neq 0$ between z-axis and beam). The center is (x_0, y_0), as the same substitution in eq. (5) can be done with the y-coordinate if the other Landau Gauge is being used. And the oscillating nature of the motion was mentioned above.

Furthermore as the motion in the xy-plane is finite (constrained) it is quantized to certain radii, where an integral part of the electron's DeBroglie wave length fits into the circumference. The rotational frequency is ω_c.

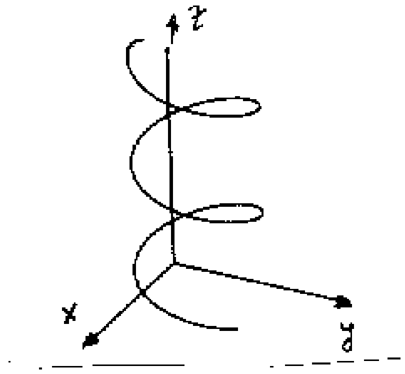

Fig. 2: *The helical motion of an electron in a static magnetic field -graph taken from Ref. [5]*

3 Basic Principle

So, now we have the electron beam with different levels. This we are going to exploit.

A laser beam (wave length λ) will shoot photons with the energy $\hbar\omega = 2\pi\hbar\frac{c}{\lambda}$ to the electron beam under a certain angle Θ (see fig. 3).

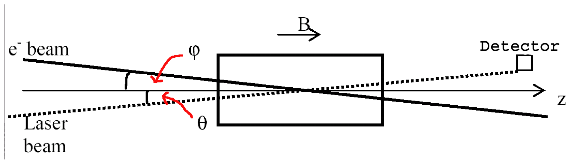

Fig. 3: *The setup with the electron beam on top and the laser beam below in a static magnetic field B still with the possible freedom of an angle between each beam and z-axis for generality - graph taken from [1]*

If under certain conditions the laser photons have the right energy, they can be resonantly absorbed by the electrons. This decrease in laser intensity can be measured and then maximum of resonance absorption allows to determine the beam energy. In order to arrive there, there are still some formulae ahead of us.

4 The Absorption Process and Conditions

As we have shown the energy of the electrons within the static magnetic field is given by eq. (6).

If a laser photon is being absorbed, energy and momentum have to be conserved:

$$E_{n,m_s} + \hbar\omega = E_{n',m_s'} \tag{7}$$

$$p_{z,0} + \frac{\hbar\omega}{c}\cos\Theta = p_z \tag{8}$$

Here $p_{z,0}$ is the original momentum of the electron in z-direction, p_z is the one after absorption. If $\varphi \neq 0$ these are just the momentum's component in z-direction under this angle.

These equations will lead us to a condition for the right photon energy $\hbar\omega$ with respect to E_{n,m_s}, B, Θ, φ. Due to the square root in equation (6), the levels are not as easily recognizable as in atom physics.

However, besides retrieving the correct photon energy for the transition $n \to n'$, we can resolve eq. (20) also for all the other parameters, especially $\gamma_0 = \frac{E_0}{m_0c^2} = \frac{E_{0,-\frac{1}{2}}}{m_0c^2}$. This will give us eventually a formula to determine the electron energy E_0 in dependence of $\hbar\omega$, B and Θ.

Taking the square of equation (7) (see (9)), substituting E_{n,m_s} and $E_{n',m_s'}$ with equation (6) (see (10)), subtracting the mass m_0 and after some shifting (see (11)) and finally replacing p_z with the expression from equation (8) we arrive at eq. (12):

$$E^2_{n,m_s} + 2\hbar\omega E_{n,m_s} + \hbar^2\omega^2 = E^2_{n',m_s'} \tag{9}$$

$$\begin{aligned} c^4m_0^2 + c^2p^2_{z,0} + |e|\hbar c^2B(2n+1+2m_s) \\ +2\cdot\hbar\omega E_{n,m_s} + \hbar^2\omega^2 \\ = c^4m_0^2 + c^2p_z^2 + |e|\hbar c^2B(2n'+1+2m_s') \end{aligned} \tag{10}$$

$$\begin{aligned} c^2p^2_{z,0} + 2|e|\hbar c^2B(n-n'+m_s-m_s') \\ +2\cdot\hbar\omega E_{n,m_s} + \hbar^2\omega^2 = c^2p_z^2 \end{aligned} \tag{11}$$

$$\begin{aligned} c^2p^2_{z,0} + 2|e|\hbar c^2B(n-n'+m_s-m_s') \\ +2\cdot\hbar\omega E_{n,m_s} + \hbar^2\omega^2 \\ = c^2(p^2_{z,0} + 2\cdot p_{z,0}\frac{\hbar\omega}{c}\cos\Theta + \frac{\hbar^2\omega^2}{c^2}\cos^2\Theta) \end{aligned} \tag{12}$$

We subtract $c^2p^2_{z,0}$ (see (13)), shift some more (see 14) and put $|e|\hbar cB$ all alone on the right hand side, to get eq. (15):

$$\begin{aligned} 2|e|\hbar c^2B(n-n'+m_s-m_s') + 2\cdot\hbar\omega E_{n,m_s} \\ +\hbar^2\omega^2 = 2\cdot cp_{z,0}\hbar\omega\cos\Theta + \hbar^2\omega^2\cos^2\Theta \end{aligned} \tag{13}$$

$$\begin{aligned} 2|e|\hbar c^2B(n-n'+m_s-m_s') + 2\cdot\hbar\omega E_{n,m_s} \\ +\hbar^2\omega^2(1-\cos^2\Theta) = 2\cdot cp_{z,0}\hbar\omega\cos\Theta \end{aligned} \tag{14}$$

$$\begin{aligned} 2\cdot\hbar\omega(E_{n,m_s} - cp_{z,0}\cos\Theta) + \hbar^2\omega^2(\sin^2\Theta) \\ = 2|e|\hbar c^2B(n'-n+m_s'-m_s) \end{aligned} \tag{15}$$

Now, by dividing with m_0c^2 and using the aforementioned cyclotron frequency $\omega_c = \frac{|e|B}{m_0}$, and also $\gamma_0 = \frac{E_{n,m_s}}{m_0c^2} = \frac{E_0}{m_0c^2}$, $cp_{z,0} = \beta_{z,0}E_0 = \beta_0\cos\varphi E_0$ (see (16)). $\beta_0 = \frac{v_0}{c}$ and $\beta_{z,0} = \frac{v_{z,0}}{c}$ are the relativistic velocities of the electron, respectively in z-direction - φ is the angle between them. Divide by 2, get γ_0 in front of the brackets and we receive eq. (17). This is the energy which the photons must have in order to be resonantly absorbed, in short the resonance condition.

$$\begin{aligned} 2\cdot\hbar\omega(\gamma_0 - \beta_{z,0}\gamma_0\cos\Theta) + \frac{\hbar^2\omega^2}{m_0c^2}(\sin^2\Theta) \\ = 2\hbar\omega_c(n'-n+m_s'-m_s) \end{aligned} \tag{16}$$

$$\boxed{\begin{aligned} \hbar\omega\gamma_0(1-\beta_0\cos\varphi\cos\Theta) + \tfrac{\hbar^2\omega^2}{2m_0c^2}(\sin^2\Theta) \\ = \hbar\omega_c(n'-n+m_s'-m_s) \end{aligned}} \tag{17}$$

So far no assumptions or approximations have been made. However, we can neglect spin-flip-transitions ...

$$m_s = m_s' \tag{18}$$

... and higher harmonics, as their transisition probability is neglectible in comparison to the first excitation,

$$(n'-n) = 1 \tag{19}$$

and we receive:

$$\boxed{\begin{aligned} \hbar\omega\left[\gamma_0(1-\beta_0\cos\varphi\cos\Theta) + \tfrac{\hbar\omega}{mc^2}\tfrac{(\sin^2\Theta)}{2}\right] \\ = \hbar\omega_c \end{aligned}} \tag{20}$$

Now we have the resonance condition in a reasonable shape and can resolve it for each parameter. In fig. 4 to 7 we have plots to give a feeling of each dependence, which is important when we come to the influence of parameter's error on the result's error.

4.1 Photon Energy

We receive the photon energy - by simply solving for $\hbar\omega$ in the quadratic equation (20):

$$\hbar\omega = \frac{-a \pm \sqrt{a^2 + 4bc}}{2b} \tag{21}$$
$$\text{with } a = \gamma_0(1 - \beta_0 \cos\varphi \cos\Theta)$$
$$b = \frac{(\sin^2\Theta)}{2m_0c^2}$$
$$c = \hbar\omega_c(n' - n)$$

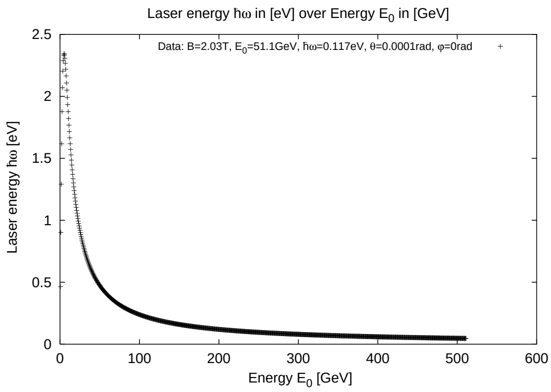

Fig. 4: *Eq.* (21)*: Laser energy $\hbar\omega$ versus energy E_0 - graph obtained with grapher.cpp*

As obvious from equation (20) rather, it goes with $\frac{1}{\gamma_0}$. Higher energies demand lower photon energies. This may translate small errors in $\hbar\omega$ to big ones in E_0 depending on the chosen $\hbar\omega$.

4.2 Magnetic Field Strength B

Resolving for B results to:

$$B = \frac{m\omega}{e} \cdot \big(\gamma_0 - \sqrt{\gamma_0^2 - 1} \cdot \cos\Theta\cos\varphi + \frac{\hbar\omega}{2m_0c^2}\sin^2\Theta\big) \tag{22}$$

Here, higher energies correlate to higher magnetic field strengths. Thus B depends basically linearly on γ_0, thus errors translate linearly as well.

4.3 Laser Beam Angle Θ

Here, in order to simplify the equations, we make the following mathematical approximations as $\Theta << 1$ and also set $\varphi = 0$:

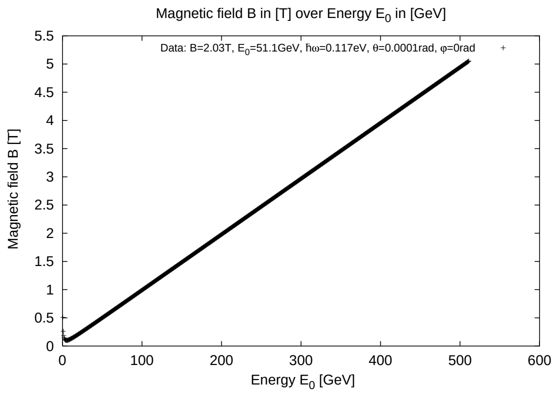

Fig. 5: *Eq.* (22)*: Magnetic field strength B versus energy E_0 - graph obtained with grapher.cpp*

$$\cos\Theta = 1 - \frac{\Theta^2}{2!} + O(\Theta^4) \tag{23}$$
$$\sin\Theta = \Theta - O(\Theta^3) \tag{24}$$
$$\sqrt{1 - \frac{1}{\gamma_0^2}} \approx \gamma_0 \cdot (1 - \frac{1}{2\gamma_0^2})$$

So that we get:

$$\gamma_0 - \sqrt{1 - \frac{1}{\gamma_0^2}} \cdot \cos\Theta + \frac{\hbar\omega}{2m_0^2}\sin^2\Theta = \Omega$$
$$\gamma_0 - \gamma_0(1 - \frac{1}{2\gamma_0^2})(1 - \frac{\Theta^2}{2!}) + \frac{\hbar\omega}{2m_0c^2}\Theta^2 = \Omega$$
$$\gamma_0\frac{\Theta^2}{2} + \frac{1}{2\gamma_0} - \frac{\Theta^2}{4\gamma_0} + \frac{\hbar\omega}{2m_0c^2}\Theta^2 = \Omega$$
$$\Theta^2(\frac{2\gamma_0^2 - 1}{4\gamma_0} + \frac{\hbar\omega}{2m_0^2}) + \frac{1}{2\gamma_0} = \Omega$$
$$\Theta^2 = \frac{\frac{\hbar\omega_c}{\hbar\omega} - \frac{1}{2\gamma_0}}{\frac{2\gamma_0^2-1}{4\gamma_0} + \frac{\hbar\omega}{2m_0c^2}}$$
$$\Theta = \sqrt{\frac{2m_0c^2}{\hbar\omega}\frac{2\gamma_0\hbar\omega_c - \hbar\omega}{2\gamma_0\hbar\omega + 2\gamma_0^2m_0c^2 - m_0c^2}} \tag{25}$$

As for small angles eq. (24) holds, the dependence is like $\frac{1}{\gamma_0}$. High energies call for low laser angles. This is critical as small changes in Θ translate to big changes in $E_0 \approx 500GeV$ as you see in the plot.

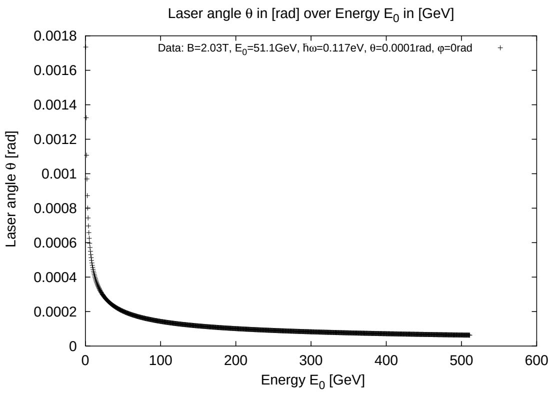

Fig. 6: *Eq. (25): Laser beam angle Θ versus energy E_0 - graph obtained with grapher.cpp*

4.4 Electron Beam Angle φ

For φ we obtain without further effort of simplifying it:

$$\varphi = \sqrt{2 - \frac{\frac{-2m_0c^2\hbar\omega_c + 2m_0c^2\gamma_0\hbar\omega + (\hbar\omega)^2 \sin^2\Theta}{\hbar\omega m_0 c^2}}{\sqrt{\gamma_0^2 - 1}\cdot\cos\Theta}} \quad (26)$$

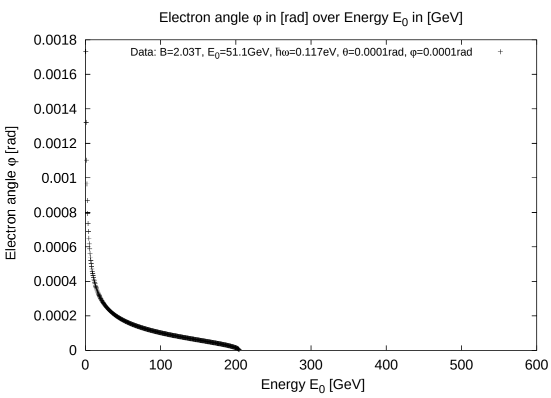

Fig. 7: *Eq. (26): Electron beam angle φ versus energy E_0 - graph obtained with grapher.cpp*

Basically the same as for Θ also applies to φ.

Remind though that here both angles are not equal to 0! (Otherwise the plot would look exactly like fig. 6) One observes that the other angle being finite severely limits the maximum energy that is measurable. Thus we find setting $\varphi = 0$ reasonable, aligning the electron beam in z-direction and regarding only one angle further on.

4.5 Electron Energy γ_0

We can make the following assumptions to drop $\frac{(\sin^2\Theta)}{2}$ in eq. (20) for reasons of simplicity:

$$\frac{(\sin^2\Theta)}{2} \le 1 - \beta_0 \cos\varphi\cos\Theta \quad (27)$$

$$\hbar\omega << \epsilon_0 \quad (28)$$

A brief proof of assummption (27):

$$\frac{\Theta^2}{2} = \frac{\Theta^2}{2}$$

$$\frac{\Theta^2}{2} = 1 - (1 - \frac{\Theta^2}{2})$$

with: $\beta_0 \le 1$, (24) and (25)

$$\frac{\Theta^2}{2} \le 1 - \beta_0(1 - \frac{\Theta^2}{2})$$

$$\frac{\sin\Theta^2}{2} \le 1 - \beta_0(\cos\Theta)$$

and: $\cos\varphi \le 1$ for $\varphi << 1$

$$\frac{\sin\Theta^2}{2} \le 1 - \beta_0\cos\Theta\cos\varphi \quad (29)$$

And we get using eq. (27) & (28):

$$\gamma_0(1 - \beta_0\cos\varphi\cos\Theta) = \frac{\omega_c}{\omega} \quad (30)$$

Furthermore, we know $\beta_0 = \sqrt{1 - \frac{1}{\gamma_0^2}} = \frac{1}{\gamma_0}\sqrt{\gamma_0^2 - 1}$ and write:

$$\gamma_0 - \sqrt{\gamma_0^2 - 1}\cdot\cos\varphi\cos\Theta \approx \frac{\omega_c}{\omega}$$

We turn equation (20) around, which is only valid if both $\Theta \neq 0$ and $\varphi \neq 0$.

$$\boxed{\gamma_0 = \frac{\frac{\omega_c}{\omega} \pm \cos\varphi\cos\Theta\cdot\sqrt{\left(\frac{\omega_c}{\omega}\right)^2 - 1 + (\cos\varphi\cos\Theta)^2}}{1 - (\cos\varphi\cos\Theta)^2}} \quad (31)$$

So that now we have a formula $f(\frac{eB}{m\omega}, \Theta, \varphi) = \gamma_0$. If we are able to find the maximum of absorption, we have found the correct parameters $B, \hbar\omega, \Theta, \varphi$ and thereby know the γ_0-value respectively the electron energy.

The dependence on each parameter you find plotted in fig. 4 to 7, just flip the x- and y-axis.

So far, we have rewritten the resonance condition to eq. (31) and if it is fulfilled we receive an equation which dependens on the parameters φ, Θ, B, $\hbar\omega$.

5 Reasonable Values and Limits

In order to be able to determine the impact of errors translating from the parameter side into the result as indicated before, we have to assume reasonable values for the parameters first.

5.1 The Magnetic Field strength B

Using an iron core in the coils a field strength of no more than $B_{max} \approx 2T$ can be reached before the iron saturates. With superconduction coils we have $B_{max} \approx 6T$. Thus B should remain below this upper limit.

Also there is a lower limit of $B_{min} = 0.004T$ which has to be considered - especially in the "straight away" setup presented later in section 5.4.2.

5.2 The electron beam

The transverse extension of the beam is $\sigma_x = 100\mu m$ in x- and $\sigma_y = 5\mu m$ in y-direction. Its divergence is typically a few nrad. Reasonable values for the electron energy are 45, 100, 250, 400, 500 GeV, translating to $\gamma_0 = 8 \cdot 10^4, 2 \cdot 10^5, 5 \cdot 10^5, 8 \cdot 10^5, 1 \cdot 10^6$.

5.3 Laser Beam

As we want to employ a reliable, easy-to-use laser, we aim for a CO_2 laser. It has a wave length of $\lambda = 10.6\mu m$. It has a power up to 50W or more and can be stabilized over time with a so called "noise eater" up to $\Delta\xi = 5 \cdot 10^{-4}$. Also, it has a manipulable divergence of typically a few mrad and a width of roughly 1mm before focusing.

Both the initial diameter and its divergence can be controlled, by focusing with a concave mirror, respectively via a telescope setup.

Thus the diameter can be decreased down to $\sigma_x = 100\mu m$.

5.4 Beam overlap

There are different approaches to overlap the electron with the laser beam. Let's go through each of them and see their (dis)advantages.

5.4.1 Intersecting

Assumung the situation as indicated in fig. 9:

There is a finite amount of space in the linear collider tunnel for such a measurement station, we can roughly expect no more than 40m. The beam pipe itself has a radius of 20mm. As the laser beam has to be fed into and out of the pipe at some point, only a certain minium angle for Θ is possible.

The electron beam size itself can be neglected here, thus the lower limit would be:

$$\sin\Theta_{limit} \approx \Theta_{limit} = \frac{20mm}{20m} = 10^{-3}\ rad \quad (32)$$

This limit is quite low and might be hard to achieve. Especially, if one takes into account that there is permanent vibration of the whole apparatus. But as you see in fig. 8, we have to get down as low as $1 \cdot 10^{-4}$ rad if we want to measure energies at the linear collider up to 500 GeV.

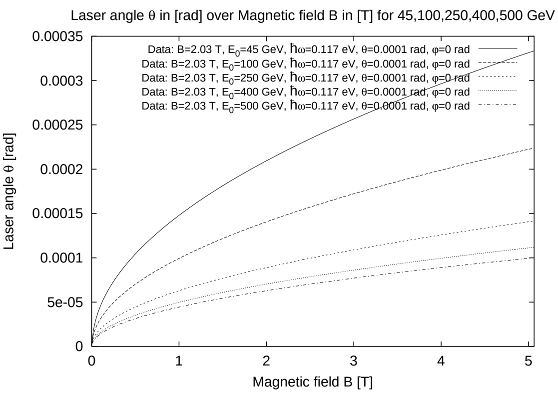

Fig. 8: *Several typical beam energies 45, 100, 250, 400, 500 GeV in a plot Θ vs. magnetic field strength B - graph obtained with grapher.cpp*

However, the advantage is that the overlap between the beams is perfect as a laser beam diameter of $\sigma_x = 100\mu m$ is achievable.

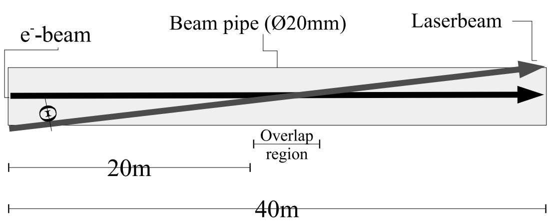

Fig. 9: *Explaining the beam overlap dependence on Θ and restrictions because of finite space in z-direction and beam pipe diameter*

5.4.2 Straight away

Here we assume that by some way it is possible to completely overlap the beams, so that both $\Theta = 0$ and $\varphi = 0$.

However eq. (31) is not valid anymore then. Instead we receive the following, starting from eq. (17) with $\cos\Theta = \cos\Phi = 1$ and $\sin\Theta = 0$:

$$\begin{aligned} \hbar\omega\gamma_0(1-\beta_0) &= \hbar\omega_c \\ \hbar\omega\gamma_0(1-\sqrt{1-\frac{1}{\gamma_0^2}}) &= \hbar\omega_c \\ \hbar\omega(\gamma_0-\gamma_0(1-\frac{1}{2\gamma_0^2})) &= \hbar\omega_c \\ \hbar\omega(\gamma_0-\gamma_0+\frac{1}{2\gamma_0}) &= \hbar\omega_c \\ \hbar\omega\frac{1}{2\gamma_0} &= \hbar\omega_c \\ \gamma_0 &= \frac{m\omega}{2eB} \end{aligned} \tag{33}$$

If we plot eq. (33), we get a correlation as seen in fig. 10.

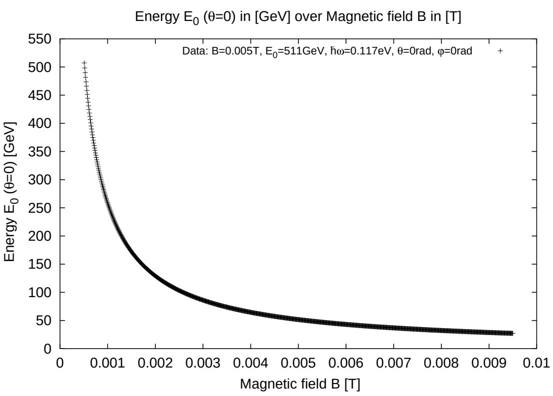

Fig. 10: *Eq. (33), E_0 vs. B plotted at $\Theta = 0$ - graph obtained with grapher.cpp*

One notices that the smaller B, the higher γ_0 become available.

However, on the one hand the overlapping of the beams itself becomes very difficult as it is impossible to reduce the diameter of the laser beam down to $\sigma_y = 5\mu m$ due to its inherent wave length $\lambda = 10.6\mu m$.

On the other hand also the necessary magnetic field strengths are too small to reach energies up to 500GeV. Thus this approach seems unsuitable.

6 On the Resonance Width

In summary we have the formalism for the level splitting. We know about the necessary laser photon energy respectively how to exploit it to obtain the energy of the electron. We have discussed typical values for the parameters involved in eq. (31).

We have stated above in section 4 that the dependencies of E_0 on $\frac{1}{\Theta}$ and $\frac{1}{\omega}$ might be critical depending on the absolute value of Θ and ω.

In order to evaluate this we use eq. (31) and a montecarlo simulation has been written - for a graphical explanation see fig. 11.

Basically, we give each parameter a gaussian distribution with a certain Full Width Half Maximum (FWHM). Also the energy itself has an uncertainty. In oder to study the effects of the error of each parameter, we set all the other errors to 0 and let just parameter vary according to its distribution. This we put into eq. (31), which gives us the resulting γ_0 or $E_0 = \gamma_0 \cdot m_0 c^2$. This value then serves as a weight for the event given by the distribution of the E_0.

So, we pick 1000 times a slightly different e.g. B. Sometimes the resulting E_0 is close, sometimes it is far away. Accordingly it gets a weight. Then we divide the sum of all events by 1000 and get a mean value of how far we were off optimum resonance condition for our given parameters.

The resulting plots serve two purposes. On the one hand by the FWHM of the plot we can tell what influence the parameter's error - in our example B - has on the energy's E_0 error. On the other hand the height of the peak shows how many of all electrons will absorb photons, a relative rate of absorption.

That's why the programme yields a curve of relative rate against the parameter as seen in fig. 12 to 14.

The FWHM of this curve gives us the uncertainty by which we can measure the energy. If the absorbed intensity suddenly drops to a half, the energy will have shifted by just this FHWM, which should be observable.

7 On the Dominating Errors

Now we have a measure of evaluating the precision by which our method would be able to determine the electron energy. Subsequently

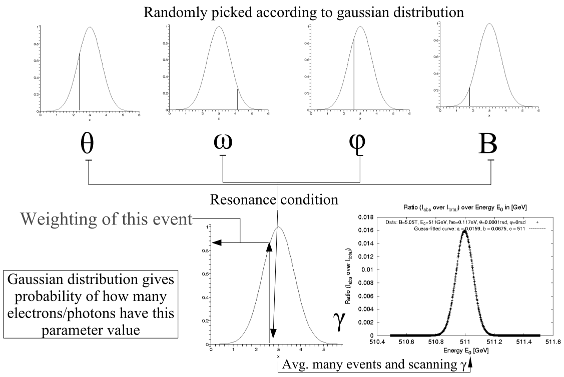

Fig. 11: *Explaining the way of the analytical error calculation. Each parameter is varied randomly according to distribution, yielding e.g. a* γ_0 *by eq.* (31). *This* γ_0 *relates to a certain weigth between 0 and 1 by the gaussian distribution of* γ_0. *This is done many times and averaged per "event"*

we want to know which parameter with its uncertainty has the greatest impact upon our measurement precision.

Thus we did multiple plots with the same parameters, increasing the error of one parameter only until it was dominating, decreasing it until it was neglectible. The remaining FWHMs were set to zero. Thereby we can obtain constrains on the parameters, e.g. how exact the laser must be adjusted in regards of its Θ-angle, what care had to be put on damping or removing vibrations.

Generally, we used the following values and errors: Each MonteCarlo run is done with 1000

parameter x	x	FWHM
λ [m]	$1.06 \cdot 10^{-5}$	0
B [T]	5.05223	0
Θ [rad]	$1 \cdot 10^{-4}$	0
γ_0 [dim.less]	10^6	10^2

randomly picked parameter sets ("particles"). φ has been neglected for reasons mentioned in section 4.4 and because of its symmetry with Θ in eq. (31). In each fig. 12 to 14 from top to bottom the resulting FHWM is given.

Remind that the *errors* $\frac{\Delta x}{x}$ are given *relative* to the values above, *FWHMs* however are given *absolute*.

All these loose constrains - rather upper limits - are gathered in tab. 1. Remember that

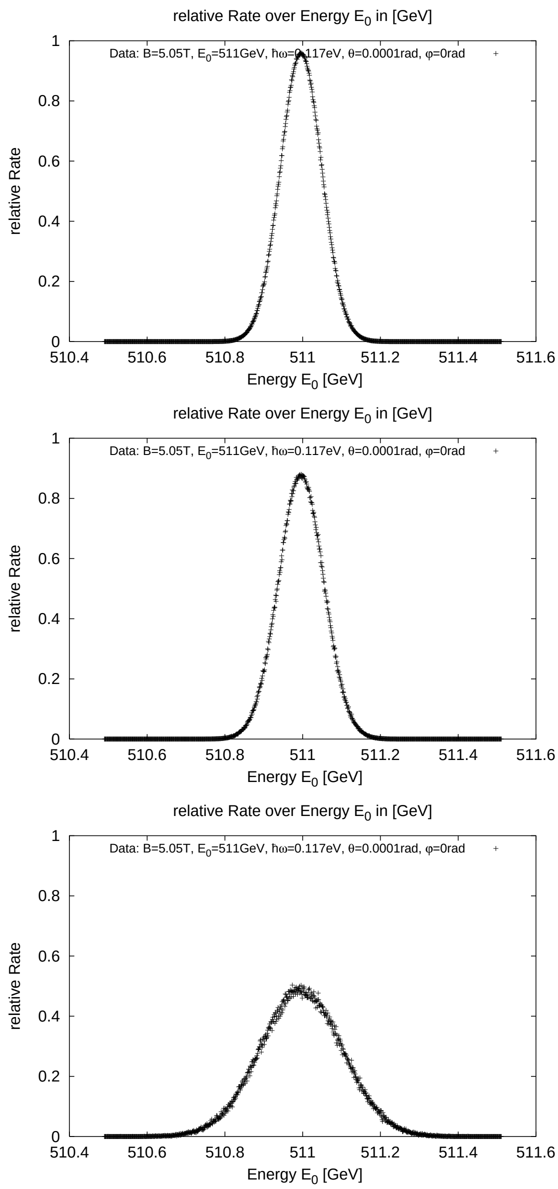

Fig. 12: $\frac{\Delta\omega}{\omega} = 3 \cdot 10^{-5}, 5.5 \cdot 10^{-5}, 1.8 \cdot 10^{-4}$ *and FWHM = 0.0533, 0.0583, 0.105 - graph obtained with montecarlo.cpp*

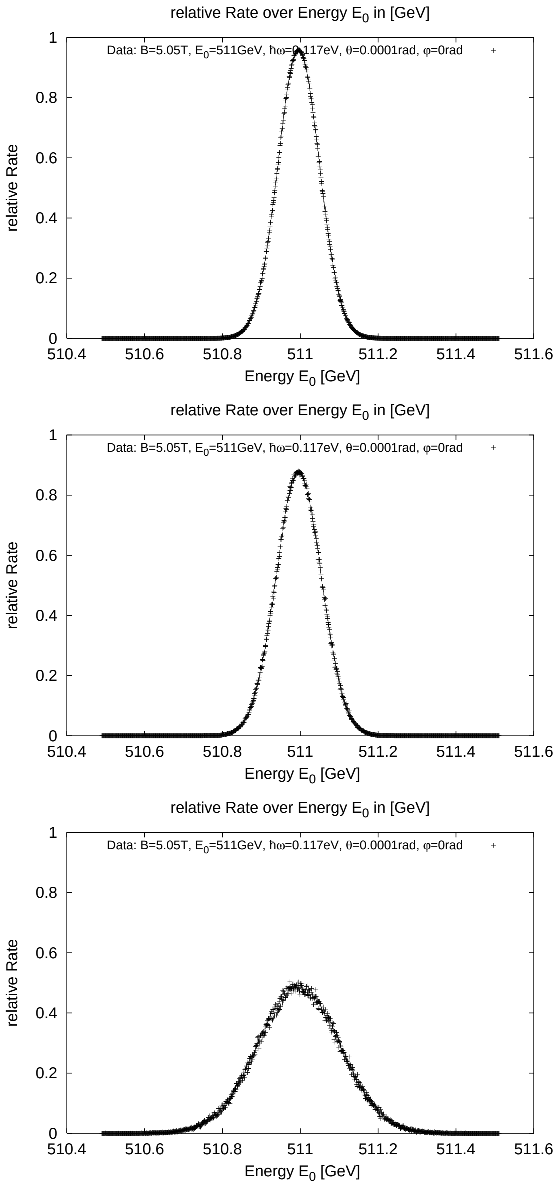

Fig. 13: $\frac{\Delta B}{B} = 3 \cdot 10^{-5}, 5.5 \cdot 10^{-5}, 1.8 \cdot 10^{-4}$ *and FWHM = 0.053, 0.0583, 0.105 - graph obtained with montecarlo.cpp*

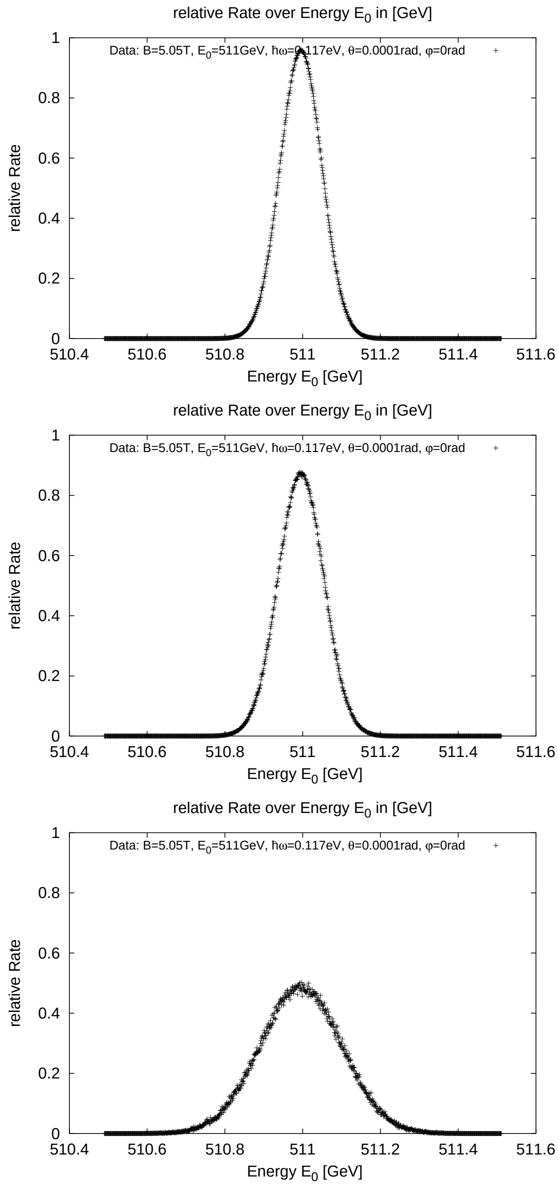

Fig. 14: $\frac{\Delta \Theta}{\Theta} = 1.5 \cdot 10^{-5}, 2.8 \cdot 10^{-5}, 9 \cdot 10^{-4}$ *and FWHM = 0.053, 0.0587, 0.105 - graph obtained with montecarlo.cpp*

energy is paramount, so all constrains are made in such a way, that the FWHM of γ_0 does not change significantly, i.e. changing γ_0's FWHM by more than 1%, 10% or 100% in the plots from top to bottom.

$\frac{\Delta x}{x}$	1%	10%	100%
B [T]	$3 \cdot 10^{-5}$	$5.5 \cdot 10^{-5}$	$1.8 \cdot 10^{-4}$
$\hbar\omega$ [eV]	$3 \cdot 10^{-5}$	$5.5 \cdot 10^{-5}$	$1.8 \cdot 10^{-4}$
Θ [rad]	$1.5 \cdot 10^{-5}$	$2.8 \cdot 10^{-5}$	$9 \cdot 10^{-5}$

Tab. 1: *Table collecting all the loose constrains from the plots 12 to 14. The relative error of each parameter other than γ_0 must have the above value or less in order not to influence the measuring of γ_0 by more than the given significance.*

As you can clearly see the dependencies from section 4 are reproduced. $B = 5.05 \pm 0.00028T$ is achievable, as is $\hbar\omega = 0.117 \pm 0.0000064eV$ as the line width of a laser is very narrow. However, besides its divergence already being far greater, stabilizing the laser angle down to $\Theta = 0.1 \pm 0.0000028mrad$ is impossible. Also the electron beam can not compensate this as its divergence is a lot smaller.

8 The Results

The suggested method is promising. However with our given parameters the error sensitivity of the laser angle Θ is unachievable, regarding natural vibrations from passing trucks and trains. As was already shown in section 5.4 regarding the overlapping of the beams, the chosen Θ is already far below our found limit.

Maybe it is possible to use another laser with a lower photon energy. Although this should be considered with care, as with lower $\hbar\omega$ its error increases in the same way as the one of Θ did (see fig. 4). Maybe there can be a balance found between a reasonably large laser angle and a small photon energy - if a laser providing such exists.

One last plot fig. 15 shall give further food for thought on this, where $\Theta = 1mrad$ was chosen and $\hbar\omega$ lowered until we reach again energies up to $E_0 = 500GeV$.

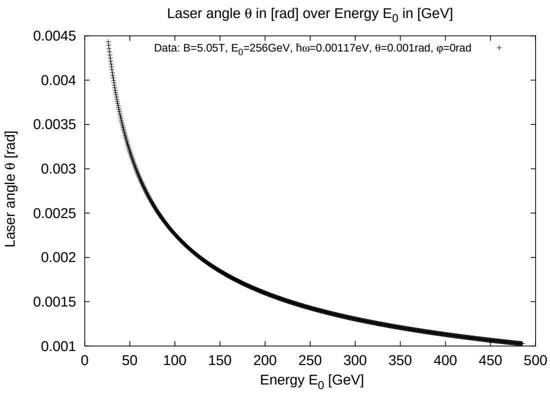

Fig. 15: *Regarding the situation for $\Theta = 1mrad$ - graph obtained with grapgher.cpp*

Also the "straight away" setup of section 5.4.2 might be more feasible here. If a higher laser energy is being used, the magnetic field strength can be increased as well, maintainting the same E_0, see eq. (33). For example a laser in the visible range $\lambda = 600nm$ gives the following E_0 over B-dependence seen in fig. 16. However eq. (33) has to be more carefully derived, giving thought to Θ varying around 0 and making error evaluation possible.

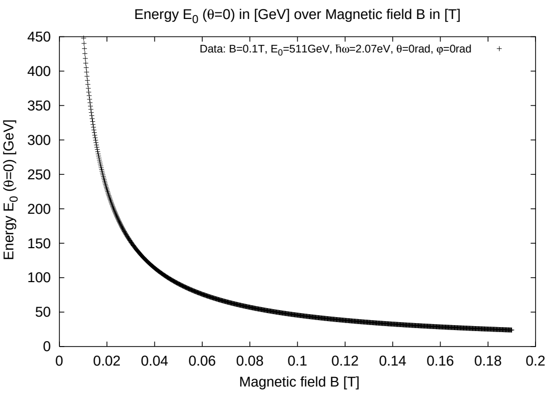

Fig. 16: *"Straight away": $\Theta = 0$ and $\lambda = 600nm$ - graph obtained with grapgher.cpp*

As one clearly notices, one parameter set has failed but there might still be others yielding feasibility.

9 Afterthoughts

9.1 The Programming

I briefly explain my unusual approach on numerical evaluation.

Basically, I evaluate the mentioned formulas analytically, using GNUlib [7]. The given x parameter of the plot is advanced through in a certain width per step, the formula evaluated and the (x,y)-pair printed to the screen. The output is captured and lateron processed with *gnuplot* ([6]), the script controlling gnuplot is generated by the programme. The parameters x and y can be given via command line ($\omega = 1$, $B = 2$, $\Theta = 3$, $\varphi = 4$, $\gamma_0 = 5$). The wrapping and putting together of all is done in a shell script. An example call would look like this:

```
  [comp's name] ~/code % ./evalm.sh 5
  Calculating ... done
{ Here the c++-programme is called }
  Plotting ... done
{ Here gnuplot makes the .ps-file }
  Wanna see (y/n)? y
{ As chosen gv displays the result}
  Displaying Result ...
  [comp's name] ~/code %
```

On the parameter side everything is controlled via a configuration file "....cfg", wherein values and their *relative* errors, step widths and so on are specified, *here montecarlo.cfg*:

```
res_steps        1.5e-6
max_res_steps    1000
max_particles    1000
gamma&error      1e+5    1E-4
lambda&error[m]  10.6E-6 0
Phi              0       0
Theta            2e-4    2e-1
B&error[T]       2.02576 1e-4
```

Here are some of the programmes with function and a brief explanation:

- *absorption* Evaluating the impact of beam divergence on the resonance width. Gives assymetrical resonance curve.
- *fitter* Taking a number of random points from *absoprtion*'s curve and trying a least square fit to get the point of maximum absorption.
- *montecarlo* Is explained in fig. 11 and section 6.
- *grapher* Using the formulas in section 4 and sweeping one specified x parameter range, yielding (x,y)-datapoints.

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References

[1] Robert A.Melikian *Energy Measurement of Relativistic Electron Beams Using Resonance Absorption of Laser Light by Electrons in a Magntic Field*

[2] The TESLA Beam Energy Spectrometer *V.N. Duginov, H. Henke, K. Hiller, A.Liapine and many others*

[3] TESLA Design Report *http://tesla.desy.de/new_pages/.. ../TDR_CD/PartII/accel.html*

[4] L.D Landau, E.M. Lifschitz *Lehrbuch der Theoretischen Physik - Quantenmechanik (Band III)*

[5] I.D. Vagner (Universite Josef Fourier - Max Planck Institute) *http://www.magniel.com/hmtj/.. ../education/lectures/v1.html*

[6] (Thomas Williams, Colin Kelley et al.) gnuplot - http://www.gnuplot.info/

[7] (Mark Galassi et al.) GNU Scientific Library - http://www.gnu.org/software/gsl/